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CYCLIC CONTRACTION RESULT IN 2-MENGER SPACE

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Abstract. In this paper we introduce and establish a cyclic contraction result in probabilistic 2-metric spaces. A control function has been utilized in our theorem. This result generalizes some existing results in 2-metric spaces. Our result is illustrated with an example.

1. Introduction

Fixed point theory has an important role in modern mathematics. In 1922, S. Banach [**1**] proved the well known Banach contraction mapping principle in metric spaces. This contraction mapping principle is one of the pivotal results of mathematical analysis. Its importance lies in its vast applications in a number of branches of modern mathematics.

The concept of metric space has been extended in various ways. One such extension has been made by Gähler [**14**] in which a positive real number is assigned to every three elements of the space. He introduced the following important definition of 2-metric space.

Definitioin 1.1. 2-metric space [14, 15]
Let X be a non empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if

(i) given distinct elements $x, y \in X$, there exists an element $z \in X$ such that

$$d(x, y, z) \neq 0,$$

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(ii) $d(x, y, z) = 0$ when at least two of x, y, z are equal,
(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z \in X$ and
(iv) $d(x, y, z) \leqslant d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

When d is a 2-metric on X, the ordered pair (X, d) is called a 2-metric space.
In 1972 Sehgal and Bharucha-Reid [**33**] generalized the Banach contraction mapping principle to probabilistic metric spaces. Probabilistic metric spaces are probabilistic generalization of metric spaces. In this space, instead of a nonnegative real number, every pair of elements is assigned to a distribution function. The inherent flexibility of these spaces allows us to extend the contraction mapping principle in more than one inequivalent ways.

Definitioin 1.2. [**18, 32**] A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t \in R} F(t) = 0$ and $\sup_{t \in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.

Definitioin 1.3. Probabilistic metric space [18, 32]
A probabilistic metric space (briefly, PM-space) is an ordered pair (X, F), where X is a non empty set and F is a mapping from $X \times X$ into the set of all distribution functions. The function $F_{x,y}$ is assumed to satisfy the following conditions for all $x, y, z \in X$,

(i) $F_{x,y}(0) = 0$,
(ii) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $F_{x,y}(t) = F_{y,x}(t)$ for all $t > 0$,
(iv) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$ then $F_{x,z}(t_1 + t_2) = 1$ for all $t_1, t_2 > 0$.

Menger space is a particular type of probabilistic metric space in which the triangular inequality is postulated with the help of a t-norm.
Shi, Ren and Wang give the following definition of n-th order t-norm.

Definitioin 1.4. n-th order t-norm [34]
A mapping $T : \Pi_{i=1}^{n}[0, 1] \to [0, 1]$ is called a n-th order t-norm if the following conditions are satisfied:

(i) $T(0, 0, ..., 0) = 0$, $T(a, 1, 1, ..., 1) = a$ for all $a \in [0, 1]$,
(ii) $T(a_1, a_{2,}, a_3, ..., a_n) = T(a_2, a_1, a_3, ..., a_n) = T(a_2, a_3, a_1, ..., a_n)$
$= .... = T(a_2, a_3, a_4, ..., a_n, a_1)$,
(iii) $a_i \geqslant b_i$, i=1,2,3,....,n implies $T(a_1, a_2, a_3, ..., a_n) \geqslant T(b_1, b_2, b_3, ..., b_n)$,
(iv) $T(T(a_1, a_2, a_3, ..., a_n), b_2, b_3, ...b_n)$
$= T(a_1, T(a_2, a_3, ..., a_n, b_2), b_3, ..., b_n)$
$= T(a_1, a_2, T(a_3, a_4 ...., a_n, b_2, b_3), b_4, ..., b_n)$
=...............................
$= T(a_1, a_2, ..., a_{n-1}, T(a_n, b_2, b_3, ..., b_n))$.

When $n = 2$, we have a binary t-norm, which is commonly known as t-norm.

Definitioin 1.5. Menger space [18, 32]
A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function

defined on $X \times X$ to the set of all distribution functions and Δ is a 2nd order t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$,
(ii) $F_{x,y}(s) = 1$ for all $s > 0$ if and only if $x = y$,
(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in X,\ s > 0$ and
(iv) $F_{x,y}(u+v) \geqslant \Delta\left(F_{x,z}(u), F_{z,y}(v)\right)$ for all $u, v \geqslant 0$ and $x, y, z \in X$.

The theory of Menger spaces is an important part of stochastic analysis. Schweizer and Sklar have given a comprehensive account of several aspects of such spaces in [**32**].
Probabilistic 2-metric space is the probabilistic generalization of 2-metric spaces.

Wen-Zhi Zeng [**37**] first introduced the concept of probabilistic 2-metric space.

Definitioin 1.6. probabilistic 2-metric space [37]
A probabilistic 2-metric space is an order pair (X, F) where X is an arbitrary set and F is a mapping from $X \times X \times X$ into the set of all distribution functions such that the following conditions are satisfied.

(i) $F_{x,y,z}(t) = 0$ for $t \leqslant 0$ and for all $x, y, z \in X$,
(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ iff at least two of x, y, z are equal,
(iii) for distinct points $x, y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ for $t > 0$,
(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$ for all $x, y, z \in X$ and $t > 0$,
(v) $F_{x,y,w}(t_1) = 1$, $F_{x,w,z}(t_2) = 1$ and $F_{w,y,z}(t_3) = 1$ then $F_{x,y,z}(t_1+t_2+t_3) = 1$, for all $x, y, z, w \in X$ and $t_1, t_2, t_3 > 0$.

A special case of the above definition is the following.

Definitioin 1.7. 2-Menger space [17]
Let X be a nonempty set. A triplet (X, F, Δ) is said to be a 2-Menger space if F is a mapping from $X \times X \times X$ into the set of all distribution functions satisfying the following conditions:

(i) $F_{x,y,z}(0) = 0$,
(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ if and only if at least two of $x, y, z \in X$ are equal,
(iii) for distinct points $x, y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ for $t > 0$,
(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$, for all $x, y, z \in X$ and $t > 0$,
(v) $F_{x,y,z}(t) \geqslant \Delta(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3))$

where $t_1, t_2, t_3 > 0$, $t_1 + t_2 + t_3 = t$, $x, y, z, w \in X$ and Δ is the 3rd order t-norm.

Definitioin 1.8. [**17**] A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be converge to a limit x if given $\epsilon > 0, 0 < \lambda < 1$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,x,a}(\epsilon) \geqslant 1 - \lambda \tag{1.1}$$

for all $n > N_{\epsilon,\lambda}$ and for every $a \in X$.

Definitioin 1.9. [**17**] A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be a Cauchy sequence in X if given $\epsilon > 0, 0 < \lambda < 1$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,x_m,a}(\epsilon) \geqslant 1 - \lambda \tag{1.2}$$

for all $m, n > N_{\epsilon,\lambda}$ and for every $a \in X$.

Definitioin 1.10. [**17**] A 2-Menger space (X, F, Δ) is said to be complete if every Cauchy sequence is convergent in X.

Several results of metric fixed point theory has been extended to these spaces. Some of the fixed point results in 2-metric spaces are [**19**, **21**, **24**, **26**, **27**, **29**] while the references [**2**, **6**, **16**, **17**, **35**] are some fixed point results in probabilistic 2-metric spaces.

In 1984 Khan, Swaleh and Sessa introduced a new category of contractive fixed point problems in metric spaces [**22**]. They introduced the concept of "altering distance function", which is a control function that alters the distance between two points in a metric space. This concept was further generalized in a number of works. There are several works in metric fixed point theory involving altering distance function, some of these are noted in [**28**, **30**] and [**31**].

Recently first two authors of the present paper had extended the concept of altering distance function to the context of Menger spaces in [**3**]. They have introduced the Φ-function. The definition is as follows:

Definitioin 1.11. Φ-function [3]

A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left continuous in $(0, \infty)$,
(iv) ϕ is continuous at 0.

With the help of Φ-function Choudhury and Das [**3**] introduced a new type of contraction mapping in Menger spaces which is known as ϕ-contraction. The idea of this control function has opened new possibilities of proving more fixed point results in Menger spaces. This concept has also applied to a coincidence point problems. Some recent results using Φ-function are noted in [**4**, **5**, **7**, **8**, **11**, **12**] and [**25**].

Recently cyclic contraction and cyclic contractive type mappings have been appeared in literature. Kirk, Srinivasan and Veeramani [**23**] initiated this line of research in metric spaces.

Definitioin 1.12. [**23**] Let A and B be two non-empty sets. A cyclic mapping is a mapping $T : A \bigcup B \to A \bigcup B$ which satisfies:

$TA \subseteq B$ and $TB \subseteq A$.

Kirk, Srinivasan and Veeramani [**23**], amongst other results, established the following generalization of the contraction mapping principle.

THEOREM 1.1. *[**23**] Let A and B be two non-empty closed subsets of a complete metric space X and suppose $f : X \to X$ satisfies:*

(1) $fA \subseteq B$ and $fB \subseteq A$,

(2) $d(fx, fy) \leqslant kd(x, y)$ forall $x \in A$ and $y \in B$ where $k \in (0, 1)$.

Then f has a unique fixed point in $A \bigcap B$.

The problems of cyclic contractions have been strongly associated with proximity point problems. Some other results dealing with cyclic contractions and proximity point problems may be noted in [**10**, **13**, **20**, **36**, **38**] and [**39**].

The present authors introduced a ϕ-contraction in the context of 2-Menger spaces for two mappings in [**9**]. The following theorem was established.

THEOREM 1.2. *[**9**] Let (X, F, Δ) be a complete 2-Menger space, where Δ is the minimum t- norm, T_1, T_2 are two self maps on X such that for all x, y, a in X and $t > 0$,*

$$F_{T_1x,T_2y,a}(\phi(t)) \geqslant F_{x,y,a}(\phi(\frac{t}{c})) \tag{1.3}$$

where $c \in (0, 1)$ and ϕ is a Φ-function. Then T_1 and T_2 have a unique common fixed point in X.

In this paper we define another contraction, namely, a cyclic contraction in 2-Menger spaces and have shown that in a 2-Menger space with minimun t-norm, the said contraction has a unique fixed point. Our theorem is supported with an example.

2. Main Result

THEOREM 2.1. *Let (X, F, Δ) be a complete 2-Menger space with the 3rd order minimum t-norm Δ and let there exist two non-empty closed subsets A and B of X such that the mapping $T : A \bigcup B \to A \bigcup B$ which satisfies the following conditions:*

$$TA \subseteq B \quad and \quad TB \subseteq A \tag{2.1}$$

$$F_{Tx,Ty,a}(\phi(t)) \geqslant F_{x,y,a}(\phi(\frac{t}{c})) \tag{2.2}$$

for all $x \in A$, $y \in B$ and $a \in X$ where $0 < c < 1$, ϕ is a ϕ-function. Then $A \bigcap B$ is non-empty and T has a unique fixed point in $A \bigcap B$.

PROOF. Let x be an arbitrary point of A. Now we construct the sequence $\{x_n\}_{n=1}^{\infty}$ in X by $x_n = T^n x$, $n \in N$, where N is the set of natural numbers. As $x \in A$, $Tx \in B$, $T^2x \in A$, $T^3x \in B$ and in general we obtain

$$T^{2n}x = x_{2n} \in A \quad and \quad T^{2n+1}x = x_{2n+1} \in B \quad for \quad all \quad n \geqslant 0. \tag{2.3}$$

For any non-negative integer n and for fixed $a \in X$, we have

$$F_{T^{2n+1}x,T^{2n+2}x,a}(\phi(t)) = F_{TT^{2n}x,TT^{2n+1}x,a}(\phi(t))$$

$$\geqslant F_{T^{2n}x,T^{2n+1}x,a}(\phi(\frac{t}{c})). \tag{2.4}$$

(by (2.2) and (2.3))

Again, for any $t > 0$, for fixed $a \in X$ and $n \geqslant 0$, we have

$$\begin{aligned} F_{T^{2n}x,T^{2n+1}x,a}(\phi(t)) &= F_{TT^{2n-1}x,TT^{2n}x,a}(\phi(t)) \\ &= F_{TT^{2n}x,TT^{2n-1}x,a}(\phi(t)) \\ &\geqslant F_{T^{2n}x,T^{2n-1}x,a}(\phi(\tfrac{t}{c})) \\ &= F_{T^{2n-1}x,T^{2n}x,a}(\phi(\frac{t}{c})). \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), for all $n \geqslant 0$, $t > 0$ and for some $a \in X$, we have

$$F_{x_n,x_{n+1},a}(\phi(t)) \geqslant F_{x_{n-1},x_n,a}(\phi(\frac{t}{c})). \tag{2.6}$$

By successive application of the above inequality for some $a \in X$, $n \geqslant 0$ and for all $t > 0$, we have

$$F_{x_n,x_{n+1},a}(\phi(t)) \geqslant F_{x_0,x_1,a}(\phi(\tfrac{t}{c^n})).$$

Taking limit on both sides as $n \to \infty$ for all $t > 0$, we have from above inequality

$$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(\phi(t)) = 1. \tag{2.7}$$

By virtue of property of ϕ and F we can choose $s > 0$ such that $s > \phi(t)$. Then, for all $a \in X$ and $t > 0$, we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(s) = 1. \tag{2.8}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then, there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find some $a \in X$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\epsilon) < 1 - \lambda. \tag{2.9}$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (2.9), so that

$$F_{x_{m(k)},x_{n(k)-1},a}(\epsilon) \geqslant 1 - \lambda. \tag{2.10}$$

If $\epsilon_1 < \epsilon$ then, we have

$$F_{x_{m(k)},x_{n(k)},a}(\epsilon_1) \leqslant F_{x_{m(k)},x_{n(k)},a}(\epsilon).$$

From the above, we conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) > k$ and satisfying (2.9), (2.10) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\phi(\epsilon_2)) < 1 - \lambda \tag{2.11}$$

and

$$F_{x_{m(k)},x_{n(k)-1},a}(\phi(\epsilon_2)) \geqslant 1-\lambda. \tag{2.12}$$

Now, we have the following possible cases.

Case-I: $m(k)$ is odd and $n(k)$ is even for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ where $m(l)$ is odd and $n(l)$ is even for all l with $n(l) > m(l) > l$ such that for some $a \in X$,

$$F_{x_{m(l)},x_{n(l)},a}(\phi(\epsilon_2)) < 1-\lambda \tag{2.13}$$

and

$$F_{x_{m(l)},x_{n(l)-1},a}(\phi(\epsilon_2)) \geqslant 1-\lambda. \tag{2.14}$$

Now, from (2.13), for some $a \in X$ and for $\epsilon_2 > 0$, we have

$$\begin{aligned} 1-\lambda &> F_{x_{m(l)},x_{n(l)},a}(\phi(\epsilon_2)) \\ &= F_{T^{m(l)}x,T^{n(l)}x,a}(\phi(\epsilon_2)) \\ &= F_{TT^{m(l)-1}x,TT^{n(l)-1}x,a}(\phi(\epsilon_2)) \\ &\geqslant F_{T^{m(l)-1}x,T^{n(l)-1}x,a}(\phi(\tfrac{\epsilon_2}{c})) \\ &\text{(by (2.2) and (2.3))} \\ &= F_{x_{m(l)-1},x_{n(l)-1},a}(\phi(\frac{\epsilon_2}{c})). \end{aligned} \tag{2.15}$$

By virtue of property of ϕ, we can choose $s_1, s_2 > 0$ such that $\phi(\frac{\epsilon_2}{c}) = \phi(\epsilon_2)+s_1+s_2$. By (2.15), for all $a \in X$ and $\epsilon_2 > 0$, we have

$$\begin{aligned} 1-\lambda &> F_{x_{m(l)-1},x_{n(l)-1},a}(\phi(\tfrac{\epsilon_2}{c})) \\ &\geqslant \Delta(F_{x_{m(l)-1},x_{n(l)-1},x_{m(l)}}(s_1), F_{x_{m(l)-1},x_{m(l)},a}(s_2), F_{x_{m(l)},x_{n(l)-1},a}(\phi(\epsilon_2))) \end{aligned}$$

$$\geqslant \Delta(F_{x_{m(l)-1},x_{m(l)},x_{n(l)-1}}(s_1), F_{x_{m(l)-1},x_{m(l)},a}(s_2), F_{x_{m(l)},x_{n(l)-1},a}(\phi(\epsilon_2))). \tag{2.16}$$

Using (2.8), for all $a \in X$ and for sufficiently large l, we have

$$F_{x_{m(l)-1},x_{m(l)},x_{n(l)-1}}(s_1) \geqslant 1-\lambda \tag{2.17}$$

and

$$F_{x_{m(l)-1},x_{m(l)},a}(s_2) \geqslant 1-\lambda. \tag{2.18}$$

Now, using (2.14), (2.17), (2.18) in (2.16), for all $a \in X$ and $\epsilon_2 > 0$, we have

$$1-\lambda > \Delta(1-\lambda, 1-\lambda, 1-\lambda) = 1-\lambda,$$

which is a contradiction.

Case-II: The integers $m(k)$ is even and $n(k)$ is odd for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ where $m(l)$ is even and $n(l)$ is odd for all l with $n(l) > m(l) > l$ such that for some $a \in X$, (2.13), (2.14) hold.
Then, we arrive at a contradiction exactly as in the Case-I above.

Case-III: The integers $m(k)$ and $n(k)$ both are even for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ where $m(l)$

and $n(l)$ both are even for all l with $n(l) > m(l) > l$ such that for some $a \in X$, (2.13), (2.14) hold.
By virtue of the property of ϕ, we can choose $\eta_1, \eta_2 > 0$ such that $\phi(\epsilon_2) > \eta_1 + \eta_2$.
Now, from (2.13) for all $a \in X$ and for $\epsilon_2 > 0$, we have
$1 - \lambda > F_{x_{m(l)},x_{n(l)},a}(\phi(\epsilon_2))$, that is,

$$1 - \lambda > \Delta(F_{x_{m(l)},x_{n(l)},x_{m(l)+1}}(\eta_1), F_{x_{m(l)},x_{m(l)+1},a}(\eta_2), F_{x_{m(l)+1},x_{n(l)},a}(\phi(\epsilon_2) - \eta_1 - \eta_2)). \tag{2.19}$$

Again, by virtue of property of ϕ, we can choose $0 < \epsilon_3 < \epsilon_2$ such that
$\phi(\epsilon_2) - \eta_1 - \eta_2 = \phi(\epsilon_3)$ and $\frac{\epsilon_3}{c} \geqslant \epsilon_2$ where $0 < c < 1$.
Now, from (2.19) for all $a \in X$, we have

$$1-\lambda > \Delta(F_{x_{m(l)},x_{m(l)+1},x_{n(l)}}(\eta_1), F_{x_{m(l)},x_{m(l)+1},a}(\eta_2), F_{x_{m(l)+1},x_{n(l)},a}(\phi(\epsilon_3))). \tag{2.20}$$

For $\epsilon_3 > 0$, for all $a \in X$, we obtain

$$\begin{aligned} F_{x_{m(l)+1},x_{n(l)},a}(\phi(\epsilon_3)) &= F_{TT^{m(l)}x,TT^{n(l)-1}x,a}(\phi(\epsilon_3)) \\ &\geqslant F_{T^{m(l)}x,T^{n(l)-1}x,a}(\phi(\tfrac{\epsilon_3}{c})) \text{ (by (2.2) and (2.3))} \\ &= F_{x_{m(l)},x_{n(l)-1},a}(\phi(\tfrac{\epsilon_3}{c})) \\ &\geqslant F_{x_{m(l)},x_{n(l)-1},a}(\phi(\epsilon_2)) \\ &\geqslant 1 - \lambda. \quad (by \quad (2.14)) \end{aligned} \tag{2.21}$$

Again, by (2.8) for sufficiently large l and for all $a \in X$, we have

$$F_{x_{m(l)},x_{m(l)+1},x_{n(l)}}(\eta_1) \geqslant 1 - \lambda \tag{2.22}$$

and

$$F_{x_{m(l)},x_{m(l)+1},a}(\eta_2) \geqslant 1 - \lambda. \tag{2.23}$$

Using (2.21), (2.22), (2.23) in (2.20) for all $a \in X$, we obtain

$$1 - \lambda > \Delta(1 - \lambda, 1 - \lambda, 1 - \lambda) = 1 - \lambda,$$

which is a contradiction.

Case-IV: The integers $m(k)$ and $n(k)$ both are odd for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ where $m(l)$ and $n(l)$ both are odd for all l with $n(l) > m(l) > l$ such that for some $a \in X$, (2.13), (2.14) hold.
Then, we arrive at a contradiction exactly as in the Case-III above.

Combining all the above four cases we can conclude that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, we have $x_n \to z$ in X for $n \to \infty$. The subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ of $\{x_n\}$ also converges to z. Now $\{x_{2n}\} \subset A$ and A is closed. Therefore $z \in A$. Similarly, $\{x_{2n-1}\} \subset B$ and B is closed. Therefore $z \in B$. Thus we have $z \in A \bigcap B$.

Now we prove that $Tz = z$.
For this, we have

$$F_{z,Tz,a}(\phi(t)) \geqslant \Delta(F_{z,Tz,x_{2n+1}}(s_1), F_{z,x_{2n+1},a}(s_2), F_{x_{2n+1},Tz,a}(\phi(t) - s_1 - s_2)). \tag{2.24}$$

(where $s_1, s_2 > 0$ and $\phi(t) > s_1 + s_2$)
Now, by the property of ϕ we can choose $\xi_1, \xi_2 > 0$ such that $s_1 = \phi(\xi_1)$ and $\phi(t) - s_1 - s_2 = \phi(\xi_2)$.
Now, from (2.24), we get

$$\begin{aligned} F_{z,Tz,a}(\phi(t)) &\geqslant \Delta(F_{z,Tz,TT^{2n}x}(\phi(\xi_1)), F_{z,x_{2n+1},a}(s_2), F_{TT^{2n}x,Tz,a}(\phi(\xi_2))) \\ &= \Delta(F_{TT^{2n}x,Tz,z}(\phi(\xi_1)), F_{z,x_{2n+1},a}(s_2), F_{TT^{2n}x,Tz,a}(\phi(\xi_2))). \end{aligned}$$

Now, using the inequality (2.2) we get

$$F_{z,Tz,a}(\phi(t)) \geqslant \Delta(F_{T^{2n}x,z,z}(\phi(\tfrac{\xi_1}{c})), F_{z,x_{2n+1},a}(s_2), F_{T^{2n}x,z,a}(\phi(\tfrac{\xi_2}{c}))).$$

By the property of ϕ and F we have

$$F_{T^{2n}x,z,z}(\phi(\tfrac{\xi_1}{c})) = 1.$$

Hence

$$F_{z,Tz,a}(\phi(t)) \geqslant \Delta(1, F_{z,x_{2n+1},a}(s_2), F_{x_{2n},z,a}(\phi(\tfrac{\xi_2}{c}))).$$

Taking limit as $n \to \infty$ and by the property of F, we get

$$F_{z,Tz,a}(\phi(t)) \geqslant \Delta(1, 1, 1) = 1.$$

Hence $z = Tz$.

To prove the uniqueness of the fixed point, let v be another fixed point of T in $A \bigcap B$, that is, $Tv = v$.
Let $a \in X$ be any element different from z and v.

Now,

$$\begin{aligned} F_{z,v,a}(\phi(t)) &= F_{Tz,Tv,a}(\phi(t)) \\ &\geqslant F_{z,v,a}(\phi(\tfrac{t}{c})) \\ &= F_{Tz,Tv,a}(\phi(\tfrac{t}{c})) \\ &\geqslant F_{z,v,a}(\phi(\tfrac{t}{c^2})). \end{aligned}$$

Repeating this process n times we get

$$F_{z,v,a}(\phi(t)) = F_{Tz,Tv,a}(\phi(t)) \geqslant F_{z,v,a}(\phi(\tfrac{t}{c^n})).$$

Letting $n \to \infty$ on both sides we get from the above inequality,

$$F_{z,v,a}(\phi(t)) \geqslant F_{z,v,a}(\phi(\tfrac{t}{c^n})) \to 1.$$

(since ϕ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$)
Hence, $F_{z,v,a}(\phi(t)) = 1$, which implies that $z = v$.

Hence the fixed point is unique. □

Example 2.1. Let $X = \{\alpha, \beta, \gamma, \delta\}$, $A = \{\alpha, \beta, \delta\}$, $B = \{\gamma, \delta\}$, the t-norm Δ is a 3rd order minimum t-norm and F be defined as

$$F_{\alpha,\beta,\gamma}(t) = F_{\alpha,\beta,\delta}(t) = \begin{cases} 0, & \text{if} \quad t \leqslant 0, \\ 0.40, & \text{if} \quad 0 < t < 4, \\ 1, & \text{if} \quad t \geqslant 4, \end{cases}$$
$$F_{\alpha,\gamma,\delta}(t) = F_{\beta,\gamma,\delta}(t) = \begin{cases} 0, & \text{if} \quad t \leqslant 0, \\ 1, & \text{if} \quad t > 0. \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. If we define $T : X \to X$ as follows: $T\alpha = \delta, T\beta = \gamma, T\gamma = \delta, T\delta = \delta$ then the mapping T satisfies all the conditions of the Theorem 2.1 where $\phi(t) = t$, $0 < c < 1$ and δ is the unique fixed point of T in $A \bigcap B$.

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