EXTERNALLY EQUITABLE COLORING IN GRAPHS

D. Lakshmanaraj and V. Swaminathan

ABSTRACT. Let $G = (V, E)$ be a simple graph. A partition of $V(G)$ into independent,externally equitable sets is called externally equitable proper color partition of G or externally equitable proper coloring of G. The minimum cardinality of an externally equitable proper coloring of G is called externally equitable chromatic number of G and is denoted by $\chi_{ee}(G)$. Since $\Pi = \{\{u_1\}, \{u_2\}, \cdots, \{u_n\}\}$ where $V(G) = \{u_1, u_2, \cdots, u_n\}$ is an externally equitable proper coloring of G, externally equitable proper color partition exists in any graph G. In this paper, this new parameter is introduced and studied.

1. Introduction

The concept of equitability has been widely studied in coloring. A proper color partition is said to be equitable if the cardinalities of the color classes differ by at most one. E. Sampathkumar introduced degree equitability in graphs. A subset S of the vertex set of a graph is said to be degree equitable if the degrees of any two vertices of S differ by at most one. Arumugam et. al. [2] studied degree equitable sets and degree equtable proper coloring of vertices of a graph. K.M.Dharmalingam defined degree equitability and out degree equitability studied dominating sets which are (i) degree equitable and (ii) out degree equitable. A subset S of the vertex set V of a graph G is said to be externally equitable, if for any $x, y \in V - S$, $||N(x) \cap S| - |N(y) \cap S|| \leqslant 1$.

2. Main Results

DEFINITION 2.1. A partition of $V(G)$ into independent,externally equitable sets is called externally equitable proper color partition of G or externally equitable proper coloring of G. The minimum cardinality of an externally equitable proper coloring of G is called externally equitable chromatic number of G and is denoted
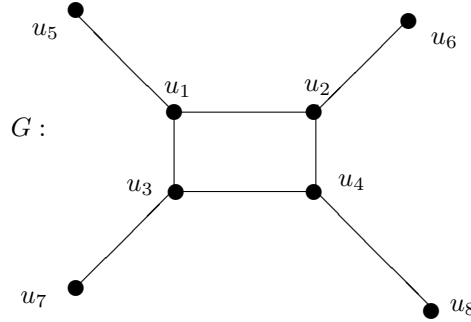
by $\chi_{ee}(G)$. Since $\Pi = \{\{u_1\}, \{u_2\}, \cdots, \{u_n\}\}$ where $V(G) = \{u_1, u_2, \cdots, u_n\}$ is an externally equitable proper coloring of G, externally equitable proper color partition exists in any graph G .

ILLUSTRATION 2.1.

G :

$\{\{u_5, u_6, u_7, u_8\}, \{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}\}$ is an externally equitable independent partition of $V(G)$.

REMARK 2.1. Since any $\chi_{ee}-$ partition of G is a proper color partition of G, $\chi(G) \leqslant \chi_{ee}(G)$ for any graph G.

2.1. $\chi_{ee}-$ proper color partition for standard graphs.

OBSERVATION 2.1.   (1) $\chi_{ee}(K_n) = n$
(2) $\chi_{ee}(P_n) = \chi(P_n) = 2.$
(3) $\chi_{ee}(C_n) = \chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$
(4) $\chi_{ee}(K_{1,n}) = \chi(K_{1,n}) = 2.$

THEOREM 2.1. $\chi_{ee}(W_{n+1}) = \begin{cases} n+1 & when\ n \geqslant 7 \\ 4 & when\ n = 3\ or\ 5 \\ 3 & when\ n = 4\ or\ 6 \end{cases}$

PROOF. When $n = 3$, $W_4 = K_4$ and hence $\chi_{ee}(W_4) = 4$.
When $n = 4$, W_5 is

G :
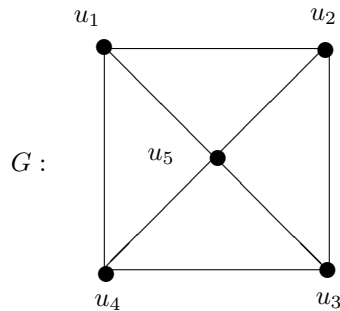
It is easily seen that $\{\{u_1, u_3\}, \{u_2, u_4\}, \{u_5\}$ is a χ_{ee}-partition of W_5. Therefore $\chi_{ee}(W_5) = 3$.

When $n = 5$, W_6 is

It is easily seen that $\{\{u_1, u_3\}, \{u_2, u_4\}, \{u_5\}, \{u_6\}\}$ is a χ_{ee}-partition of W_6. Therefore $\chi_{ee}(W_6) = 4$.

When $n = 6$, W_7 is

It is easily seen that $\{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}, \{u_7\}\}$ is a χ_{ee}-partition of W_7. Therefore $\chi_{ee}(W_6) = 3$. Let $n \geqslant 7$. Consider W_{n+1}. let $\{u_1, u_2, \ldots, u_n\}$ be the vertices on the cycle of W_{n+1} and v be the central vertex. In any χ_{ee}-partition of W_{n+1}, $\{v\}$ is an element of the partition. Let V_1 be any other element of the partition. Then $|N(v) \cap V_1| = |V_1|$. Since for any $x \in V(W_{n+1}) - V_1$, $x \neq v$, $|N(x) \cap V_1| \leqslant 2$, we get that $|V_1| \leqslant 3$. If V_1 contains $u_{i+1}, u_{i+3}, u_{i+5}$, then $|N(u_{i+7} \cap V_1| = 0$. A similar argument shows that $|V_1| \neq 2$. Therefore $|V_1| = 1$. Therefore $\chi_{ee}(W_{n+1}) = n + 1$. $\qquad\square$

THEOREM 2.2. $\chi_{ee}(D_{r,s}) = \begin{cases} \max\{r, s\} + 2 & \text{when } |r - s| \geqslant 2 \\ 3 & \text{when } |r - s| \leqslant 1 \, and \, r, s \geqslant 3 \\ 2 & \text{when } r = 2, s = 1 \text{ or } 2 \\ & \text{or } r = 1, s = 1 \text{ or } s = 2 \end{cases}$

PROOF. **Case(i):** $|r - s| \geqslant 2$. Let $r = max\{r, s\}$. Let u, v be the centers of $D_{r,s}$ and let $u_1, u_2, \ldots, u_r$ be the pendant vertices at u and $v_1, v_2, \ldots, v_s$ be the pendant vertices at v. Let $\pi = \{V_1, V_2, \ldots, V_t\}$ be a χ_{ee}-partition of $D_{r,s}$. If V_1 contains two pendants at u, then any pendant at v not in V_1 will have no neighbours in V_1 and $u \notin V_1$ has two neighbours in V_1, a contradiction. Therefore V_1 contains all pendant vertices at v. Suppose V_1 does not contain a pendant at u. Then that pendant at u will have no neighbour in V_1, a contradiction since u has two neighbours in V_1. Therefore V_1 contains all pendants at u. Then $|N(u) \cap V_1| = r$ and $|N(v) \cap V_1| = s$ and $|r - s| \geqslant 2$, a contradiction. Therefore, V_1 cannot contain two pendants at u. Similarly, V_1 cannot contain two pendant at v. Suppose $V_1 = \{u_i, v_j\}$ where $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$. Then V_1 is externally equitable independent. Thus, $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$, $V_3 = \{u_3, v_3\}, \cdots, V_s = \{u_s, v_s\}$ are elements of any χ_{ee}-partition of $D_{r,s}$. The remaining pendants at u and the two centers must appear as singletons in π. Therefore $\chi_{ee}(D_{r,s}) = max\{r, s\} + 2$.
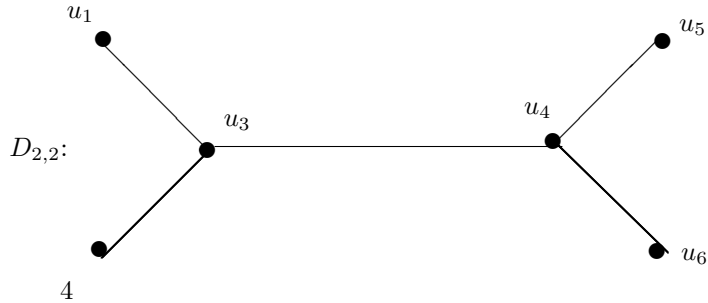
Case (ii): $|r - s| \leqslant 1$ and $r, s \geqslant 3$. Let $\pi = \{\{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}, \{u\}, \{v\}\}$. Then π is an externally equitable independent partition of $D_{r,s}$. Therefore $\chi_{ee}(D_{r,s}) \leqslant 3$. Suppose $\pi = \{V_1, V_2\}$ be a χ_{ee}-partition of $D_{r,s}$. Clearly $u \in V_1$ and $v \in V_2$. Therefore V_2 contains all the pendants at u and V_1 contains all the pendants at v. Then $|N(u_1) \cap V_1| = 1$ and $|N(v) \cap V_1| = r$, a contradiction, since $r \geqslant 3$. Therefore $\chi_{ee}(D_{r,s}) \geqslant 3$. Therefore $\chi_{ee}(D_{r,s}) = 3$.

Case(iii): Suppose $r = 2$ and $s = 1$

$D_{2,1}$:
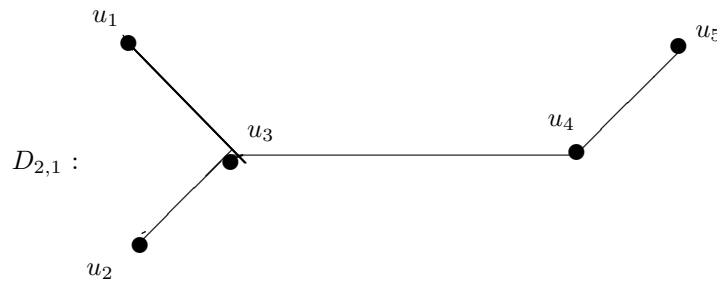
Here $\pi = \{\{u, v_1\}, \{v, u_1, u_2\}\}$ is a χ_{ee}-partition of $D_{2,1}$.Therefore $\chi_{ee}(D_{2,1}) = 2$.
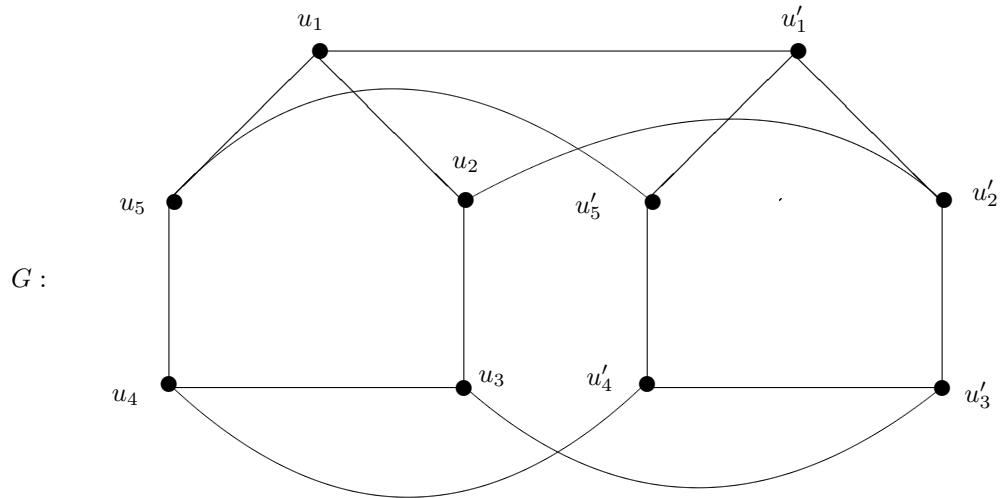
Suppose $r = 2$ and $s = 2$

$D_{2,2}$:

Then $\pi = \{\{u, v_1, v_2\}, \{v, u_1, u_2\}\}$ is a χ_{ee}-partition of $D_{2,2}$. Therefore $\chi_{ee}(D_{2,2}) = 2$. When $r = 1$, $s = 1$ then $\chi_{ee}(D_{1,1}) = P_4$ and $\chi_{ee}(P_4) = 2$.   □

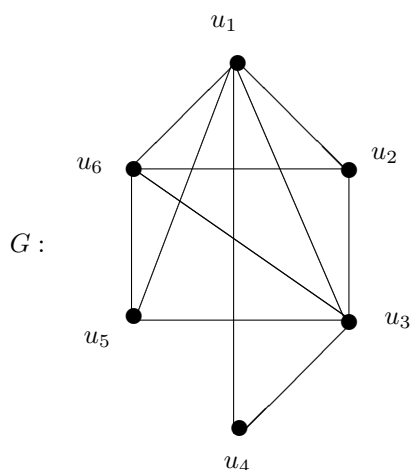REMARK 2.2. There exists regular graph G such that $\chi_{ee}(G) > \chi(G)$.
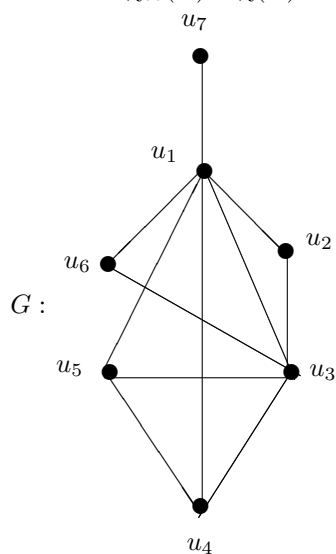
For: let

G :

$\pi = \{\{u_1, v_3\}, \{u_2, v_4\}, \{u_3, v_5\}, \{u_4, v_1\}, \{u_5, v_2\}\}$ is an externally equitable independent partition of $V(G)$. $\chi(G) = 3$, $\chi_{ee}(G) = 5$, since $\beta_0^{ee}(G) = 2$.

PROPOSITION 2.1. *Given any positive integer k there exists a graph G such that $\chi_{ee}(G) - \chi(G) = k$.*

PROOF. Let $k \geqslant 6$. Let $G = D_{2,k}$. Then $\chi_{ee}(G) = max\{2, k\} + 2 = k + 2$, $\chi(G) = 2$. Therefore $\chi_{ee}(G) - \chi(G) = k$. $\chi(D_{3,4}) = 2$ *and* $\chi_{ee}(D_{3,4}) = 3$. Therefore $\chi_{ee}(G) - \chi(G) = 1$. □

$\chi(G) = 4$, $\chi_{ee}(G) = 6$. Therefore $\chi_{ee}(G) - \chi(G) = 2$.

$\chi(G) = 4$, $\chi_{ee}(G) = 7$. Therefore $\chi_{ee}(G) - \chi(G) = 3$.

G :
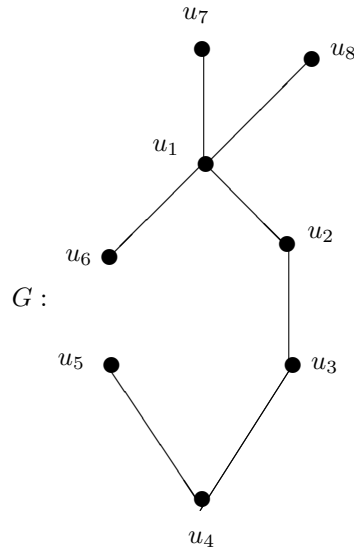
$\chi(G) = 4$, $\chi_{ee}(G) = 8$. Therefore $\chi_{ee}(G) - \chi(G) = 4$.

G :

$\chi(G) = 4$, $\chi_{ee}(G) = 9$. Therefore $\chi_{ee}(G) - \chi(G) = 5$.

THEOREM 2.3. $\chi_{ee}(G) = 1$ if and only if G is $\overline{K_n}$.

PROOF. Suppose $\chi_{ee}(G) = 1$. Then no two vertices of G are adjacent. Therefore $G = \overline{K_n}$. Converse is obvious.   □

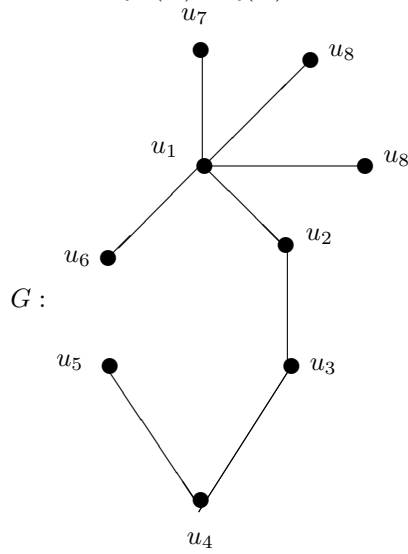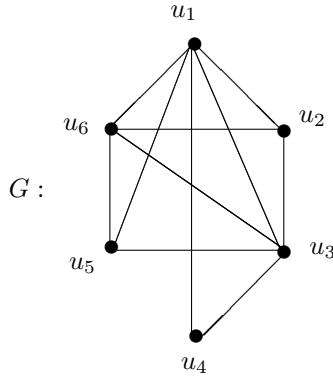THEOREM 2.4. $\chi_{ee}(G) = 2$ *if and only if G is bipartite and the degrees of any two* *elements of the same partition differ by at most one.*

PROOF. Suppose $\chi_{ee}(G) = 2$. Then $V(G) = X \cup Y$ where X and Y are independent and any two vertices of X have almost equal number of neighbors in Y and vice versa .(that is, for every $u, v \in X$, $||N(u) \cap Y| - |N(v) \cap Y|| \leqslant 1$ and for every $u, v \in Y$, $||N(u) \cap X| - |N(v) \cap X|| \leqslant 1$). The converse is obvious. $\square$

EXAMPLE 2.1.

$G :$

G is a bipartite Graph. Therefore $\chi(G) = 2$. But $\chi_{ee}(G) = 3$ and the color classes are $\{\{u_1, u_2\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3\}\}$.

THEOREM 2.5. $\frac{n}{\beta_0^{ee}(G)} \leqslant \chi_{ee}(G) \leqslant n - \beta_0^{ee}(G) + 1$.

PROOF. Let $\chi_{ee}(G) = k$. Let π be a partition of $V(G)$ into k externally equitable independent sets $V_1, V_2, \cdots, V_k$. Then $|V_i| \leqslant \beta_0^{ee}(G) \ \vee \ i, 1 \leqslant i \leqslant k$.
$n = |V_1| + |V_2| + |V_3| + \cdots + |V_k| \leqslant k\beta_0^{ee}(G)$. Therefore $\frac{n}{\beta_0^{ee}(G)} \leqslant k$. Let S be a β_0^{ee} - set of G. Let $\pi = \{S, \{v_1\}, \{v_2\}, \ldots, \{v_t\}\}$ where
$V - S = \{v_1, v_2, \ldots, v_t\}$ and $t = n - \beta_0^{ee}(G)$. Then π is an externally equitable independent partition of G. Therefore $\chi_{ee}(G) \leqslant n - \beta_0^{ee}(G) + 1$. $\square$

PROPOSITION 2.2. *Let S be any externally equitable independent set of C_n^+. Let* $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n\}$ *and* $V(C_n^+) = \{u_1, u_2, u_3, \ldots, u_n, u_1', u_2', u_3', \ldots, u_n'\}$ *where u_i' is the pendant vertex of C_n^+ adjacent with u_i, $1 \leqslant i \leqslant n$. Then S cannot contain any pair of vertices of the form u_i, u_{i+2}, $1 \leqslant i \leqslant n$ ($i + 2$ taken mod n).*

PROOF. Suppose $u_i, u_{i+2} \in S$. Then $|N(u_{i+1}) \cap S| \geqslant 2$. Therefore, $|N(x) \cap S| \geqslant 1$, for every $x \in V - S$. Therefore S is a dominating set of C_n^+. Since S is independent, $u_{i+1} \notin S$ and as S is a dominating set of C_n^+ we get that $u_{i+1}' \in S$. Therefore $|N(u_{i+1}) \cap S| = 3$. Hence $|N(x) \cap S| \geqslant 2$, for every $x \in V - S$. Since $V - S$ contains pendant vertices, this is not possible. $\square$

PROPOSITION 2.3. *When $n \cong 1, 2 \ (mod \ 3)$, $\chi_{ee}(C_n^+) \geqslant 4$.*

PROOF. Suppose $n \cong 1 \ (mod \ 3)$. Let $\pi = \{V_1, V_2, V_3\}$ be a χ_{ee}- partition of C_n^+. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. By Proposition 2.2, $|V_i \cap V(C_n)| \leqslant \frac{n-1}{3}$, $1 \leqslant i \leqslant 3$. Therefore $\sum\limits_{i=1}^{i=3} |V_i \cap V(C_n)| \leqslant n-1$ a contradiction, since $V_1 \cup V_2 \cup V_3 = V(C_n^+)$. Suppose $n \cong 2 (mod \ 3)$. Arguing as before $\sum\limits_{i=1}^{i=3} |V_i \cap V(C_n)| \leqslant n-2$ a contradiction. Therefore $\chi_{ee}(C_n^+) \geqslant 4$, when $n \cong 1, 2 (mod \ 3)$. $\square$

PROPOSITION 2.4. *Let $n \cong 0 \ (mod \ 3)$. Let V_1 be an externally equitable independent set of C_n^+. Let $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n\}$. Suppose $|V_1 \cap V(C_n)| = \frac{n}{3}$ and suppose $u_j^{'} \in V_1$ for some j such that $u_j \notin V_1$. Then $|V_1| = n$.*

PROOF. Let $n \cong 0 (mod \ 3)$. Let without loss of generality $u_1, u_4, \ldots, u_{n-2} \in V_1$ and $u_2^{'} \in V_1$. Therefore Then $|N(u_2) \cap V_1| = 2$. Therefore, V_1 is a dominating set of C_n^+. Therefore $u_3^{'}, u_5^{'}, u_6^{'} \ldots, u_{n-1}^{'}, u_n^{'} \in V_1$ and hence $|V_1| = n$. $\square$

PROPOSITION 2.5. *Let $n \cong 0 (mod \ 3)$. Suppose $\pi = \{V_1, V_2, V_3\}$ be a χ_{ee}- partition of C_n^+. Then $|V_i \cap V(C_n)| = \frac{n}{3}$, for all $i = 1, 2$ and 3.*

PROOF. Clearly $|V_i \cap V(C_n)| \leqslant \frac{n}{3}$, for all $i = 1, 2$ and 3. Suppose $|V_i \cap V(C_n)| < \frac{n}{3}$. Therefore for some $j \neq i, 1 \leqslant j \leqslant 3, |V_i \cap V(C_n)| > \frac{n}{3}$ (since $\sum\limits_{i=1}^{i=3} |V_i \cap V(C_n)| = n$) a contradiction. Therefore, $|V_i \cap V(C_n)| = \frac{n}{3}$, for all $i = 1, 2$ and 3. $\square$

PROPOSITION 2.6. *Let $n \cong 0 (mod \ 3)$. Then $\chi_{ee}(C_n^+) > 3$.*

PROOF. Suppose $\chi_{ee}(C_n^+) = 3$. Let $\pi = \{V_1, V_2, V_3\}$ be a χ_{ee}- partition of C_n^+. Then $|V_i \cap V(C_n)| = \frac{n}{3}$, for all $i = 1, 2$ and 3. Since $V_1 \cup V_2 \cup V_3 = V(C_n^+)$ for any $j, 1 \leqslant j \leqslant n, u_j^{'} \in V_i$, for some $i, 1 \leqslant i \leqslant 3$. Suppose $u_j^{'} \in V_1$. Then by Lemma 2.32, $|V_1| = n$. Also, there exists $u_r^{'} \notin V_i$, for some r, $1 \leqslant r \leqslant n$. Therefore $u_r^{'} \in V_2 \ or \ V_3$. Therefore by lemma 2.2.32, $|V_2| = n$ or $|V_3| = n$. Since $V(C_n^+) = 2n$, one of V_2, V_3 is empty a contradiction. Therefore $\chi_{ee}(C_n^+) \geqslant 4$. $\square$

PROPOSITION 2.7. $\chi_{ee}(C_n^+) \leqslant 4$.

PROOF. **Case(i):** Let $n = 3k$. Then $\{\{u_1, u_4, u_2^{'}, u_3^{'}, u_5^{'}, u_6^{'}, \ldots, u_{3k}^{'}\},$ $\{u_2, u_5, u_8, \ldots, u_{3k-1}\}, \{u_3, u_7, u_{10}, \ldots, u_{3k-2}, u_1^{'}\}, \{u_6, u_9, u_{12}, \ldots, u_{3k}, u_4^{'}\}\}$ is an externally equitable independent partition of C_n^+. Therefore, $\chi_{ee}(C_n^+) \leqslant 4$.

Case(ii): Let $n = 3k + 1$. Then $\{\{u_1, u_4, u_2^{'}, u_3^{'}, u_5^{'}, u_6^{'}, \ldots, u_{3k+1}^{'}\},$ $\{u_2, u_6, u_9, \ldots, u_{3k}, u_4^{'}\}, \{u_3, u_7, u_{10}, \ldots, u_{3k+1}\}, \{u_5, u_8, u_{11}, \ldots, u_{3k-1}, u_1^{'}\}\}$ is an externally equitable independent partition of C_n^+.
Therefore $\chi_{ee}(C_n^+) \leqslant 4$.

Case(iii): Let $n = 3k + 2$. Then $\{u_1, u_4, u_2^{'}, u_3^{'}, u_5^{'}, u_6^{'}, \ldots, u_{3k+2}^{'}\},$ $\{u_2^{'}, u_6, u_9, \ldots, u_{3k}, u_4^{'}\}, \ \{u_3, u_7, u_{10}, \ldots, u_{3k+1}, u_1^{'}\}, \ \{u_5, u_8, u_{11}, \ldots, u_{3k+2}\}$ is an externally equitable independent partition of C_n^+.
Therefore $\chi_{ee}(C_n^+) \leqslant 4$. $\square$

THEOREM 2.6. $\chi_{ee}(C_n^+) = 4$.

PROOF. Follows from proposition 2.3 to 2.7.

$\square$

REMARK 2.3. Let $G = (V, E)$ be a simple graph. Let S be an externally equitable independent set of G and for any $u \in V - S$, $|N(u) \cap S| \geqslant 2$. Then S is a dominating set of G. In particular, if $|N(u) \cap S| = r \geqslant 2$ then S is a $(r - 1)-$ dominating set of G.

REMARK 2.4. If S is an externally equitable independent set of G and for any $u \in V - S$ $|N(u) \cup S| \geqslant 3$ then S does not contain any pendant vertex. ((ie) $deg_G(v) \geqslant 2$, for any $v \in V - S$)

OBSERVATION 2.2. Let G and H be two vertex disjoint graphs. Any externally equitable independent set S of $G + H$ contains either vertices from $V(G)$ or $V(H)$ and not from both. Also, if $S \subseteq V(G)$, then for any $u \in V(H)$, $|N(u) \cap S| = |S|$. Hence, $|N(v) \cap S| = |S| - 1$ or $|S|$ for any $v \in V - S$. Therefore S is a $(|S| - 1)$-dominating set of G. If for any $x \in V(G) - S$, $deg_G(x) \leqslant |S| - 2$ then S is not an externally equitable independent set of $G + H$.

OBSERVATION 2.3. Let G and H be two vertex disjoint graphs. Let S be an externally equitable independent set of G. Then G is an externally equitable independent set of $G \cup H$ if and only if for every $x \in V - S$, $|N(x) \cup S| \leqslant 1$. Similar results holds for H also. Also, if S is an externally equitable independent set of $G \cup H$ and S is a dominating set of G, then $|N(x) \cap S| = 1$, for every $x \in V - S$.

OBSERVATION 2.4. Let G and H be two vertex disjoint graphs. Let S be an externally equitable independent set of $G + H$. Then S is an externally equitable independent set of G if $S \subseteq V(G)$ or an externally equitable independent set of H if $S \subseteq V(H)$.

OBSERVATION 2.5. Let $\pi = \{S_1, S_2, \ldots, S_r\}$ be a χ_{ee} partition of G. $S_1, S_2, \ldots, S_r$ are externally equitable independent partition of $G + H$ if and only if every $u \in S_i$, $1 \leqslant i \leqslant r$, $n - |S_i| \geqslant deg_G(u) \geqslant n - |S_i| - (r - 1)$.

OBSERVATION 2.6. Let $\pi = \{S_1, S_2, \ldots, S_t\}$ be externally equitable independent partition of $G + H$. Let $S_1, S_2, \ldots, S_r$ be an externally equitable independent sets of G and $S_{r+1}, S_{r+2}, \ldots, S_t$ be externally equitable independent sets of H. Then $\pi_1 = \{S_1, S_2, \ldots, S_r\}$ is an externally equitable independent partition of G and $\pi_2 = \{S_{r+1}, S_{r+2}, \ldots, S_t\}$ be an externally equitable independent partition of H . Also for every $u \in S_i$, $1 \leqslant i \leqslant r$, $n - |S_i| \geqslant deg_G(u) \geqslant n - |S_i| - (r - 1)$ and $n - |S_j| \geqslant deg_H(u) \geqslant n - |S_j| - (t - r - 1)$ for every $u \in S_j$, $r + 1 \leqslant j \leqslant t$.

OBSERVATION 2.7. Let $\pi_1 = \{S_1, S_2, \ldots, S_r\}$ be an externally equitable independent partition of G and $\pi_2 = \{S_{r+1}, S_{r+2}, \ldots, S_t\}$ is an externally equitable independent partition of H . Then $\pi_1 \cup \pi_2$ is an externally equitable independent partition of $G + H$ if and only if for every i, $1 \leqslant i \leqslant r$ $n - |S_i| \geqslant deg_G(u) \geqslant n - |S_i| - (r - 1)$ for every $u \in S_i$ and $n - |S_j| \geqslant deg_H(u) \geqslant n - |S_j| - (t - r - 1)$ for every $u \in S_j$, $r + 1 \leqslant j \leqslant t$. Let $k = max\{|S_i|, 1 \leqslant i \leqslant r\}$ and $l = min\{|S_i|, 1 \leqslant i \leqslant r\}$.

$n - k - (r-1) \leqslant deg_G(u) \leqslant n - l$, for every $u \in V(G)$. Therefore $\delta(G) \geqslant n - k - (r-1)$ and $\Delta(G) \leqslant n - l$.

ILLUSTRATION 2.2. Consider $G = K_3$ and $H = C_8$. Let $V(K_3) = \{u_1, u_2, u_3\}$ and $V(C_8) = \{v_1, v_2, \ldots v_8\}$. $\pi_1 = \{\{u_1\}, \{u_2\}, \{u_3\}\}$ and $\pi_2 = \{\{v_1, v_3, v_5, v_7\}, \{v_2, v_4, v_6, v_8\}\}$ are χ_{ee} partitions of G and H respectively. Let $S_1 = \{v_1, v_3, v_5, v_7\}$ and $S_2 = \{v_2, v_4, v_6, v_8\}$. If $\pi_1 \cup \pi_2$ is an externally equitable independent partition of $G + H$, then S_1 and S_2 are 3 dominating sets of H which is not possible since $\Delta(H) = 2$. A similar argument shows that a χ_{ee} partition of H cannot contain an element of cardinality three since any externally equitable independent set of cardinality three in C_8 is not a independent 2-dominating set of C_8. If π_2 contains an element of cardinality two then it is not a dominating set of C_8 since $\gamma(C_8) = 3$. Therefore, every element of π_2 is a singleton and hence $\chi_{ee}(K_3 + C_8) = 11$.

OBSERVATION 2.8. $\chi_{ee}(G + H) \leqslant |V(G)| + |V(H)|$ and the upper bound is sharp as seen in the above example.

OBSERVATION 2.9. Given any positive integer k, there exist graph G and H such that $\chi_{ee}(G + H) - (\chi_{ee}(G) + \chi_{ee}(H)) = k$.

PROOF. **Case 1:** k is even. Let $G = K_3$ and $H = C_{2k+2}$. $\chi_{ee}(G+H) = 2k+5$. $\chi_{ee}(G) = 3$ and $\chi_{ee}(H) = 2$. Hence the observation.
Case 2: k is odd. Consider $G = K_4$, $H = D_{k+2,k}$. $\chi_{ee}(G + H) = 2k + 4 + 4 = 2k + 8$, $\chi_{ee}(G) = 4$, $\chi_{ee}(H) = k + 4$. Therefore, $\chi_{ee}(G + H) - (\chi_{ee}(G) + \chi_{ee}(H)) = k$. □

OBSERVATION 2.10. Let G and H be two vertex disjoint graphs. Let $\pi_1 = \{V_1, V_2, \ldots V_k\}$ and $\pi_2 = \{W_1, W_2, \ldots W_r\}$ be χ_{ee}- partitions of G and H respectively. Let $r_i = \min_{x \in V(G) - V_i} \{N(x) \cap V_i\}$, $1 \leqslant i \leqslant k$. Let $s_j = \min_{x \in V(H) - W_j} \{N(y) \cap W_j\}$, $1 \leqslant j \leqslant r$. If $r = k$ and $r_i = s_i$ for every $i = 1$ to k, then $\{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_k \cup W_k\}$ is an externally equitable independent partition of $G \cup H$. Hence $\chi_{ee}(G \cup H) \leqslant k$.

PROPOSITION 2.8. *Let G and H be two vertex disjoint graphs. Then* $max\{\chi_{ee}(G), \chi_{ee}(H)\} \leqslant \chi_{ee}(G \cup H)$.

PROOF. Let $\pi = \{V_1, V_2, \ldots V_k\}$ be a $\chi_{ee}-$ partition of $G \cup H$. Let $\pi_1 = \{V_1 \cap V(G), V_2 \cap V(G), \ldots V_k \cap V(G)\}$ and $\pi_2 = \{V_1 \cap V(H), V_2 \cap V(H), \ldots V_k \cap V(H)\}$. Clearly π_1 and π_2 are independent partitions of G and H respectively. Let $x, y \in V(G) - (V_i \cap V(G))$, $1 \leqslant i \leqslant k$. Therefore $x, y \in V(G \cup H) - V_i$. Therefore $|(N(x) \cap V_i) - (N(y) \cap V_i)| \leqslant 1$. Since x and y are not adjacent with any vertex of H, $N(x) \cap V_i = N(x) \cap (V_i \cap V(G))$ and $N(y) \cap V_i = N(y) \cap (V_i \cap V(G))$. Therefore $|N(x) \cap (V_i \cap V(G)) - N(y) \cap (V_i \cap V(G))| \leqslant 1$ for every i, $1 \leqslant i \leqslant k$. Therefore π_1 is an externally equitable independent partition of G. Similarly π_2 is an externally equitable independent partition of H. Therefore $\chi_{ee}(G) \leqslant k$ and $\chi_{ee}(H) \leqslant k$. Therefore $max\{\chi_{ee}(G), \chi_{ee}(H)\} \leqslant \chi_{ee}(G \cup H)$. □

REMARK 2.5. The bound is sharp as seen from the following example: Let $G = K_3$ and $H = C_8$. $V(G) = \{u_1, u_2, u_3\}$ and $V(H) = \{v_1, v_2, \ldots v_8\}$. $\chi_{ee}(G) = 3$ and

$\chi_{ee}(H) = 2$. $\{\{u_1, v_1, v_3, v_5, v_7\}, \{u_2, v_2, v_4, v_6, v_8\}, \{u_3\}\}$ is an externally equitable independent partition of $G \cup H$. Therefore $\chi_{ee}(G \cup H) \leqslant 3$. But $\chi_{ee}(G \cup H) \geqslant 3$, $\chi_{ee}(G \cup H) = 3 = \max \{\chi_{ee}(G), \chi_{ee}(H)\}$.

THEOREM 2.7. *Let G and H be two vertex disjoint graphs. Let π_1 and π_2 be two partitions of G and H respectively satisfying the following. Let $\pi_1 = \{V_1, V_2, \ldots V_k\}$, $\pi_2 = \{W_1, W_2, \ldots W_r\}$. Any vertex in $V(G) - V_i$ is adjacent with either a or $a + 1$ vertices of V_i $1 \leqslant i \leqslant k$ and any vertex in $V(H) - W_j$ is adjacent with either a or $a + 1$ vertices of W_j $1 \leqslant i \leqslant r$. Then $\chi_{ee}(G \cup H) \leqslant max\{\chi_{ee}(G), \chi_{ee}(H)\}$.*

PROOF. Let with out loss of generality $k \geqslant r$. Consider $\pi = \{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_r \cup W_r, \ldots, V_k\}$. Clearly π is an externally equitable independent partition of $G \cup H$. Therefore $\chi_{ee}(G \cup H) \leqslant k = \max\{\chi_{ee}(G), \chi_{ee}(H)\}$. □

REMARK 2.6. Let G and H be two vertex disjoint graphs. Let π_1 and π_2 be two partitions of G and H respectively satisfying the following. Let $\pi_1 = \{V_1, V_2, \ldots V_k\}$, $\pi_2 = \{W_1, W_2, \ldots W_r\}$. Any vertex in $V(G) - V_i$ is adjacent with either a or $a + 1$ vertices of V_i $1 \leqslant i \leqslant k$ and any vertex in $V(H) - W_j$ is adjacent with either a or $a + 1$ vertices of W_j $1 \leqslant i \leqslant r$. Then $\chi_{ee}(G \cup H) \leqslant \max\{k, r\}$.

References

[1] R. B. Allan and R. C. Laskar, *On domination and independent domination numbers of a graph*, Discrete Math. 23 (1978), 73-76.

[2] A. Anitha, S. Arumugam and E. Sampathkumar, *Degree Equitable Sets in a Graph*, International J.Math. Combin., 3(2009), 32-47.

[3] A. Anitha, S. Arumugam, S. B. Rao and E. Sampathkumar, *Degree Equitable Chromatic number of a a Graph*, JCMCC, 75(2010), 187-199.

[4] C. Berge, *Theory of graphs and its applications*, Dunod, Paris, 1958.

[5] K. M. Dharmalingam, *Studies in Graph Theory - Equitable domiantion and Bottleneck domination*, Ph.D Thesis, 2006.

[6] K. M. Dharmalingam and V.Swaminathan, *Degree Equitable domiantion in Graphs*, Kragujevac J. Math., 35(1)(2011),191-197.

[7] G. H. Fricke, Teresa W. Haynes, S. T. Hetniemi, S. M. Hedetniemi and R. C. Laskar, *Excellent trees*, Bull. Inst. Combin. Appl., 34 (2002), 27-38.

[8] F. Harary, *Graph Theory*, ddison Wesley, Reading Mass, 1972.

[9] S. T. Hedetniemi and R. C. Laskar, *Bibliography on domination in graphs and some basic definitions of domination parameters*, Discrete Math. 86(1990) 257-277.

[10] Peter Dankelmann Durban Gayla S. Domke, Wayne Goddard, Paul Grobler, Johannes H. Hattingh and Henda C. Swart, *Maximum Sizes of Graphs with Given Domination*, ParametersPreprint submitted to Elsevier Science 15 October 2004.

[11] Terasa W. Haynes, Stephen T. Hedetneimi, Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., 1998.

[12] W.Meyer, *Equitable coloring*, Amer.Math. Monthly, 80 (1973), 920-922.

Associate Professor, Department of Mathematics, Sethu Institute of Technology, Virudhunagar District, Madurai, India

Ramanujan Research Center in Mathematics, Saraswathi Narayanan College, Madurai, India
    E-mail address: sulanesri@yahoo.com