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APPROXIMATING COMMON FIXED POINTS OF FINITE FAMILY OF ASYMPTOTICALLY NONEXPANSIVE NON-SELF MAPPINGS

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ABSTRACT. The aim of this paper is to study the strong convergence of an implicit iteration process to a common fixed point for a finite family of asymptotically nonexpansive nonself mappings in a uniformly convex Banach spaces.

1. Introduction and Preliminaries

Let K be a nonempty closed convex subset of a Banach space E. A self mapping $T: K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$; $u_n \to 0 \; as \; n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$\|T^n x - T^n y\| \leqslant (1 + u_n)\|x - y\| \quad \forall\, n \geqslant 1 \tag{1.1}$$

T is called uniformly $L-$Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leqslant L\|x - y\| \quad \forall\, n \geqslant 1 \tag{1.2}$$

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk ([**9**]), as an important generalization of the class of nonexpansive maps, who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self mapping of K, then T has a fixed point. Iterative techniques for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authors (See [**19, 2, 3, 4, 17, 13, 5, 18, 1**]) using the Mann iteration method (See e.g.[**19**]) or the Ishikawa iteration method (See e.g. [**15**]).
In 1978, Bose ([**15**]) proved that if K is a bounded closed convex nonempty subset of a uniformly convex Banach space E satisfying Opial's ([**22**]) condition and $T : K \to$

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K is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at $x \in K$, i.e. $\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0$. Passty ([**5**]) and also Xu ([**7**]) proved that the requirement that T satisfies Opial's condition can be replaced by the condition that E has a Frechet differentiable norm. Furthermore, Tan and Xu ([**10, 11**]) later proved that the asymptotic regularity of T can be weakend to the weakly asymptotic regularity of T at x *i.e.* $\omega - \lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$.
In all the above results, the operator T remains a self mapping of a nonempty closed convex subset K of a uniformly convex Banach space E. If, however, the domain of T, $D(T)$ is a proper subset of E, and T maps $D(T)$ into E, then the iteration process of Mann and Ishikawa studied by these authors.
The purpose of this paper is to construct a multistep iterative scheme with errors for approximating common fixed point of a finite family of asymptotically nonexpansive nonself mappings and to prove strong convergence theorems for such maps.
Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Then for arbitrary $x_1 \in K$, we define the sequence $\{x_n\}$ iteratively as follows:

$$(1.3) \qquad \begin{cases} x_n^1 = P(\alpha_n^1 x_n + \beta_n^1 T_1^n x_n + \gamma_n^1 u_n^1) \\ x_n^2 = P(\alpha_n^2 x_n + \beta_n^2 T_2^n x_n^1 + \gamma_n^2 u_n^2) \\ \cdots = \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots = \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_{n+1} = x_n^{(N)} = P(\alpha_n^N x_n + \beta_n^N T_N^n x_n^{N-1} + \gamma_n^N u_n^N) \qquad \forall\, n \geqslant 1 \end{cases}$$

where $\{\alpha_n^1\}, \{\alpha_n^2\}, \cdots\cdots, \{\alpha_n^N\}, \{\beta_n^1\}, \{\beta_n^2\}, \cdots\cdots, \{\beta_n^N\}, \{\gamma_n^1\}, \{\gamma_n^2\}, \cdots\cdots, \{\gamma_n^N\}$ are sequences in $[0, 1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, 3, \cdots\cdots N$ and $\{u_n^1\}, \{u_n^2\}, \cdots\cdots, \{u_n^N\}$ are bounded sequences in K.

Definition 1.1. Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to E$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

Recall that the following:

(1) A mapping $T : K \to K$ with $F(T) \neq \phi$ is said to satisfy condition (A) [**6**] on K if there exists a non decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in K$, $\|x - Tx\| \geqslant f(d(x, F))$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.
(2) A family $\{T_1, T_2, \cdots\cdots T_n\}$ of N self-mappings on K with $F = \bigcap_{i=1}^N F(T_i) \neq \phi$ is said to satisfy condition (B) on K if there exists f *and* d as in (i) such that

$$max_{1\leqslant i\leqslant N}\{\|x - T_i x\|\} \geqslant f(d(x, F)),$$

for all $x \in K$.
When $T_i = T$ for all $i = 1, 2, \cdots\cdots N$, then condition (B) reduces to condition (A).

Lemma 1.1. ([**12**]) *Let* $\{a_n\}, \{\beta_n\}$ *and* $\{r_n\}$ *be non-negative sequences satisfying* $a_{n+1} \leqslant (1+r_n)a_n + \beta_n$, $\forall\, n \in N$. *If* $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, *then* $\lim_{n\to\infty} a_n\}$ *exist. Moreover, if* $\liminf_{n\to\infty} a_n = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. ([**14**])
Let $p > 1$ *and* $R > 1$ *be two fixed numbers and* E *be a Banach space. Then* E *is uniformly convex if and only if there exists a continuous, strictly increasing and convex function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$ *such that*

$$\|\lambda x + (1-\lambda)y\|^p \leqslant \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_R(0) = \{x \in E : \ \|x\| \leqslant R\}$, *and* $\lambda \in [0,1]$, *where* $w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

2. Main Results

Before proving our main result we shall prove the following crucial lemmas.

Lemma 2.1. *Let* E *be a real uniformly convex Banach space and* K *be a nonempty closed convex subset which is also a nonexpansive retract of* E. *Let* $T_1, T_2, \cdots\cdots, T_N : K \to K$ *be* N *asymptotically nonexpansive nonself mappings with sequences* $\{r_n^i\}$ *such that* $\sum_{n=1}^{\infty} r_n^i < \infty$, *for all* $1 \leqslant i \leqslant N$ *and* $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. *Let* $\{\alpha_n^i\}$, $\{\beta_n^i\}$, $\{\gamma_n^i\}$ *are sequences in [0,1] with* $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ *for all* $i = 1, 2, 3, \cdots\cdots N$. *From arbitrary* $x_1 \in K$ *define the sequence* $\{x_n\}$ *iteratively by* (1.3), *where* $\{u_n^i\}$ *are bounded sequences in* K *with* $\sum_{n=1}^{\infty} u_n^i < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n^i < \infty$. *Then*

$$\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \leqslant (1 + b_n^{N-1})\|x_n - x^*\| + d_n^{N-1},$$

for all $n \geqslant 1$, $x^* \in F$ *and for some sequence* $\{d_n^i\}$ *for all* $i = 1, 2, 3, \cdots\cdots N$ *of numbers such that* $\sum_{n=1}^{\infty} d_n^i < \infty$.

Proof. Let $x^* \in F$, then from (1.3) we get

$$\begin{aligned}
\|x_n^1 - x^*\| &= \|P(\alpha_n^1 x_n + \beta_n^1 T_1^n x_n + \gamma_n^1 u_n^1) - Px^*\| \\
&\leqslant \alpha_n^1\|x_n - x^*\| + \beta_n^1\|T_1^n x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\| \\
&\leqslant \alpha_n^1\|x_n - x^*\| + \beta_n^1(1 + r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\| \\
&\leqslant \alpha_n^1(1 + r_n^1)\|x_n - x^*\| + \beta_n^1(1 + r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\| \\
&\leqslant (1 - \beta_n^1)(1 + r_n^1)\|x_n - x^*\| + \beta_n^1(1 + r_n^1)\|x_n - x^*\| \\
&\quad + \gamma_n^1\|u_n^1 - x^*\| \\
&\leqslant (1 + r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\| \\
&\leqslant (1 + r_n^1)\|x_n - x^*\| + d_n^0
\end{aligned}$$

where $d_n^0 = \gamma_n^1\|u_n^1 - x^*\|$. Since $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, then $\sum_{n=1}^{\infty} d_n^0 < \infty$.

Next we note that,

$$\begin{aligned}
\|x_n^2 - x^*\| &= \|P(\alpha_n^2 x_n + \beta_n^2 T_2^n x_n^1 + \gamma_n^2 u_n^2) - Px^*\| \\
&\leqslant \alpha_n^2\|x_n - x^*\| + \beta_n^2\|T_2^n x_n^1 - x^*\| + \gamma_n^2\|u_n^2 - x^*\| \\
&\leqslant \alpha_n^2\|x_n - x^*\| + \beta_n^2(1+r_n^2)\|x_n^1 - x^*\| + \gamma_n^2\|u_n^2 - x^*\| \\
&\leqslant \alpha_n^2\|x_n - x^*\| + \beta_n^2(1+r_n^2)[(1+r_n^1)\|x_n - x^*\| + d_n^0] \\
&\quad + \gamma_n^2\|u_n^2 - x^*\| \\
&\leqslant [\alpha_n^2 + \beta_n^2(1+r_n^2)(1+r_n^1)]\|x_n - x^*\| + \beta_n^2(1+r_n^2)d_n^0 \\
&\quad + \gamma_n^2\|u_n^2 - x^*\| \\
&\leqslant (\alpha_n^2 + \beta_n^2)(1+r_n^2)(1+r_n^1)\|x_n - x^*\| + \beta_n^2(1+r_n^2)d_n^0 \\
&\quad + \gamma_n^2\|u_n^2 - x^*\| \\
&\leqslant (1 + r_n^1 + r_n^2 + r_n^1 r_n^2)\|x_n - x^*\| + \beta_n^2(1+r_n^2)d_n^0 + \gamma_n^2\|u_n^2 - x^*\| \\
&\leqslant (1+b_n^1)\|x_n - x^*\| + d_n^1
\end{aligned}$$

where, $d_n^1 = \beta_n^2(1+r_n^2)d_n^0 + \gamma_n^2\|u_n^2 - x^*\|$ and $b_n^1 = (r_n^1 + r_n^2 + r_n^1 r_n^2)$
Since $\sum_{n=1}^{\infty} d_n^0 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, $\sum_{n=1}^{\infty} r_n^i < \infty$, for $i = 1, 2$
and so $\sum_{n=1}^{\infty} d_n^1 < \infty$, and $\sum_{n=1}^{\infty} b_n^1 < \infty$.

$$\|x_n^i - x^*\| \leqslant (1 + b_n^{i-1})\|x_n - x^*\| + d_n^{i-1} \quad \forall\, n \geqslant 1,\ \forall\, i = 1, 2, \cdots N$$

Thus , $\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \leqslant (1 + b_n^{N-1})\|x_n - x^*\| + d_n^{N-1}$ for all $n \geqslant 1$.
This completes the proof of the lemma. □

REMARK 2.1. If we put $P = I$ (Identity mapping) in Lemma (2.1), then it generalizes the corresponding lemma of Schu [**8**] for one mapping. Further, if $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ and $\lim_{n\to\infty}\|x_n - T_i^n x_n\| = 0$ for all $i = 1, 2, \cdots N$, then we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

LEMMA 2.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2, \cdots\cdots, T_N : K \to K$ be N uniformly continuous asymptotically nonexpansive nonself mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leqslant i \leqslant N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{x_n\}$ be a sequence defined by* (1.3) *with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\beta_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \cdots N$ & for some $\varepsilon \in (0, 1)$. Then $\|x_n - T_i x_n\| = 0$, for all $i = 1, 2, \cdots N$.*

PROOF. Let $x^* \in F = \bigcap_{i=1}^{N} F(T_i)$. Then by Lemma (2.1) and Lemma (1.1) $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Let $\lim_{n\to\infty}\|x_n - x^*\| = a$. If $a = 0$, then by the continuity of each T_i the conclusion follows. Now suppose that $a > 0$. First, we will show that $\lim_{n\to\infty}\|T_N^n x_n - x_n\| = 0$. Since $\{x_n\}$ and $\{u_n^i\}$ bounded for all $i = 1, 2, \cdots N$, there exist $R > 0$ such that $x_n - x^* + \gamma_n^i(u_n^i - x_n)$, $T_i^n x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n) \in B_R(0)$ for all $n \geqslant 1$ and for all $i = 1, 2, \cdots N$.

Now using lemma (1.3), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|x_n^N - x^*\|^2 \\
&= \|P\left(\alpha_n^N x_n + \beta_n^N T_N^n x_n^{N-1} + \gamma_n^N u_n^N\right) - Px^*\|^2 \\
&= \|\alpha_n^N x_n + \beta_n^N T_N^n x_n^{N-1} + \gamma_n^N u_n^N - x^*\|^2 \\
&= \|\beta_n^N (T_N^n x_n^{N-1} - x^* + \gamma_n^N (u_n^N - x_n)) + (1 - \beta_n^N)(x_n - x^* + \\
&\quad \gamma_n^N (u_n^N - x_n))\|^2 \\
&\leqslant \beta_n^N \|T_N^n x_n^{N-1} - x^* + \gamma_n^N (u_n^N - x_n)\|^2 \\
&+ (1 - \beta_n^N)\|x_n - x^* + \gamma_n^N (u_n^N - x_n)\|^2 \\
&\quad - w_2(\beta_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leqslant \beta_n^N (\|T_N^n x_n^{N-1} - x^*\| + \gamma_n^N \|u_n^N - x_n\|)^2 \\
&\quad + (1 - \beta_n^N)(\|x_n - x^*\| + \gamma_n^N \|u_n^N - x_n\|)^2 \\
&\quad - w_2(\beta_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leqslant \beta_n^N (\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \|u_n^N - x_n\|)^2 \\
&\quad + (1 - \beta_n^N)(\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \|u_n^N - x_n\|)^2 \\
&\quad - w_2(\beta_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leqslant (\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \|u_n^N - x_n\|)^2 \\
&\quad - w_2(\beta_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leqslant (\|x_n - x^*\| + \lambda_n^{N-2})^2 - w_2(\beta_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \qquad (2.1)
\end{aligned}$$

where $\lambda_n^{N-2} = d_n^{N-2} + \gamma_n^N \|u_n^N - x_n\|$

Observe that $\varepsilon^3 \leqslant w_2(\beta_n^N)$. Now (2.1) implies that

$$\varepsilon^3 g(\|T_N^n x_n^{N-1} - x_n\|) \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-2}$$

where $\rho_n^{N-2} = 2\lambda_n^{N-2} + \left(\lambda_n^{N-2}\right)^2$

Since $\sum_{n=1}^{\infty} d_n^{N-2} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{N-2} < \infty$, we get $\sum_{n=1}^{\infty} \rho_n^{N-2} < \infty$, which implies that

$$\lim_{n\to\infty} g(\|T_N^n x_n^{N-1} - x_n\|) = 0$$

Since g is strictly increasing and continuous at 0, it follows that

$$\lim_{n\to\infty} \|T_N^n x_n^{N-1} - x_n\| = 0$$

Since for all N, T_N is asymptotically nonexpansive, note that

$$\begin{aligned}
\|x_n - x^*\| &\leqslant \|x_n - T_N^n x_n^{N-1}\| + \|T_N^n x_n^{N-1} - x^*\| \\
&= \|x_n - T_N^n x_n^{N-1}\| + (1 + r_n^N)\|x_n^{N-1} - x^*\|
\end{aligned}$$

for all $n \geqslant 1$

Thus

$$a = \lim_{n\to\infty} \|x_n - x^*\| \leqslant \liminf \|x_n^{N-1} - x^*\| \leqslant \limsup \|x_n^{N-1} - x^*\| \leqslant a,$$

and therefore

$$\lim_{n\to\infty} \|x_n^{N-1} - x^*\| = a.$$

Using the same argument in the proof above, we have

$$\begin{aligned}
\|x_n^{N-1} - x^*\|^2 &\leqslant \beta_n^{N-1}\|T_{N-1}^n x_n^{N-2} - x^* + \gamma_n^{N-1}(u_n^{N-1} - x_n)\|^2 \\
&\quad + (1-\beta_n^{N-1})\|x_n - x^* + \gamma_n^{N-1}(u_n^{N-1} - x_n)\|^2 \\
&\quad - w_2(\beta_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
&\leqslant \beta_n^{N-1}(\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|)^2 \\
&\quad + (1-\beta_n^{N-1})(\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|)^2 \\
&\quad - w_2(\beta_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
&\leqslant (\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|)^2 \\
&\quad - w_2(\beta_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
&\leqslant (\|x_n - x^*\| + \lambda_n^{N-3})^2 - w_2(\beta_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \qquad (2.2)
\end{aligned}$$

$$\text{where } \lambda_n^{N-3} = d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|$$

This implies that

$$\varepsilon^3 g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-3},$$

$$\text{where } \rho_n^{N-3} = 2\lambda_n^{N-3} + (\lambda_n^{N-3})^2.$$

Therefore

$$\lim_{n\to\infty} \|T_{N-1}^n x_n^{N-2} - x_n\| = 0.$$

Thus we have

$$\begin{aligned}
\|x_n - T_N^n x_n\| &\leqslant \|x_n - T_N^n x_n^{N-1}\| + \|T_N^n x_n^{N-1} - T_N^n x_n\| \\
&\leqslant \|x_n - T_N^n x_n^{N-1}\| + (1 + r_n^N)\|x_n^{N-1} - x_n\| \\
&\leqslant \|x_n - T_N^n x_n^{N-1}\| \\
&\quad + (1 + r_n^N)\|\alpha_n^{N-1}x_n + \beta_n^{N-1}T_{N-1}^n x_n^{N-2} + \gamma_n^{N-1}u_n^{N-1} - x_n\| \\
&\leqslant \|x_n - T_N^n x_n^{N-1}\| \\
&\quad + (1 + r_n^N)[\beta_n^{N-1}\|T_{N-1}^n x_n^{N-2} - x_n\| + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|]
\end{aligned}$$

.

Since $\lim_{n\to\infty} \|T_N^n x_n^{N-1} - x_n\| = 0$ and $\lim_{n\to\infty} \|T_{N-1}^n x_n^{N-2} - x_n\| = 0$,also $\sum_{n=1}^{\infty} \gamma_n^{N-1} < \infty$ and $\sum_{n=1}^{\infty} r_n^N < \infty$, it follows that $\lim_{n\to\infty} \|x_n - T_N^n x_n\| = 0$. Similarly

$$\lim_{n\to\infty} \|x_n - T_{N-2}^n x_n^{N-3}\| = \lim_{n\to\infty} \|x_n - T_{N-3}^n x_n^{N-4}\| = \cdots$$

$$\cdots\cdots\cdots = \lim_{n\to\infty} \|x_n - T_2^n x_n^1\| = 0$$

This implies that

$$\lim_{n\to\infty} \|x_n - T_{N-1}^n x_n\| = \lim_{n\to\infty} \|x_n - T_{N-2}^n x_n\| = \cdots = \lim_{n\to\infty} \|x_n - T_3^n x_n\| = 0$$

It remains to show that

$$\lim_{n\to\infty} \|x_n - T_1^n x_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - T_2^n x_n\| = 0$$

Note that

$$\begin{aligned}
\|x_n^1 - x^*\|^2 &\leqslant \beta_n^1(\|T_1^n x_n - x^*\| + \gamma_n^1\|u_n^1 - x_n\|)^2 \\
&\quad + (1-\beta_n^1)(\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x_n\|)^2 \\
&\quad - w_2(\beta_n^1) g(\|T_1^n x_n - x_n\|) \\
&\leqslant \beta_n^1(\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x_n\|)^2 \\
&\quad + (1-\beta_n^1)(\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x_n\|)^2 \\
&\quad - w_2(\beta_n^1) g(\|T_1^n x_n - x_n\|) \\
&\leqslant (\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x_n t\|)^2 - w_2(\beta_n^1) g(\|T_1^n x_n - x_n\|)
\end{aligned}$$

Thus we have

$$\varepsilon^2 g(\|T_1^n x_n - x_n\|) \leqslant (\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x_n\|)^2 - \|x_n^1 - x^*\|^2$$

and therefore $\lim_{n\to\infty}\|x_n - T_1^n x_n\| = 0$.
Since

$$\begin{aligned}
\|x_n - T_2^n x_n\| &\leqslant \|x_n - T_2^n x_n^1\| + \|T_2^n x_n^1 - T_2^n x_n\| \\
&\leqslant \|x_n - T_2^n x_n^1\| + (1 + r_n^2)\|x_n^1 - x_n\| \\
&\leqslant \|x_n - T_2^n x_n^1\| + (1 + r_n^2)\|\alpha_n^1 x_n + \beta_n^1 T_1^n x_n + \gamma_n^1 u_n^1 - x_n\| \\
&\leqslant \|x_n - T_2^n x_n^1\| + (1 + r_n^2)[\beta_n^1\|T_1^n x_n - x_n\| + \gamma_n^1\|u_n^1 - x_n\|],
\end{aligned}$$

Which implies that

$$\lim_{n\to\infty} \|x_n - T_2^n x_n\| = 0.$$

Therefore

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0,$$

for all $i = 1, 2, \cdots N$. On the other hand, by Remark (2.1), it is clear that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Therefore by Lemma (2.1), we conclude that $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$. □

THEOREM 2.1. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2, \cdots\cdots, T_N : K \to K$ be N uniformly continuous asymptotically nonexpansive nonself mappings with sequences $\{r_n^i\}$ such that $\sum_{n=1}^{\infty} r_n^i < \infty$, for all $1 \leqslant i \leqslant N$ and $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Suppose $\{T_1, T_2, \cdots\cdots, T_N\}$ satisfies condition (B). Let $\{x_n\}$ be a sequence defined by (1.3) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\beta_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \cdots N$ and for some $\varepsilon \in (0,1)$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots\cdots, T_N\}$.*

PROOF. From Lemma (2.1) and (1.1), we see that $\lim_{n\to\infty} \|x_n - x^*\|$ exist for all $x^* \in F = \bigcap_{i=1}^{N} F(T_i)$. Let $\lim_{n\to\infty} \|x_n - x^*\| = a$ for all $a \geqslant 0$. Without loss of generality, if $a = 0$, then there is nothing to prove. So that we assume that $a > 0$, as proved in Lemma (2.1), we have

$$\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \leqslant (1 + b_n^{N-1})\|x_n - x^*\| + d_n^{N-1},$$

for all $n \geqslant 1$,
where $\{d_n^i\}_{n=1}^{\infty}$, for all $i = 1, 2, \cdots N$, is non-negative real sequences such that $\sum_{n=1}^{\infty} d_n^i < \infty$ for all $i = 1, 2, \cdots N$.
This gives that

$$d(x_{n+1}, F) \leqslant (1 + b_n^{N-1}) d(x_n, F) + d_n^{N-1} \quad forall$$

n$\in N$.

Applying Lemma (1.1) to the above inequality, we obtained that $\lim_{n\to\infty} d(x_n, F)$ exist.
Also by Lemma (2.2) $\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0$ for all $i = 1, 2, \cdots N$. Since $\{T_1, T_2, \cdots\cdots, T_N\}$ satisfies condition (B), we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$.
Next we show that $\{x_n\}$ is a Cauchy sequence.
Since $\lim_{n\to\infty} d(x_n, F) = 0$, then given any $\varepsilon > 0$ there exist a natural number n_0 such that $d(x_n, F) < \frac{\varepsilon}{3}$ for all $n \geqslant n_0$.
So we can find $p^* \in F$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}$
For all $n \geqslant n_0$ and $m \geqslant 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - p^*\| + \|p^* - x_n\| \\ &\leqslant \|x_{n_0} - p^*\| + \|x_{n_0} - p^*\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent ,since E is complete.
Let $\lim x_n = q^*$. Then $q^* \in K$.
It remains to show that $q^* \in F$. Let $\varepsilon_1 > 0$ be given, then there exists a natural number n_1 such that $\|x_n - x^*\| < \frac{\varepsilon_1}{4}$ for all $n \geqslant n_1$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a natural number $n_2 \geqslant n_1$ such that , for all $n \geqslant n_2$, we have $d(x_n, F) < \frac{\varepsilon_1}{5}$ and in particular, we have $d(x_{n_2}, F) < \frac{\varepsilon_1}{5}$.
Therefore, there exists $w^* \in K$ such that $\|x_{n_2} - w^*\| < \frac{\varepsilon_1}{4}$

For any $i \in I$ and $n \geqslant n_2$, we have

$$\begin{aligned}\|T_i q^* - q^*\| &\leqslant \|T_i q^* - w^*\| + \|w^* - q^*\| \\ &\leqslant 2\|q^* - w^*\| \\ &\leqslant 2(\|q^* - x_{n_2}\| + \|x_{n_2} - w^*\|) \\ &< 2(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}) < \varepsilon_1\end{aligned}$$

This implies that $T_i q^* = q^*$. Hence $q^* \in F(T_i)$ for all $i \in I$ and so $q^* \in F = \bigcap_{i=1}^{N} F(T_i)$. Thus $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \cdots\cdots, T_N\}$. □

REMARK 2.2. Theorem (2.1) extend the corresponding result of Su and Qin [**21**] to the case of multistep iterative sequences with errors for a finite family of asymptotically nonexpansive nonself mappings.

REMARK 2.3. Our result also extend the corresponding result of Shahzad [**14**] to the case of multistep iterative sequences with errors for a finite family of more general class of nonexpansive mappings.

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