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A CONSTRUCTION OF A QUASI-ANTIORDER BY ANTI-CONGRUENCE ON SEMIGROUP WITH APARTNESS

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Abstract. Each quasi-antiorder τ on anti-ordered semigroup S induces anti-congruence q on S such that S/q is an ordered semigroup under anti-order induced by τ. In this note we prove that the converse of this statement also holds: Each anti-congruence q on a semigroup $(S, =, \neq, \cdot)$ such that S/q is an anti-ordered semigroup induces a quasi-antiorder on S.

1. Introduction and Preliminaries

This short investigation, in Bishop's constructive algebra in sense of well-known books [**1**]- [**3**], [**6**] and Romano's papers [**4**] and [**5**], is a continuation of the author's paper [**4**]. Bishop's constructive mathematics is develop on Constructive logic - logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in the Minimal logic.

Let $(S, =, \neq)$ be a set. The relation '$\neq$' is a binary relation on S, which satisfies the following properties:

$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq z \Longrightarrow x \neq y \vee y \neq z,\ x \neq y \wedge y = z \Longrightarrow x \neq z.$

Follows Heyting, it called apartness. Let Y be a subset of S and $x \in S$. A relation q on S is a coequality relation on S if and only if it is consistent, symmetric and cotransitive (see, for example, [**4**], [**5**]). Let $(S, =, \neq, \cdot)$ be a semigroup with an apartness (see, for example [**4**] or [**5**]). As in [**4**], a coequality relation q on S is an anti-congruence if and only if it is cancellative with the semigroup operation in the following sense:

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$$(\forall x, y, z \in S)(((xz, yz) \in q \Longrightarrow (x, y) \in q) \wedge ((zx, zy) \in q \Longrightarrow (x, y) \in q)).$$

A relation α on S is anti-order (see: [**4**], [**5**]) on S if and only if

$$\alpha \subseteq \neq,$$
$$(\forall x, y, z \in S)((x, z) \in \alpha \Longrightarrow ((x, y) \in \alpha \vee (y, z) \in \alpha)),$$
$$(\forall x, y \in S)(x \neq y \Longrightarrow ((x, y) \in \alpha \vee (y, x) \in \alpha)), \text{ (linearity) and}$$
$$(\forall x, y, z \in S)(((xz, yz) \in \alpha \Longrightarrow (x, y) \in \alpha) \wedge ((zx, zy) \in \alpha \Longrightarrow (x, y) \in \alpha)).$$

A relation σ on S is a quasi-antiorder (see: [**4**], [**5**]) on S if

$$\sigma \subseteq \neq,$$
$$(\forall x, y, z \in S)((x, z) \in \alpha \Longrightarrow ((x, y) \in \sigma \vee (y, z) \in \sigma)),$$
$$(\forall x, y, z \in S)(((xz, yz) \in \sigma \Longrightarrow (x, y) \in \sigma) \wedge ((zx, zy) \in \sigma \Longrightarrow (x, y) \in \sigma)).$$

Let x be an element of S and A a subset of S. We write $x \bowtie A$ if and only if $(\forall a \in A)(x \neq a)$ holds, and $A^C = \{x \in S : x \bowtie A\}$. If τ is a quasi-antiorder on S, then (see: [**4**], Lemma 0) the relation $q = \tau \cup \tau^{-1}$ is an anti-congruence on S. Firstly, the relation $q^C = \{(x, y) \in S \times S : (x, y) \bowtie q\}$ is a congruence on S compatible with q, in the following sense

$$(\forall a, b, c \in S)(((a, b) \in q^C \wedge (b, c) \in q) \Longrightarrow (a, c) \in q).$$

We can construct semigroup $S/(q^C, q) = \{aq^C : a \in S\}$, where $aq^C = \{x \in S : (a, x) \in q\}$ is the class of relation q^C generated by the element a. If q is an anti-congruence on a semigroup S with apartness, then (see: [**4**], [**5**]) the set $S/(q, q^C)$ is a semigroup with $aq^C =_1 bq^C \Longleftrightarrow (a, b) \bowtie q$, $aq^C \neq_1 bq^C \Longleftrightarrow (a, b) \in q$, $aq^C \cdot bq^C = (ab)q^C$. We can also construct the semigroup $S/q = \{aq : a \in S\}$ where $aq = \{x \in S : (x, a) \in q\}$ is the class of relation q generated by the element a. Let q be anti-congruence on a semigroup S with apartness. Then the set S/q is a semigroup with $aq =_1 bq \Longleftrightarrow (a, b) \bowtie q$, $aq \neq_1 bq \Longleftrightarrow (a, b) \in q$, $aq \cdot bq = (ab)q$.

2. The Result

For a given ordered semigroup $(S, =, \neq, \cdot, \alpha)$ under anti-order α is essential to know if there exists a anti-congruence q on S such that S/q be an anti-ordered semigroup. This plays an important role for studying the structure of anti-ordered semigroups. The following question is natural: If $(S, =, \neq, \cdot, \alpha)$ is an anti-ordered semigroup and q an anti-congruence on S, is the set S/q anti-ordered semigroup? A probable anti-order on S/q could be the relation θ on S/q defined by means of the anti-order on S, that is $\theta = \{(xq, yq) \in S/q \times S/q : (x, y) \in \alpha\}$ is not an anti-order, in general case. The following question arises: Is there anti-congruence q on S for which S/q is anti-ordered semigroup? The concept of quasi-antiorder relation was introduced by this author in his paper [**4**]. According to [**4**], if $(S, =, \neq, \cdot, \alpha)$ is an anti-ordered semigroup and σ a quasi-antiorder on S, then the relation q on S, defined by $q = \sigma \cup \sigma^{-1}$ is an anti-congruence on S and the set S/q is an anti-ordered semigroup under anti-order θ defined by $(xq, yq) \in \theta \Longleftrightarrow (x, y) \in \sigma$. So, according to results of [**?**], each quasi-antiorder σ on an ordered semigroup S under anti-order α induces an anti-congruence $q = \sigma \cup \sigma^{-1}$ on S such that S/q is an ordered semigroup under anti-order θ. (For a further study of quasi-antiorder on anti-ordered semigroups we refer to paper [**4**] and [**5**].) In this paper we prove that the converse of this statement also holds. If $(S, =, \neq, \cdot)$ is a semigroup and q anti-congruence on S and if there exists an order relation θ on S/q such that the $(S/q, =_1, \neq_1, \circ, \theta)$ is an ordered semigroup under anti-order θ, then there exists a quasi-antiorder τ on S such that $q = \tau \cup \tau^{-1}$. So, each anti-congruence q on a semigroup $(S, =, \neq, \cdot, \alpha)$ such that S/q is an ordered semigroup under anti-order θ induces a quasi-antiorder τ on S.

THEOREM 2.1. *Theorem Let q be an anti-congruence on S and suppose there exists an anti-order relation Θ_1 on S/q such that $(S/q, =_1, \neq_1, \circ, \Theta_1)$ is an anti-ordered semigroup. Then there exists a quasi-antiorder σ on S such that $\sigma \cup \sigma^{-1} = q$ and $\Theta_1 = \theta$.*

Proof: Let q be an anti-congruence on semigroup $(S, =, \neq, \cdot)$ and let Θ_1 be an anti-order relation on S/q such that $(S/q, =_1, \neq_1, \circ, \Theta_1)$ is an anti-ordered semigroup. Let σ be a relation on S defined by $(x, y) \in \sigma \Longleftrightarrow (xq, yq) \in \Theta_1$. Then:
(1) The relation σ is a quasi-antiorder relation on S compatible with the semigroup operation:

1.1 $(x, y) \in \sigma \Longleftrightarrow (xq, yq) \in \Theta_1$
$\Longrightarrow xq \neq_1 yq$
$\Longleftrightarrow (x, y) \in q$
$\Longrightarrow x \neq y.$

1.2 $(x, z) \in \sigma \Longleftrightarrow (xq, zq) \in \Theta_1$
$\Longrightarrow (\forall yq \in S/q)((xq, yq) \in \Theta_1 \vee (yq, zq) \in \Theta_1)$
$\Longrightarrow (\forall y \in S)((x, y) \in \sigma \vee (y, z) \in \sigma);$

1.3 $(ax, by) \in \sigma \Longleftrightarrow ((ax)q, (by)q) \in \Theta 1$
$\Longleftrightarrow ((aq) \circ (xq), (bq) \circ (yq)) \in \Theta_1$
$\Longrightarrow ((aq, bq) \in \Theta_1 \vee (xq, yq) \in \Theta_1)$
$\Longleftrightarrow ((a, b) \in \sigma \vee (x, y) \in \sigma).$

(2) $q = \sigma \cup \sigma^{-1}$. Indeed:

2.1 $(a, b) \in q \Longleftrightarrow aq \neq_1 bq$
$\Longrightarrow ((aq, bq) \in \Theta_1 \vee (bq, aq) \in \Theta_1)$
$((a, b) \in \sigma \vee (b, a) \in \sigma)$
$\Longleftrightarrow (a, b) \in \sigma \cup \sigma - 1;$

2.2 $(x, y) \in \sigma \cup \sigma^{-1} \Longleftrightarrow ((x, y) \in \sigma \vee (y, x) \in \sigma)$
$\Longleftrightarrow ((xq, yq) \in \Theta_1 \vee (yq, xq) \in \Theta_1)$
$\Longrightarrow xq \neq_1 yq$
$\Longleftrightarrow (x, y) \in q;$

(3) $\Theta_1 = \Theta$. In fact:

$(aq, bq) \in \Theta_1 \Longleftrightarrow (a, b) \in \sigma$
$\Longleftrightarrow (a(\sigma \cup \sigma^{-1}), b(\sigma \cup \sigma^{-1}) \in \theta$
$\Longleftrightarrow (aq, bq) \in \theta.$ □

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