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On (γ, γ')-connected spaces

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Abstract. In this paper, we define (γ, γ')-connected spaces and study their properties in topological spaces.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topic of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. Kasahara [**5**] defined the concept of an operation on topological spaces. Umehara et. al. [**7**] introduced the notion of $\tau_{(\gamma,\gamma')}$ which is the collection of all (γ, γ')-open sets in a topological space (X, τ). Recently, G. S. S. Krishnan and K. Balachandran (see [**1**], [**3**], [**2**]) studied in this field. In this paper, we introduce and study the concepts of minimal (γ, γ')-open and maximal (γ, γ')-closed sets in topological spaces. In this paper, we define (γ, γ')-connected spaces and study their properties in topological spaces.

2. preliminaries

Definition 2.1. Let (X, τ) be a topological space. An operation γ [**5**] on the topology τ is function $\gamma : \tau \to P(X)$ such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V.

Definition 2.2. A subset A of a topological space (X, τ) is said to be (γ, γ')-open set [**7**] if for each $x \in A$ there exist open neighbourhoods U and V of x such that $U^\gamma \cup V^{\gamma'} \subset A$. The complement of (γ, γ')-open set is called (γ, γ')-closed.

Definition 2.3. [**4**] Let A be a subset of a topological space (X, τ). A point $x \in A$ is said to be (γ, γ')-interior point of A if there exist open neighbourhods U and V

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of x such that $U^\gamma \cup V^{\gamma'} \subset A$ and we denote the set of all such points by $\text{Int}_{(\gamma,\gamma')}(A)$. Thus $\text{Int}_{(\gamma,\gamma')}(A) = \{x \in A : x \in U \in \tau, V \in \tau$ and $U^\gamma \cup V^{\gamma'} \subset A$. Note that A is (γ,γ')-open if and only if $A = \text{Int}_{(\gamma,\gamma')}(A)$. A set A is called (γ,γ')-closed if and only if $X \backslash A$ is (γ,γ')-open.

Definition 2.4. [**7**] A point $x \in X$ is called a (γ,γ')-closure point of $A \subset X$, if $(U^\gamma \cup V^{\gamma'}) \cap A \neq \emptyset$, for any open neighbourhoods U and V of x. The set of all (γ,γ')-closure points of A is called (γ,γ')-closure of A and is denoted by $\text{Cl}_{(\gamma,\gamma')}(A)$. A subset A of X is called (γ,γ')-closed, if $\text{Cl}_{(\gamma,\gamma')}(A) \subset A$. Note that $\text{Cl}_{(\gamma,\gamma')}(A)$ is contained in every (γ,γ')-closed superset of A.

Definition 2.5. [**6**] An operation γ on τ is said to be regular if for any open neighbourhoods U, V of $x \in X$, there exits an open neighbourhood W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition 2.6. [**6**] An operation γ on τ is said to be open if for any open neighbourhood U of each $x \in X$, there exists (γ,γ')-open set B such that $x \in B$ and $U^\gamma \supseteq B$.

3. Properties of (γ,γ')-connected spaces

Definition 3.1. A topological space (X,τ) is said to be (γ,γ')-connected if there does not exist a pair A, B of nonempty disjoint (γ,γ')-open subset of X such that $X = A \cup B$, otherwise X is called (γ,γ')-disconnected. In this case, the pair (A, B) is called a (γ,γ')-disconnection of X. A subset A of a space (X,τ) is (γ,γ')-connected if it is (γ,γ')-connected as a subspace.

Example 3.1. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}$. For $b \in X$, defined an operation $\gamma : \tau \to P(X)$ such that

$$\gamma(A) = \begin{cases} A & \text{if a} \in \text{A}, \\ \text{Cl}(A) & \text{if a} \notin \text{A}, \end{cases}$$

and

$$\gamma'(A) = \begin{cases} A & \text{if A} = \{\text{a, c}\}, \\ A \cup \{b\} & \text{if A} \neq \{\text{a, c}\}. \end{cases}$$

It is clear that X is (γ,γ')-connected but not connected.

Theorem 3.1. *A topological space (X,τ) is (γ,γ')-disconnected (resp. (γ,γ')-connected) if and only if there exists a (resp. does not exist) nonempty subset A of X which is both (γ,γ')-open and (γ,γ')-closed in X.*

Proof. The proof is clear. □

Definition 3.2. A mapping $f : (X,\tau_1) \to (Y,\tau_2)$ is said to be $((\gamma,\gamma'),(\beta,\beta'))$-continuous if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U such that $x \in U$ and $f(U^\gamma) \subset V^\beta$, where $\gamma : \tau_1 \to P(X); \beta : \tau_2 \to P(Y)$ are operations on τ_1 and τ_2, respectively.

A $((\gamma, \gamma'), (\beta, \beta'))$-continuous mapping has be charachterized as:
If $f : (X, \tau_1) \to (Y, \tau_2)$ is a mapping and (β, β') is open, then f is $((\gamma, \gamma'), (\beta, \beta'))$-continuous if and only if for each (β, β')-open set V in Y, $f^{-1}(V)$ is (γ, γ')-open in X. We use this characterization and prove:

Theorem 3.2. *The $((\gamma, \gamma'), (\beta, \beta'))$-continuous image of (γ, γ')-connected spce is (γ, γ')-connected, where (β, β') is open.*

Proof. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be $((\gamma, \gamma'), (\beta, \beta'))$-continuous from a (γ, γ')-connected space (X, τ_1) onto a space (Y, τ_2). Suppose that Y is (γ, γ')-disconnected and (A, V) is a (γ, γ')-disconnection of Y. Since f is $((\gamma, \gamma'), (\beta, \beta'))$-continuous, therefore $f^{-1}(A), f^{-1}(B)$ are both (γ, γ')-open in X. Clearly $f^{-1}(A), f^{-1}(B)$ is a pair of (γ, γ')-disconnection of X a contradiction. Hence Y is (γ, γ')-connected. □

Definition 3.3. The (γ, γ')-boundary of a subset A of (X, τ) is defined as

$$\mathrm{Cl}_{(\gamma,\gamma')}(A) \cap \mathrm{Cl}_{(\gamma,\gamma')}(X \backslash A).$$

Next we characterize (γ, γ')-connectedness in terms of (γ, γ')-boundary as.

Theorem 3.3. *A topological space (X, τ) is (γ, γ')-connected if and only if every nonempty proper subspace has a nonempty (γ, γ')-boundary.*

Proof. Suppose that a nonempty proper subspace A of a (γ, γ') - connected space (X, τ) has empty (γ, γ') - boundary. Then A is (γ, γ')-open and $\mathrm{Cl}_{(\gamma,\gamma')}(A) \cap \mathrm{Cl}_{(\gamma,\gamma')}(X \backslash A) = \emptyset$. Let p be a (γ, γ')-limit point of A. Then $p \in \mathrm{Cl}_{(\gamma,\gamma')}(A)$ but $p \notin \mathrm{Cl}_{(\gamma,\gamma')}(X \backslash A)$. In particular, $p \notin X \backslash A$ and so $p \in A$. Thus A is (γ, γ')-closed and (γ, γ')-open. By theorem 3.1, X is (γ, γ')-disconnected. This contradicition proves that A has a nonempty (γ, γ')-boundary. Conversely, suppose X is (γ, γ')-disconnected. Then by Theorem 3.1, X has a proper subspace A which is both (γ, γ')-closed and (γ, γ')-open. Then $\mathrm{Cl}_{(\gamma,\gamma')}(A) = A, \mathrm{Cl}_{(\gamma,\gamma')}(X \backslash A) = (X \backslash A)$ and $\mathrm{Cl}_{(\gamma,\gamma')}(A) \cap \mathrm{Cl}_{(\gamma,\gamma')}(X \backslash A) = \emptyset$. So A has empty (γ, γ')-boundary, a contradiction. Hence X is (γ, γ')-connected. □

Definition 3.4. A two point discrete space $D = \{a, b\}$ is called (γ, γ')-discrete if $\tau_{(\gamma,\gamma')} = \tau$.

Theorem 3.4. *If a space (X, τ) is (γ, γ')-connected, then there does not exist a surjective $((\gamma, \gamma'), (\beta, \beta'))$ - continuous function f from X onto two point (γ, γ') - discrete space, where (β, β') is open.*

Proof. Suppose there exits a $((\gamma, \gamma'), (\beta, \beta'))$ - continuous from a (γ, γ') - connected space (X, τ) onto a two point (γ, γ')-discrete space $D = \{a, b\}$. Then $((\gamma, \gamma'), (\beta, \beta'))$-continuity of f impolies $A = f^{-1}\{a\}$ and $B = f^{-1}\{b\}$ are (γ, γ')-open in X. Clearly (A, B) is a (γ, γ')-disconnection of X. This contradiction proves the theorem. □

Definition 3.5. Let X be a space and $A \subset X$. Then the class of (γ, γ')-open sets in A is defined in a natural way as: $\tau_{(\gamma,\gamma')A} = \{A \cap O : O \in \tau_{(\gamma,\gamma')}\}$, where $\tau_{(\gamma,\gamma')}$ is the class of (γ, γ')-open sets of X. That is, G is (γ, γ')-open in A if and only if $G = A \cap O$, where O is a (γ, γ')-open set in X.

THEOREM 3.5. *Let (A, B) be a (γ, γ')-disconnection of a space (X, τ) and C be a (γ, γ')-connected subspace of X. Then C is contained in A or B.*

PROOF. Suppose that C is neither contained in A nor in B. Then $C \cap A, C \cap B$ are both nonempty (γ, γ')-open subsets of C such that $(C \cap A) \cap (C \cap B) = \emptyset$ and $(C \cap A) \cup (C \cap B) = C$. This gives that $(C \cap A, C \cap B)$ is a (γ, γ')-disconnection of C. This contradiction proves the theorem. □

THEOREM 3.6. *Let $X = \underset{\alpha \in I}{\cup} \{X_\alpha\}$, where each X_α is (γ, γ')-connected and $\underset{\alpha \in I}{\cap} \{X_\alpha\} \neq \emptyset$. Then X is (γ, γ')-connected.*

PROOF. Suppose on the contrary that (A, B) is a (γ, γ')-disconnection of X. Since each X_α is (γ, γ')-connected, therefore by Theorem 3.5, $X_\alpha \subset A$ or $X_\alpha \subset B$. Since $\cap X_\alpha \neq \emptyset$, thereore all X_α are contained in A or in B. This gives that, if $X \subset A$, then $B = \emptyset$ or if $X \subset B$, then $A = \emptyset$. This contradictions proves that X is (γ, γ')-connected. □

Using Theorem 3.6, we characterize (γ, γ')-connectedness as:

THEOREM 3.7. *A space (X, τ) is (γ, γ')-connected if and only if for every pair of points x, y in X, there is a (γ, γ')-connected subset of X which contains both x and y.*

PROOF. The necesity is immediate since the (γ, γ')-connected space itself contains these two points. For the sufficiency, suppose that for any two points x, y; there is a (γ, γ')-connected subspace $C_{x,y}$ of X such that $x, y \in C_{x,y}$. Let $a \in X$ be a fixed point and $\{C_{a,c}, x \in X\}$ be a class of all (γ, γ')-connected subsets of X which contain a and $x \in X$. Then $X = \underset{x \in X}{\cup} \{C_{a,x}\}$ and $\underset{x \in X}{\cap} \{C_{a,x}\} = \emptyset$. Therefore by Theorem 3.6, X is (γ, γ')-connected. □

THEOREM 3.8. *Let C be a (γ, γ')-connected subset of a space (X, τ) and $A \subset X$ such that $C \subset A \subset \mathrm{Cl}_{(\gamma, \gamma')}(C)$. Then A is (γ, γ')-connected.*

PROOF. It is sufficient to show that $\mathrm{Cl}_{(\gamma, \gamma')}(C)$ is (γ, γ')-connected. On the contrary, suppose that $\mathrm{Cl}_{(\gamma, \gamma')}(C)$ is (γ, γ')-disconnected. Then there exists a (γ, γ')-disconnection (H, K) of $\mathrm{Cl}_{(\gamma, \gamma')}(C)$. That is, there are $H \cap C, K \cap C$ (γ, γ')-open sets in C such that $(H \cap C) \cap (K \cap C) = (H \cap K) \cap C = \emptyset$, and $(H \cap C) \cup (K \cap C) = (H \cup K) \cap C = C$. This gies that $(H \cap C, K \cap C)$ is a (γ, γ')-disconnection of C, a contradiction. This proves that $\mathrm{Cl}_{(\gamma, \gamma')}(C)$ is (γ, γ')-connected. □

DEFINITION 3.6. A maximal (γ, γ')-connected subset of a space (X, τ) is called a (γ, γ')-component of X. If X is itself (γ, γ')-connected, then X is the only (γ, γ')-component of X.

EXAMPLE 3.2. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}$. For $b \in X$, defined an operation $\gamma : \tau \to P(X)$ such that

$$\gamma(A) = \begin{cases} A & \text{if A = \{a\}}, \\ \mathrm{Cl}(A) & \text{if A} \neq \text{\{a\}}, \end{cases}$$

and

$$\gamma'(A)=\begin{cases} A & \text{if A} \neq \{\text{b}\},\\ \mathrm{Cl}(A) & \text{if A} = \{\text{b}\}.\end{cases}$$

It is clear that $\{a,c\}$ is a maximal (γ,γ')-connected set.

THEOREM 3.9. *Let X be a topological space. Then we have the following*

(1) *For each $x\in X$, there is exactly one (γ,γ')-component of X containing x.*
(2) *Each (γ,γ')-connected subset of X is contained in exactly one (γ,γ')-component of X.*
(3) *A (γ,γ')-connected subset of X which is both (γ,γ')-open and (γ,γ')-closed is a (γ,γ')-connected, if γ and γ' are regular.*
(4) *Every (γ,γ')-component of X is (γ,γ')-closed in X.*

PROOF. (1) Let $x\in X$ and $\{C_\alpha:\alpha\in I\}$ a class of all (γ,γ')-connected subsets of X containing x. Put $C=\underset{\alpha\in I}{\cup}C$, then by Theorem 3.6, C is (γ,γ')-connected and $x\in X$. Suppose $C\subset C^*$ for some (γ,γ')-connected subset C^* of X. Then $x\in C^*$ and hence C^* is one of the C_α's and hence $C^*\subset C$. Consequently $C=C^*$. This proves that C is a (γ,γ')-component of X which contains x. (2). Let A be a (γ,γ')-connected subset of X which is not a (γ,γ')-component of X. Suppose that C_1,C_2 are (γ,γ')-components of X such that $A\subset C_1, A\subset C_2$. Since $C_1\cap C_2=\emptyset, C_1\cup C_2$ is another (γ,γ')-connected set which contains C_1 as well as that C_2, a cotradiction to the fact that C_1 and C_2 are (γ,γ')-components. This proves that A is contained in exactly one (γ,γ')-component of X. (3) Suppose that A is (γ,γ')-connected subset of X which is both (γ,γ')-open and (γ,γ')-closed. By (2), A is contained is exactly one (γ,γ')-component C of X. If A is a proper subset of C, and since (γ,γ') is regular, therefore $C=(C\cap A)\cup(C\cap(X\backslash A))$ is a (γ,γ')-disconnection of C, a contradiction. Thus $A=C$. (4) Suppose a (γ,γ')-component C of X is not (γ,γ')-closed. Then by Theorem 3.8, $\mathrm{Cl}_{(\gamma,\gamma')}(C)$ is (γ,γ')-connected containing (γ,γ')-component C of X. This implies $C=\mathrm{Cl}_{(\gamma,\gamma')}(C)$ and hence C is (γ,γ')-closed. □

4. (γ,γ')-Locally connected spaces

DEFINITION 4.1. A space (X,τ) is said to be (γ,γ')-locally connected if for any point $x\in X$ and any (γ,γ')-open set U containing x, there is a (γ,γ')-connected (γ,γ')-open set V such that $x\in V\subset U$.

EXAMPLE 4.1. Let $X=\{a,b,c\},\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{a,c\}\}$. For $a\in X$, define an operation $\gamma:\tau\to P(X)$ such that

$$\gamma(A)=\begin{cases} A & \text{if a} \in \text{A},\\ \mathrm{Cl}(A) & \text{if a} \notin \text{A},\end{cases}$$

and

$$\gamma'(A)=\begin{cases} A & \text{if A} \neq \{\text{b}\},\\ \mathrm{Cl}(A) & \text{if A} = \{\text{b}\}.\end{cases}$$

It is clear that $\emptyset, X, \{a\}, \{b\}, \{a, b\}$ are the only (γ, γ')-open sets. Clearly X is (γ, γ')-locally connected but not locally connected.

THEOREM 4.1. *If X is a (γ, γ')-locally connected space, then X has a (γ, γ')-neighbourhood base comprising (γ, γ')-connected (γ, γ')-open sets.*

PROOF. Let (β, β') be the class of all (γ, γ')-connected (γ, γ')-open subsets of a (γ, γ')-locally connected space (X, τ). We show that (β, β') is a (γ, γ')-neighbourhood base for a topology τ on X. Let U be (γ, γ')-open subset on X and $x \in U$. Since X is (γ, γ')-locally connected space, therefore there exists a (γ, γ')-connected (γ, γ')-open set $B \in \beta$ such that $x \in B \subset U$. This implies that each (γ, γ')-open set in X is the union of members of (β, β'). Consequently (β, β') is a (γ, γ')-neighbourhood base for τ. □

The following theorem shows that (γ, γ')-locally connectedness is a (γ, γ')-open hereditary property.

THEOREM 4.2. *A (γ, γ')-open subset of (γ, γ')-locally connected space is (γ, γ')-locally connected.*

PROOF. Let U be a (γ, γ')-open subset of a (γ, γ')-locally connected space (X, τ). Let $x \in U$ and V be a (γ, γ')-open neighbourhood of x in U. Then V is a (γ, γ')-open neighbourhood of x in X. Since X is (γ, γ')-locally connected, therefore there exists a (γ, γ')-connected, (γ, γ')-open neighbourhood W of x such that $x \in W \subset V$. In this way W is also a (γ, γ')-connected (γ, γ')-open neighbourhood x in U such that $x \in W \subset U \subset V$ or $x \in W \subset V$. This proves that U is (γ, γ')-locally connected. □

DEFINITION 4.2. A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be $((\gamma, \gamma'), (\beta, \beta'))$-closed (resp. $((\gamma, \gamma'), (\beta, \beta'))$-open) if for any (γ, γ')-closed ((γ, γ')-open) set A of X, $f(A)$ is (β, β')-closed (resp. (β, β')-open) in Y.

THEOREM 4.3. *A $((\gamma, \gamma'), (\beta, \beta'))$-continuous, $((\gamma, \gamma'), (\beta, \beta'))$-open surjective image of (γ, γ')-locally connected space is (γ, γ')-locally connected space.*

PROOF. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be $((\gamma, \gamma'), (\beta, \beta'))$-continuous, $((\gamma, \gamma'), (\beta, \beta'))$ - open from a (γ, γ')-locally connected space (X, τ) to a space Y. We show that $Y = f(X)$ is (γ, γ')-locally connected space. Let $y \in Y$ and choose $x \in X$ such that $f(x) = y$. Let U be (β, β')-open set containing x. Since X is (γ, γ')-locally connected, there exists a (γ, γ')-connected, (γ, γ')-open set V containing x such that $x \in V \subset f^{-1}(U)$. This gives that $f(x) \in f(V) \subset f(f^{-1}(U)) = U$ or $y \in f(V) \subset U$. Since f is $((\gamma, \gamma'), (\beta, \beta'))$-continuous, $f(V)$ is (γ, γ')-open. Moreover $f(V)$ is (γ, γ')-connected. This proves that Y is (γ, γ')-locally connected. □

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