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ON WEAK CONVERGENCE THEOREM FOR NONSELF I-QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we construct Ishikawa iteration scheme with error for nonself I-quasi nonexpansive maps and establish the weak convergence of a sequence of Ishikawa iteration of nonself I-quasi nonexpansive maps in a Banach space which satisfies Opial's condition.

1. Introduction and Preliminaries

Let K be a nonempty convex subset of a real Banach space E. The map $T : K \to K$ is nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$. Nonexpansive selfmaps ever since their introduction, remained a papular area of research in various fields. Iterative construction of fixed points of these maps is a fascinating field of research. In 1967, Browder [**3**] studied the iterative construction of fixed points of nonexpansive self maps on closed and convex subset of a Hilbert space.

Two most popular iteration procedure for obtaining fixed points of T, if they exists, are : Mann iteration [**12**], defined by

$$\begin{aligned} &x_1 \in K, \\ &x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T x_n \qquad n \geqslant 1 \end{aligned} \tag{1.1}$$

and, Ishikawa Iteration [**8**], defined by

$$\begin{aligned} &x_1 \in K, \\ &x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T y_n \\ &y_n = (1-\beta_n)\, x_n + \beta_n T x_n, \qquad n \geqslant 1 \end{aligned} \tag{1.2}$$

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for certain choices of $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. If we take $\beta_n = 0$ in (1.2) then we obtain iteration (1.1). In sequel, let $F(T) = \{x \in K : Tx = x\}$ be the set of fixed points of a mapping T.

The first nonlinear ergotic theorem was proved by Baillon [**5**] for general nonexpansive mappings in Hilbert space H: If K is a closed and convex subset of $H$and T has a fixed point, then for all $x \in K$, $\{T^n x\}$is weakly almost convergent, as $n \to \infty$, to a fixed point of T. It was also shown by Pazy [**1**] that if $H$is a real Hilbert space and $\left(\frac{1}{n}\right) \sum_{i=0}^{n-1} T^i x$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$.

The concept of quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf ([**6**]) and Dotson ([**11**]) studied quasi-nonexpansive mappings in Banach spaces. Kirk ([**10**]) gave this concept in metric spaces which we adopt to a normed space as follows: $T$is called a quasi-nonexpansive mapping provided $\|Tx - p\| \leqslant \|x - p\|$ for all $x \in K$ and $p \in F(T)$.

Recall that a Banach space $E$is said to be uniformly convex if for each r with $0 \leqslant r \leqslant 2$, the modulus of convexity of Egiven by

$$\delta(r) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant r\right\}$$

satisfies the inequality $\delta(r) > 0$.

The space $E$is said to satisfy Opial's condition ([**14**]) if, for each sequence $\{x_n\}$in E, the condition $x_n \to x$ implies that $\overline{\lim_{n\to\infty}} \|x_n - x\| < \overline{\lim_{n\to\infty}} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

The following definitions and Lemma will be needed for the proof of our result.

Let Kbe a subset of a normed space $E = (E, \|.\|)$ and T and I are self mappings of K. Then T is called I-nonexpansive on K if $\|Tx - Ty\| \leqslant \|Ix - Iy\|$.

T is called I- quasi-nonexpansive on K if $\|Tx - p\| \leqslant \|Ix - p\|$ for all $x, y \in K$ and $p \in F(T) \bigcap F(I)$.

Let E be a real Banach space and K be a closed convex subset of E. A mapping $T : K \to K$ is said to be demi-closed at the origin if, for any sequence $\{x_n\}$ in K, the condition $x_n \to x_0$ weakly $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

Remark 1.1. If I is an identity map then I- nonexpansive maps and I-quasi nonexpansive mappings reduces to nonexpansive and quasi nonexpansive mappings.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all$x \in K$. A map $P : E \to E$ is a retraction if $P^2 = P$. It easily follows that if a map $P$is a retraction, then $Py = y$ for all y in the range of P. A set K is optimal if each point outside Kcan be moved to be closer to all points of K. Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Lemma 1.1. ([**15**]) *Let* $\{s_n\}$ and $\{t_n\}$ *be two nonnegative real sequences satisfying* $s_{n+1} \leqslant s_n + t_n$ *for all* $n \geqslant 1$. *If* $\sum_{n=1}^{\infty} t_n < \infty$, *then* $\lim_{n\to\infty} s_n$ *exists.*

LEMMA 1.2. ([**3**]) *Let K be a nonempty closed convex subset of a uniformly convex Banach space and let $T : K \to E$ be a nonexpansive map. Then $I - T$ is demi-closed at 0.*

LEMMA 1.3. ([**16**]) *Suppose that E is a uniformly convex Banach space and $0 < p \leqslant t_n \leqslant q < 1$ for all $n \in N$. Suppose further that* $\{x_n\}$ and $\{y_n\}$ *are sequences in E such that* $\limsup_{n\to\infty} \|x_n\| \leqslant r$, $\text{limsup}_{n\to\infty} \|y_n\| \leqslant r$ *and* $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ *hold for some $r \geqslant 0$. Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$

There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example Petryshyn and Williamson ([**13**]) studied the weak and strong convergence to a fixed points of quasi-nonexpansive maps. Their analysis was related to the convergence of Mann iterates studied by Dotson ([**11**]). Subsequently, Ghosh and Debnath ([**7**]) discussed the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces. In [**9**], the weak convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved.

In [**2**], Rhoades and Temir considered T and I self mappings of K, where $T$is an I-nonexpansive mapping. They established the weak convergence of sequence of Mann iterates to a common fixed point of T and I. Subsequently, Kiziltunc and Ozdemir [**4**] considered T and I be nonself mappings of K with T is I-nonexpansive mapping and establish the weak convergence theorem of the sequence of Ishikawa iterates to a common fixed point of T and I.

In this paper, we consider T and Inonself mappings of K, where T is an I-quasi nonexpansive mapping and establish the weak convergence of the sequence of Ishikawa iterates with error to a common fixed point of T and I.

Iteration Scheme 1.4 [Ishikawa Iteration with error]: Let E be a uniformly convex Banach space, let K be a nonempty convex subset of E with Pas a nonexpansive retraction. Let $T : K \to E$ be a given nonself mapping. The Ishikawa iterative scheme with error is defined as follows:

$$\begin{cases} x_1 \in K \\ x_{n+1} = P\left(\alpha_n x_n + \beta_n T y_n + \gamma_n u_n\right) \\ \quad y_n = P\left(\alpha_n' x_n + \beta_n' T x_n + \gamma_n' v_n\right), \quad n \geqslant 1 \end{cases} \tag{1.3}$$

Where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}$ and $\{\gamma_n'\}$ are real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n'$; and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K.

2. Main Results

Before proving our main result we begin with the following lemmas.

LEMMA 2.1. *Let Kbe a closed convex bounded subset of a uniformly convex Banach space $E$and let T, I be two nonself mappings with T be I-quasi-nonexpansive mapping, I a nonexpansive mapping on K. If $\{x_n\}$is defined as in* (1.3) *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}$ and $\{\gamma_n'\}$ *are real sequences in* $[0,1]$ *such that* $\alpha_n +$

$\beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ *;* $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \gamma'_n < \infty$ *;* $\{u_n\}$ and $\{v_n\}$ *are bounded sequences in* K*, then* $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. For $p \in F(T) \cap F(I)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|P(\alpha_n x_n + \beta_n T y_n + \gamma_n u_n) - p\| \\ &\leqslant \alpha_n \|x_n - p\| + \beta_n \|T y_n - p\| + \gamma_n \|u_n - p\| \\ &\leqslant \alpha_n \|x_n - p\| + \beta_n \|I y_n - p\| + \gamma_n \|u_n - p\| \\ &\leqslant \alpha_n \|x_n - p\| + \beta_n \|y_n - p\| + \gamma_n \|u_n - p\| \qquad (2.1) \end{aligned}$$

where

$$\begin{aligned} \|y_n - p\| &= \|P(\alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n) - p\| \\ &\leqslant \alpha'_n \|x_n - p\| + \beta'_n \|T x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leqslant \alpha'_n \|x_n - p\| + \beta'_n \|I x_n - p\| + \gamma'_n \|v_n - p\| \\ &\leqslant \alpha'_n \|x_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \qquad (2.2) \end{aligned}$$

Substituting the value of (2.2) into (2.1) we obtain,

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant (\alpha_n + \alpha'_n \beta_n + \beta_n \beta'_n) \|x_n - p\| + \gamma_n \|u_n - p\| + \beta_n \gamma'_n \|v_n - p\| \\ &\leqslant ((1 - \beta_n) + (1 - \beta'_n)\beta_n + \beta_n \beta'_n) \|x_n - p\| + \gamma_n \|u_n - p\| + \beta_n \gamma'_n \|v_n - p\| \\ &\leqslant \|x_n - p\| + d_n \end{aligned}$$

where $d_n = \gamma_n \|u_n - p\| + \beta_n \gamma'_n \|v_n - p\|$
Since $\sum_{n=1}^{\infty} \gamma_n < \infty$, and $\sum_{n=1}^{\infty} \gamma'_n < \infty$ implies that $\sum_{n=1}^{\infty} d_n < \infty$ and by Lemma (1.1) $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof of the lemma. □

Lemma 2.2. *Let* E *be a uniformly convex Banach space and let* K *be a nonempty closed convex subset of* E *. Let* $T : K \to E$ *be a I-quasi-nonexpansive mapping with* $F(T) \cap F(I) \neq \phi$*and* I *a nonexpansive mapping. Let* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ $\{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ *are real sequences in* $[0,1]$ *such that* $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ *and* $\varepsilon \leqslant \beta_n, \beta'_n \leqslant 1 - \varepsilon$ *for all* $n \in N$ *and some* $\varepsilon > 0$ *;*$\{u_n\}$ and $\{v_n\}$ *are bounded sequences in* K*. Then for the sequence* $\{x_n\}$ *given by* (1.3) *, we have* $\lim_{n\to\infty} \|x_n - T x_n\| = 0$.

Proof. For any $p \in F(T) \cap F(I)$, set

$$\begin{aligned} r_1 &= \sup\{\|u_n - p\| : n \geqslant 1\}, \\ r_2 &= \sup\{\|v_n - p\| : n \geqslant 1\}, \\ r_3 &= \sup\{\|x_n - p\| : n \geqslant 1\}, \\ r &= \max\{r_i : 1 \leqslant i \leqslant 3\} \end{aligned}$$

Now consider

$$\begin{aligned}
\|y_n - p\| &= \|P(\alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n) - p\| \\
&\leqslant \alpha'_n \|x_n - p\| + \beta'_n \|T x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leqslant \alpha'_n \|x_n - p\| + \beta'_n \|I x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leqslant \alpha'_n \|x_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leqslant (\alpha'_n + \beta'_n) \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\leqslant \|x_n - p\| + \gamma'_n r \qquad (2.3)
\end{aligned}$$

Since by Lemma (2.1) $\lim_{n\to\infty} \|x_n - p\|$exists. Let $\lim_{n\to\infty} \|x_n - p\| = c$, then by the continuity of T the conclusion follows.
Now, let $c > 0$. We claim that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.
Since $\{u_n\}$ and $\{v_n\}$ are bounded, it follows that $\{u_n - x_n\}$ and $\{v_n - x_n\}$ are bounded.
Taking limit sup on both sides in the inequality (2.3), we have

$$\limsup_{n\to\infty} \|y_n - p\| < c \qquad (2.4)$$

Next consider,

$$\begin{aligned}
\|Ty_n - p + \gamma_n (u_n - x_n)\| &\leqslant \|Ty_n - p\| + \gamma_n \|u_n - x_n\| \\
&\leqslant \|Iy_n - p\| + \gamma_n r \\
&\leqslant \|y_n - p\| + \gamma_n r
\end{aligned}$$

Taking limit sup on both sides in the above inequality and using (2.4), we get

$$\limsup_{n\to\infty} \|Ty_n - p + \gamma_n (u_n - x_n)\| \leqslant c$$

Then $\|x_n - p + \gamma_n (u_n - x_n)\| \leqslant \|x_n - p\| + \gamma_n \|u_n - x_n\| \leqslant \|x_n - p\| + \gamma_n r$ yields

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n (u_n - x_n)\| \leqslant c$$

Again $\lim_{n\to\infty} \|x_{n+1} - p\| = c$ means that

(2.5)

$$\liminf_{n\to\infty} \|\beta_n (Ty_n - p + \gamma_n (u_n - x_n)) + (1 - \beta_n)(x_n - p + \gamma_n (u_n - x_n))\| \geqslant c$$

On the other hand we have

$$\begin{aligned}
&\|\beta_n (Ty_n - p + \gamma_n (u_n - x_n)) + (1 - \beta_n)(x_n - p + \gamma_n (u_n - x_n))\| \\
&\leqslant \beta_n \|Ty_n - p\| + (1 - \beta_n) \|x_n - p\| + \gamma_n \|u_n - x_n\| \\
&\leqslant \beta_n \|Iy_n - p\| + (1 - \beta_n) \|x_n - p\| + \gamma_n \|u_n - x_n\| \\
&\leqslant \beta_n \|y_n - p\| + (1 - \beta_n) \|x_n - p\| + \gamma_n \|u_n - x_n\| \\
&\leqslant \beta_n (\|x_n - p\| + \gamma'_n r) + (1 - \beta_n) \|x_n - p\| + \gamma_n \|u_n - x_n\| \\
&\leqslant \|x_n - p\| + \gamma'_n r + \gamma_n r
\end{aligned}$$

Therefore we obtain
(2.6)
$$\limsup_{n\to\infty} \|\beta_n (Ty_n - p + \gamma_n (u_n - x_n)) + (1-\beta_n)(x_n - p + \gamma_n (u_n - x_n))\| \leqslant c$$

From (2.5) and (2.6) we get
$$\lim_{n\to\infty} \|\beta_n (Ty_n - p + \gamma_n (u_n - x_n)) + (1-\beta_n)(x_n - p + \gamma_n (u_n - x_n))\| = c$$

Hence applying Lemma (1.3) we have $\lim_{n\to\infty} \|Ty_n - x_n\| = 0$.

Since P is a nonexpansive retraction we have
$$\begin{aligned}
\|x_n - Tx_n\| &\leqslant \|x_n - Ty_n\| + \|Tx_n - Ty_n\| \\
&\leqslant \|x_n - Ty_n\| + \|Ix_n - Iy_n\| \\
&\leqslant \|x_n - Ty_n\| + \|x_n - y_n\| \\
&\leqslant \|x_n - Ty_n\| + \|Px_n - P(\alpha'_n x_n + \beta'_n Tx_n + \gamma'_n v_n)\| \\
&\leqslant \|x_n - Ty_n\| + \|x_n - (\alpha'_n x_n + \beta'_n Tx_n + \gamma'_n v_n)\| \\
&\leqslant \|x_n - Ty_n\| + \beta'_n \|x_n - Tx_n\| + \gamma'_n \|x_n - v_n\| \\
&\leqslant \|x_n - Ty_n\| + \beta'_n \|x_n - Tx_n\| + \gamma'_n r
\end{aligned}$$

That is $(1-\beta'_n)\|x_n - Tx_n\| \leqslant \|x_n - Ty_n\| + \gamma'_n r$
On taking limit as $n \to \infty$ both sides we get $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.This completes the proof of the lemma. □

Now we prove our main result.

Theorem 2.1. *Let E be a uniformly convex Banach space satisfying the Opial's property and let K, T and $\{x_n\}$ be as in Lemma (2.2). If $F(T) \cap F(I) \neq \phi$, then $\{x_n\}$ converges weakly to a fixed point of $F(T) \cap F(I)$.*

Proof. For any $p \in F(T) \cap F(I)$, it follows from Lemma (2.1) that
$$\lim_{n\to\infty} \|x_n - p\|$$
exists. We now prove that $\{x_n\}$has a unique weak sub sequential limit in $F(T)$. By Lemmas (1.2) and(2.2), we know that $p \in F(T)$.

Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$be two sub sequences of $\{x_n\}$ which converges weakly to p and q, respectively. We will show that $p = q$.

Suppose that E satisfies Opial's property and that $p \neq q$ is in weak limit set of the sequence $\{x_n\}$. Then $\{x_{n_k}\} \to p$ and $\{x_{m_k}\} \to q$, respectively. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T) \cap F(I)$, then by Opial's property we conclude that
$$\begin{aligned}
&\lim_{n\to\infty} \|x_n - p\| = \\
&\lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| < \lim_{j\to\infty} \left\|x_{m_j} - p\right\| = \\
&\lim_{n\to\infty} \|x_n - p\|
\end{aligned}$$
a contradiction. This proves that $\{x_n\}$ converges weakly to a fixed point of $F(T) \cap F(I)$. This completes the proof of the theorem. □

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