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About functional equation $f(st) = f(s) + f(t)\,(s,\, t \in \mathbf{R} \setminus \{\mathbf{0}\})$

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Abstract. We analyse continuously differentiable functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, that are the solutions of functional equation $f(st) = f(s) + f(t)$. We prove that $f \equiv 0$, and logarithmic functions $f(t) = \log_a |t|$, $(0 < a \neq 1)$ are the only solutions of the equation above.

1. Introduction

The motivation for this paper is the property $\log_a(xy) = \log_a x + \log_a y \quad (x, y > 0;\ \ 0 < a \neq 1)$, of logarithmic function. We want these functions to explain as the solutions of functional equation. Also, we obtain some properties of these solutions.

2. Continuously differentiable solutions of equation $f(st) = f(s) + f(t)$

At first we consider the equation above for $s, t \in \mathbb{R}^+$. Hence, we analyse the equation

$$f(st) = f(s) + f(t)\ (s, t > 0) \ \ (1)$$

We are interesting for continuously differentiable functions $f : \mathbb{R}^+ \to \mathbb{R}$, that are solutions of equation (1). Let S denotes the set of all solutions of (1), and let $f \in S$ be a concrete solution. Obviously, $f(1) = 0$.

The relation $0 = f(1) = f(t \cdot \frac{1}{t}) = f(t) + f(\frac{1}{t})$ implies that $f(\frac{1}{t}) = -f(t)$ for every $t > 0$. Hence, for any fixed $f \in S$, the structure $(\{f(s) : s > 0\}, +)$ is an Abelian group. Note that

$$f(\frac{s}{t}) = f(s \cdot \frac{1}{t}) = f(s) + f(\frac{1}{t}) = f(s) - f(t).$$

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Also, for every $k \in \mathbb{N}$ and $t > 0$, it holds $f(t^k) = kf(t)$. For a fixed $t > 0$ we have

$$\begin{aligned} f'(t) &= \lim_{h\to 0+} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0+} \frac{f\left[t\left(1+\frac{h}{t}\right)\right] - f(t)}{h} \\ &= \lim_{h\to 0+} \frac{f(t) + f\left(1+\frac{h}{t}\right) - f(t)}{h} = \lim_{h\to 0+} \frac{f\left(1+\frac{h}{t}\right)}{h}. \end{aligned}$$

Because $f(1) = 0$ and the function f is continuously differentiable at every point $t > 0$, we have (by L'Hospital's rule) :

$$f'(t) = \lim_{h\to 0+} \frac{f\left(1+\frac{h}{t}\right)}{h} = \lim_{h\to 0+} f'\left(1+\frac{h}{t}\right)\frac{1}{t} = f'(1)\frac{1}{t}.$$

Hence,

$$f'(t) = \frac{f'(1)}{t} \ (t > 0) \ (2)$$

Now, we have

$$f'(st) = \frac{1}{f'(1)} \frac{f'(1)}{s} \frac{f'(1)}{t} = \frac{1}{f'(1)} f'(s) f'(t)$$

For $s = \frac{1}{t}$ we obtain

$$f'(t) f'\left(\frac{1}{t}\right) = [f'(1)]^2$$

The relation $f(t) = -f\left(\frac{1}{t}\right)$ implies $f'(t) = \frac{1}{t^2} f'\left(\frac{1}{t}\right)$, i.e.,

$$f'\left(\frac{1}{t}\right) = t^2 f'(t)$$

Therefore,

$$f'(t) f'\left(\frac{1}{t}\right) = t^2 \left[f'(t)\right]^2 = \left[f'(1)\right]^2$$

From relation (2) we see that the behaviour of f on interval $(0, \infty)$ is dependent of value and sign of number $f'(1)$. Hence, we have the following discussion

a) If $f'(1) = 0$, then $f'(t) = 0$ for all $t > 0$, i.e., $f(t) \equiv const. = C$. From equation (1) we conclude that $C = 0$.

b) If $f'(1) > 0$, then $f'(t) > 0$ for all $t > 0$. Then, f is monotonous increasing function on $(0, \infty)$.

c) If $f'(1) < 0$, then $f'(t) < 0$ for all $t > 0$. Then, f is monotonous decreasing function on $(0, \infty)$.

If $f'(1) > 0$, then, f is strict monotonous increasing on $(0, \infty)$. Therefore, $\lim_{t\to\infty} f(t) = \infty$, and

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} f\left(\frac{1}{s}\right) = -\lim_{s\to\infty} f(s) = -\infty.$$

In the same manner we conclude : If $f'(1) < 0$, then, $\lim_{t\to\infty} f(t) = -\infty$, and $\lim_{t\to 0+} f(t) = +\infty$.

Remark 1
From (2) we see that : If $f'(1) = 0$, then $f \equiv 0$ on $(0, \infty)$. If $f'(1) \neq 0$, then, we have a differential equation $\frac{f'(t)}{f'(1)} = \frac{1}{t}$, and we obtain the solution $f(t) = f'(1)\ln t + C$. Using the fact that $f(1) = 0$ we conclude that $C = 0$, i.e., $f(t) = f'(1)\ln t$. Because $f'(1) \neq 0$, we can write $f'(1) = \frac{1}{\ln a}$, for some $a \neq 1$. If $f'(1) > 0$, then $a \in (1, +\infty)$, if $f'(1) < 0$, then $a \in (0, 1)$. Hence, $f(t) = \frac{\ln t}{\ln a} = \log_a t$.

Conclusion The only continuously differentiable solutions of equation (1) are the functions : $f \equiv 0$, and the logarithmic functions $f(t) = \log_a t \quad (0 < a \neq 1)$.

Remark 2 If we consider the solutions of functional equation (1) that are defined on $\mathbb{R}\setminus\{0\}$, then $0 = f(1) = f((-1)(-1)) = f(-1) + f(-1) = 2f(-1)$ implies $f(-1) = 0$. Also, from $0 = f(-1) = f(\frac{-s}{s}) = f(-s) - f(s)$, we conclude that $f(-s) = f(s)$ for every $s \in \mathbb{R}\setminus\{0\}$. Hence, f is a even function.

Corollary The set of all solutions of functional equation $f(st) = f(s) + f(t)$, that are defined and continuously differentiable on $\mathbb{R}\setminus\{0\}$, consists of the following functions : a) $f \equiv 0$, b) $f(t) = \log_a |t|$ $(a > 1)$, c) $f(t) = \log_a |t|$ $(0 < a < 1)$.

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