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ROBUST FINITE-TIME STABILITY OF CONTINUOUS AND DISCRETE-TIME SYSTEMS WITH INTERVAL TIME-VARYING DELAY AND NONLINEAR PERTURBATIONS

Abstract. In this paper, we present one approach in finite-time stability analysis of continuous and discrete-time systems with interval time-varying delay, nonlinear perturbations and parameter uncertainties. The new integral inequality for continuous quadratic function with exponential weights and new finite sum inequality for discrete quadratic function with the power weights are derived. By using these inequalities and convenient continuous and discrete Lyapunov-Krasovskii-like functionals with exponential and power weights, respectively, some sufficient conditions of finite-time stability are obtained in form of linear matrix inequalities. Some numerical examples are presented to illustrate the proposed methodology.

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1. Introduction

The phenomenon of time-delay is very common in practical engineering systems, such as chemical systems, biological systems, mechanical systems, and networked control systems. Frequently, the time-delay is not constant but time-varying. The existence of time-delay might result to the performance deterioration or even instability of system. A large number of researchers were involved in the study of time-delay systems (see [1-8] and references therein).

In practice, due to environmental noise, uncertain or slowly varying parameters and time-delay, the most real problems can be modelled by systems with interval time-varying delay, nonlinear perturbations and parameter uncertainties. The nonlinearities and/or parameter uncertainties also can cause instability and poor performance of practical systems. Many nonlinearities can be approximated by a function that satisfy the growth Lipschitzian condition, while the most uncertainties can be characterized by norm-bounded time-varying structured perturbations. Consequently, the stability problem of time-delay systems with nonlinear perturbations and parameter uncertainties has received increasing attention (see [9-19] and reference therein).

The most of the existing literature related to system stability focus upon Lyapunov asymptotic stability, which is defined over an infinite-time interval. However, in many practical applications, this concept is often insufficient to study the transient performances of a system. A system can be Lyapunov stable but completely useless because it possesses undesirable transient performances. For example, the temperature, pressure or some other parameters in industrial processes should be kept within specified bounds in a prescribed time interval. In order to study these problems, the concept of finite-time stability (FTS) was introduced in [20] and [21]. In that sense, a system is said to be FTS during a fixed time interval if its state do not exceed some prescribed bound for bounded initial states. Initially, the concept of FTS had only academic significance. With the development of the theory of linear matrix inequalities (LMIs), this stability concept has attracted remarkable attention of researchers, see for instance [22], [23] for continuous and [24-26] for discrete-time regular systems. Recently, the method of FTS is applied to various systems, such as nonlinear systems, neural network systems, fuzzy systems, switched systems and uncertain systems. Also, FTS and finite-time H_∞ control problems have attracted great attention from both academic and industrial community (see [27] and [28]).

The concept of finite-time stability can be also applied to time-delay systems. Some early results of FTS for constant time-delay systems can be found in [29]. The results of these investigations are conservative since they are based on restrictive algebraic inequalities. Recently, using LMIs, less conservative results are obtained for FTS of time-delay systems [30-39]. In many practical systems, time delay is not constant but time-varying. In particular, many researchers pay attention to the systems with interval time-varying delay, which means the lower bound of time delay is not restricted to zero (see [13-18] for Lyapunov stability and [19], [35-39] for FTS).

The FTS for continuous-time systems with time-varying delay and the nonlinear perturbations and/or parameter uncertainties are discussed in [19, 31, 38] and [40-49]. However, to the best of our knowledge, a few works, which refer to the FTS of discrete-time systems with interval time-varying delay, nonlinear perturbations and parameter uncertainties, have been published up to date. These articles study the following systems and their properties: neural networks with Markovian jumps [50], uncertain discrete jump systems [51], and discrete-time switched nonlinear systems [52].

The goal of this paper is to present the main authors' results in the FTS analysis of continuous [44] and discrete-time [47] systems with interval time-varying delay, nonlinear perturbations and parameter uncertainties. The new integral inequality (II) for continuous quadratic function with exponential weights and new finite sum inequality (FSI) for discrete quadratic function with the power weights are derived. It has shown that the II and FSI are less conservative then the corresponding continuous and discrete Jensen's inequality. Further, the new continuous Lyapunov-Krasovskii like functional (CLKLF) with exponential weights $e^{\gamma(t-s)}$ (for continuous-time systems) and discrete Lyapunov-Krasovskii like functional (DLKLF) with power weights γ^{k-j-1} (for discrete-time systems) are proposed. By using II (FSI), the inequality $V(x(t)) < e^{\gamma t}V(x(0))$ $(V(k) < \gamma^k V(0))$ is obtained and more precisely estimations of upper bound of $V(x(0))$ $(V(0))$ and lower bound of $V(x(t))$ $(V(k))$ are estimated. As special cases, the problems of FTS for nominal systems with constant or time-varying delay are considered. Finally, the numerical examples are presented to illustrate the effectiveness of the developed results and their improvement over the existing literature.

The rest of the paper is organized as follows. In Section 2, the problem formulation is given and new integral inequality and new finite sum inequality are derived. Sections 3 and 4 present FTS for continuous and discrete-time delay systems with nonlinear perturbations and parameter uncertainties, respectively. Finally, some numerical examples with system simulations is presented in Section 5 to show the effectiveness of the proposed criteria.

Notations. Z^+ denotes the set of all real non-negative numbers. The matrix transposition was denoted by a superscript 'T'. $\Re^n$ and $\Re^{n\times m}$ are the n-dimensional Euclidean spaces and the set of all real matrices having dimension n × m, respectively. $X > 0$ $(X \geq 0)$ denotes a real positive definite (semi-definite) matrix, while $X > Y$ $(X \geq Y)$ implies that the matrix $X - Y$ is a positive definite (semi-definite) matrix. $\lambda_{\max}(X)$ $(\lambda_{\min}(X))$ denotes the maximum (minimum) of eigenvalues of a real symmetric matrix X. The symbol * within a matrix represents the symmetric term of the matrix. NF is short for "it is not feasible".

2. Problem formulation and preliminaries

2.1. Continuous-time systems. Consider the following continuous-time system with time-varying delay and nonlinear perturbations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d(t)) + f(x(t),t) + g(x(t-d(t)),t), \\ x(\theta) &= \phi(\theta), \quad \theta \in [-d_M, 0] \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $A \in \Re^{n\times n}$ and $A_d \in \Re^{n\times n}$ are known constant matrices. The time-varying delay function $d(t)$ satisfies

$$d_m \leq d(t) \leq d_M, \quad \dot{d}(t) \leq \rho < 1 \tag{2}$$

The initial condition, $\phi(\theta)$, is a continuous and differentiable vector-valued function of $\theta \in [-d_M, 0]$ whose the first derivative satisfies

$$\sup_{\theta\in[-d_M,0]} \dot{\phi}^T(\theta)\dot{\phi}(\theta) \leq \delta \tag{3}$$

$f(x(t),t)$ and $g(x(t-d(t)),t)$ are unknown functions, which represent nonlinear perturbations with respect to the current state $x(t)$ and delay state $x(k-d(t))$, respectively. In this paper, we assume the following restrictions on the perturbations

$$\begin{aligned} f^T(x(t),t)f(x(t),t) &\leq \varepsilon x^T(t)F^TFx^T(t) \\ g^T(x(t-d(t)),t)g(x(t-d(t)),t) &\leq \varepsilon_d x^T(t-d(t))F_d^TF_dx^T(t-d(t)) \end{aligned} \tag{4}$$

where F and F_d are known real constant matrices, and ε, ε_d are known positive scalars [13-16].

In case when the perturbations $f(x(t),t)$ and $g(x(t-d(t)),t)$ can be described as linear vector functions,

$$f(x(t),t) = \Delta A(t)x(t), \quad g\left(x(t-d(t)),t\right) = \Delta A_d(t)x(t-d(t)) \tag{5}$$

where $\Delta A(t)$ and $\Delta A_d(t)$ are parametric structured uncertainties, the system (1) becomes [16-18]:

$$\dot{x}(t) = \left(A + \Delta A(t)\right)x(t) + \left(A_d + \Delta A_d(t)\right)x(t - d(t)), \quad x(\theta) = \phi(\theta), \quad \theta \in [-d_M, 0] \quad (6)$$

In practice, $\Delta A(t)$ and $\Delta A_d(t)$ are assumed to be norm-bounded as

$$\left[\Delta A(t) \quad \Delta A_d(t)\right] = G\Delta(t)\left[H \quad H_d\right] \quad (7)$$

where G, H and H_d are known real constant matrices, and $\Delta(t)$ is an uncertain matrix function which satisfies

$$\Delta^T(t)\Delta(t) \le I \quad (8)$$

By introducing a new variable $z(t) = \Delta(t)\left(Hx(t) + H_d x(t - d(t))\right)$, the system (6) can be expressed as follows:

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Gz(t), \quad z(t) = \Delta(t)\left(Hx(t) + H_d x(t - d(t))\right),$$
$$x(\theta) = \phi(\theta), \quad \theta \in [-d_M, 0] \quad (9)$$

In the case where the system (1) does not contain the perturbations, i.e. $f(x(t),t) = 0, \quad g(x(t - d(t)),t) = 0$, then the following nominal system is obtained:

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)), \quad x(\theta) = \phi(\theta), \quad \theta \in [-d_M, 0] \quad (10)$$

To study the finite-time stability of the systems (1), (6) and (10), we introduce the following definition.

Definition 1. [44] The systems (1), (6) and (10) are said to be finite-time stable (FTS) with respect to (α, β, T), where $0 \le \alpha < \beta$, if

$$\sup_{t \in [-d_M, 0]} \phi^T(t)\phi(t) \le \alpha \quad \Rightarrow \quad x^T(t)x(t) < \beta, \forall t \in [0, T] \quad (11)$$

The following lemma will be used for the derivation of the main results.

Lemma 1. [44] For any appropriately dimensioned matrices $Z = Z^T > 0$, $Z \in \Re^{n \times n}$, $M \in \Re^{m \times n}$ and positive scalars d_1, $d_2 > d_1$ and γ, the following inequality holds

$$-\int_{t-d_2}^{t-d_1} e^{\gamma(t-s)}\dot{x}^T(s)Z\dot{x}(s)ds \le \xi^T(t)\varepsilon MZ^{-1}M^T\xi(t) + 2\xi^T(t)M\left(x(t - d_1) - x(t - d_2)\right) \quad (12)$$

where $\dot{x}(t) = dx(t)/dt$ and

$$\varepsilon = \left(e^{-\gamma d_1} - e^{-\gamma d_2}\right)/\gamma \quad (13)$$

$\xi(t) \in \Re^{m \times 1}$ is the vector function which contains suitably selected variables and/or functions.

Proof. Using the following inequality:

$$\int_{t-d_2}^{t-d_1} \left(M^T\xi(t) + e^{\gamma(t-s)}Z\dot{x}(s)\right)^T \left(e^{\gamma(t-s)}Z\right)^{-1} \left(M^T\xi(t) + e^{\gamma(t-s)}Z\dot{x}(s)\right)ds \ge 0 \quad (14)$$

we obtain:

$$\xi^T(t)MZ^{-1}M^T\xi(t)\int_{t-d_2}^{t-d_1} e^{-\gamma(t-s)}ds + 2\xi^T(t)MZ^{-1}Z\int_{t-d_2}^{t-d_1}\dot{x}(s)ds$$
$$+\int_{t-d_2}^{t-d_1} e^{\gamma(t-s)}\dot{x}^T(s)Z\dot{x}(s)ds \ge 0 \quad (15)$$

which implies

$$-\int_{t-d_2}^{t-d_1} e^{\gamma(t-s)}\dot{x}^T(s)Z\dot{x}(s)ds \le \xi^T(t)\varepsilon MZ^{-1}M^T\xi(t) + 2\xi^T(t)M\left(x(t-d_1)-x(t-d_2)\right) \quad (16)$$

This completes the proof. □

Remark 1. In this lemma, dimension and structure of the vector $\xi(t)$ can be adopted according to considered stability problem. Lemma 1 represents a new result that has been utilized for the estimation of the upper bound of the integral term with exponential function, which appears in derivative of LKLF.

2.2. Discrete-time systems.

Consider a discrete-time uncertain system with interval time-varying delay and nonlinear perturbations:

$$\begin{aligned} x(k+1) &= \left(A+\Delta A(k)\right)x(k) + \left(A_d + \Delta A_d(k)\right)x(k-d(k)) \\ &\quad + f\left(x(k),k\right) + g\left(x(k-d(k)),k\right) \\ x(j) &= \phi(j), \quad j \in \{-d_M, -d_M+1, \cdots, -1, 0\} \end{aligned} \quad (17)$$

where $k \in Z^+$, $x(k) \in \Re^n$ is the state vector, $A, A_d \in \Re^{n\times n}$ are known real constant matrices and $d(k)$ is time-varying delay satisfying

$$0 < d_m \le d(k) \le d_M, \quad d_m < d_M \quad (18)$$

where d_m and d_M are known positive constants. The $\phi(j)$ denotes a vector-valued initial function which satisfies

$$\sup_{j\in\{-d_M,-d_M+1,\cdots,-1\}} \left(\phi(j+1)-\phi(j)\right)^T\left(\phi(j+1)-\phi(j)\right) \le \delta \quad (19)$$

$f\left(x(k),k\right)$ and $g\left(x(k-d(k)),k\right)$ are unknown nonlinear perturbations with respect to the current state $x(k)$ and discrete delay state $x(k-h)$, respectively, which satisfy the following conditions [40-42]

$$\begin{aligned} f^T\left(x(k),k\right)f\left(x(k),k\right) &\le x^T(k)F^TFx(k) \\ g^T\left(x(k-d(k)),k\right)g\left(x(k-d(k)),k\right) &\le x^T(k-d(k))F_d^TF_dx(k-d(k)) \end{aligned} \quad (20)$$

where and F_d are known real constant matrices.

The parameter structured uncertainties $\Delta A(k)$ and $\Delta A_d(k)$ are assumed to be the form [19, 31, 38, 43]:

$$\left[\Delta A(k) \quad \Delta A_d(k)\right] = G\Delta(k)\left[H \quad H_d\right] \quad (21)$$

where G, H and H_d are known real constant matrices, and $\Delta(k)$ is unknown real time-varying matrix satisfying

$$\Delta^T(k)\Delta(k) \le I \quad (22)$$

By introducing a new variable $z(k)$, the system (17) can be expressed as follows:

$$\begin{aligned} x(k+1) &= Ax(k) + A_dx(k-d(k)) + Gz(k) + f\left(x(k),k\right) + g\left(x(k-d(k)),k\right) \\ z(k) &= \Delta(k)\left(Hx(k) + H_dx(k-d(k))\right), \\ x(j) &= \phi(j), \quad j \in \{-d_M, -d_M+1, \cdots, -1, 0\} \end{aligned} \quad (23)$$

If the system (17) does not contain the perturbations and uncertainties, then the following nominal system is obtained:

$$x(k+1) = Ax(k) + A_d x(k-d(k)), \quad x(j) = \phi(j), \quad j \in \{-d_M, -d_M+1, \cdots, -1, 0\} \tag{24}$$

To study the finite-time stability of the discrete-time systems with time-varying delay, we introduce the following definition and lemma.

Definition 2. [47] The discrete-time systems (17) with time varying delay are said to be finite-time stable (FTS) with respect to (α, β, N), where $0 \le \alpha < \beta$, if

$$\sup_{j\in\{-d_M, -d_M+1, \cdots, 0\}} \phi^T(j)\phi(j) \le \alpha \quad \Rightarrow \quad x^T(k)x(k) < \beta, \quad \forall k \in \{1, 2, \cdots, N\} \tag{25}$$

Lemma 2. [47] For any appropriately dimensioned matrices $Z > 0$, $Z \in \Re^{n\times n}$, $M \in \Re^{m\times n}$, positive integers d_1, $d_2 > d_1$ and positive scalar γ, the following inequality holds

$$-\sum_{j=k-d_2}^{k-d_1-1} \gamma^{k-j} y^T(j) Z y(j)$$
$$\le \xi^T(k)\rho M Z^{-1} M^T \xi(k) + 2\xi^T(k) M \left(x(k-d_1) - x(k-d_2) \right) \tag{26}$$

where $y(k) = x(k+1) - x(k)$, $\xi(k) \in \Re^{m\times 1}$ is suitably selected vector function of the state vector and ρ is positive constant which is defined by

$$\rho = \begin{cases} d_2 - d_1, & \gamma = 1 \\ (\gamma^{-d_1} - \gamma^{-d_2}) / (\gamma - 1), & \gamma \ne 1 \end{cases} \tag{27}$$

Proof. The proof is similar to the proof given in [53]. □

Remark 2. Lemma 2 represents an extension of Lemma 1 from [53].

In this paper, we extend the existing results of FTS problems to a class of continuous and discrete-time systems with interval time-varying delay, nonlinear perturbations and parameter uncertainties.

3. FTS for continuous-time systems

3.1. FTS for nominal time-delay systems.

Theorem 1. [44] Nominal system (10) is finite-time stable with respect to (α, β, T), $\alpha < \beta$, if there exist positive scalars γ, $\lambda_i, i = 1, 2, \cdots, 7$, positive define matrices $P, Q_1, Q_2, Q_3, R_1, R_2$, matrices $M = \left[M_1^T M_2^T \; M_3^T M_4^T \right]^T$, $L = \left[L_1^T \; L_2^T \; L_3^T \; L_4^T \right]^T$ and $S = \left[S_1^T \; S_2^T \; S_3^T \; S_4^T \right]^T$, such that the following conditions hold

$$\begin{bmatrix} \left[\Omega_{ij} \right]_{i,j=1,2\cdots,4} & c_1 M & c_2 L & c_3 S \\ * & -c_1 R_1 & 0 & 0 \\ * & * & -c_2 R_2 & 0 \\ * & * & * & -c_3 R_2 \end{bmatrix} < 0 \tag{28}$$

$$\Omega_{11} = A^T P + PA - \gamma P + Q_1 + Q_2 + Q_3 + d_m A^T R_1 A + d_M A^T R_2 A + M_1 + M_1^T + S_1 + S_1^T,$$

$$\Omega_{12} = PA_d + d_m A^T R_1 A_d + d_M A^T R_2 A_d + M_2^T + L_1 - S_1 + S_2^T,$$
$$\Omega_{13} = -M_1 + M_3^T + S_3^T, \ \ \Omega_{14} = M_4^T - L_1 + S_4^T,$$
$$\Omega_{22} = -e^{\gamma d_m}(1-\rho)Q_3 + d_m A_d^T R_1 A_d + d_M A_d^T R_2 A_d + L_2 + L_2^T - S_2 - S_2^T,$$
$$\Omega_{23} = -M_2 + L_3^T - S_3^T, \ \ \Omega_{24} = -L_2 + L_4^T - S_4^T,$$
$$\Omega_{33} = -e^{\gamma d_m} Q_1 - M_3 - M_3^T, \ \ \Omega_{34} = -M_4^T - L_3, \quad \Omega_{44} = -e^{\gamma d_M} Q_2 - L_4 - L_4^T,$$
$$\lambda_1 I < P < \lambda_2 I, \quad Q_1 < \lambda_3 I, \quad Q_2 < \lambda_4 I, \quad Q_3 < \lambda_5 I, \quad R_1 < \lambda_6 I, \quad R_2 < \lambda_7 I \tag{29}$$
$$\alpha e^{\gamma T}\left(\lambda_2 + \varepsilon_1\lambda_3 + \varepsilon_2\lambda_4 + \varepsilon_2\lambda_5\right) + \delta e^{\gamma T}\left(\varepsilon_3\lambda_6 + \varepsilon_4\lambda_7\right) - \beta\lambda_1 < 0 \tag{30}$$

where

$$\begin{gathered}\varepsilon_1 = \left(e^{\gamma d_m} - 1\right)/\gamma, \quad \varepsilon_2 = \left(e^{\gamma d_M} - 1\right)/\gamma, \quad \varepsilon_3 = \left(e^{\gamma d_m} - \gamma d_m - 1\right)/\gamma^2, \\ \varepsilon_4 = \left(e^{\gamma d_M} - \gamma d_M - 1\right)/\gamma^2, \quad c_1 = \left(1 - e^{-\gamma d_m}\right)/\gamma, \\ c_2 = \left(e^{-\gamma d_m} - e^{-\gamma d_M}\right)/\gamma, \quad c_3 = \left(1 - e^{-\gamma d_M}\right)/\gamma\end{gathered} \tag{31}$$

Proof. Let us consider the following LKLF

$$V\left(x(t)\right) = V_1\left(x(t)\right) + V_2\left(x(t)\right) + V_3\left(x(t)\right) + V_4\left(x(t)\right) + V_5\left(x(t)\right) + V_6\left(x(t)\right)$$
$$V_1\left(x(t)\right) = x^T(t)Px(t), \quad V_2\left(x(t)\right) = \int_{t-d_m}^{t} e^{\gamma(t-s)} x^T(s) Q_1 x(s) ds$$
$$V_3\left(x(t)\right) = \int_{t-d_M}^{t} e^{\gamma(t-s)} x^T(s) Q_2 x(s) ds, \quad V_4\left(x(t)\right) = \int_{t-d(t)}^{t} e^{\gamma(t-s)} x^T(s) Q_3 x(s) ds$$
$$V_5\left(x(t)\right) = \int_{-d_m}^{0}\int_{t+\theta}^{t} e^{\gamma(t-s)} \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta,$$
$$V_6\left(x(t)\right) = \int_{-d_M}^{0}\int_{t+\theta}^{t} e^{\gamma(t-s)} \dot{x}^T(s) R_2 \dot{x}(s) ds d\theta \tag{32}$$

The derivative of $V\left(x(t)\right)$ along solution of gives

$$\begin{aligned}\dot{V}(x(t)) \le\ & \gamma V(x(t)) + x^T(t)\left[A^T P + PA - \gamma P\right]x(t) + 2x^T(t)PA_d x(t-d(t)) \\ & + x^T(t)Q_1 x(t) - e^{\gamma d_m} x^T(t-d_m)Q_1 x(t-d_m) + x^T(t)Q_2 x(t) \\ & - e^{\gamma d_M} x^T(t-d_M)Q_2 x(t-d_M) + x^T(t)Q_3 x(t) \\ & - e^{\gamma d_m}(1-\rho)x^T(t-d(t))Q_3 x(t-d(t)) + d_m \dot{x}^T(t)R_1\dot{x}(t) \\ & + d_M \dot{x}^T(t)R_2\dot{x}(t) - \int_{t-d_m}^{t} e^{\gamma(t-s)}\dot{x}^T(s)R_1\dot{x}(s)ds - \int_{t-d_M}^{t} e^{\gamma(t-s)}\dot{x}^T(s)R_2\dot{x}(s)ds\end{aligned} \tag{33}$$

By using Lemma 1, we have

$$-\int_{t-d_m}^{t} e^{\gamma(t-s)}\dot{x}^T(s)R_1\dot{x}(s)ds \le \xi^T(t)\left[c_1 M R_1^{-1} M^T + \Sigma_1\right]\xi(t) \tag{34}$$

$$-\int_{t-d_M}^{t} e^{\gamma(t-s)}\dot{x}^T(s)R_2\dot{x}(s)ds \le \xi^T(t)\left[c_2LR_2^{-1}L^T + c_3SR_2^{-1}S^T + \Sigma_2 + \Sigma_3\right]\xi(t) \tag{35}$$

where

$$\xi(t) = \left[x(t)^T \quad x^T(t-d(t)) \quad x^T(t-d_m) \quad x^T(t-d_M)\right]^T$$

$$\Sigma_1 = \begin{bmatrix} M_1+M_1^T & M_2^T & -M_1+M_3^T & M_4^T \\ * & 0 & -M_2 & 0 \\ * & * & -M_3-M_3^T & -M_4^T \\ * & * & * & 0 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 0 & L_1 & 0 & -L_1 \\ * & L_2+L_2^T & L_3^T & -L_2+L_4^T \\ * & * & 0 & -L_3 \\ * & * & * & -L_4-L_4^T \end{bmatrix},$$

$$\Sigma_3 = \begin{bmatrix} S_1+S_1^T & -S_1+S_2^T & S_3^T & S_4^T \\ * & -S_2-S_2^T & -S_3^T & -S_4^T \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix},$$

$$\hat{c}_2 = \frac{1}{\gamma}\left(e^{-\gamma d(t)} - e^{-\gamma d_M}\right), \quad \hat{c}_3 = \frac{1}{\gamma}\left(1 - e^{-\gamma d(t)}\right),$$

Further, we have

$$\dot{V}(x(t)) \le \gamma V(x(t)) + \xi^T(t)\left[\Omega + c_1MR_1^{-1}M^T + c_2LR_2^{-1}L^T + c_3SR_2^{-1}S^T\right]\xi(t) \tag{36}$$

If

$$\Omega + c_1MR_1^{-1}M^T + c_2LR_2^{-1}L^T + c_3SR_2^{-1}S^T < 0 \tag{37}$$

then

$$\dot{V}\left(x(t)\right) < \gamma V\left(x(t)\right) \tag{38}$$

By using Schur complement, from (37) (28) follows. Integrating (38) from 0 to t, with $t \in [0, T]$, we get

$$V\left(x(t)\right) < e^{\gamma t} V\left(x(0)\right) \tag{39}$$

The initial value of LKLF can be written as

$$\begin{aligned} V\left(x(0)\right) < \alpha&\left[\lambda_{\max}(P) + \varepsilon_1\lambda_{\max}(Q_1) + \varepsilon_2\lambda_{\max}(Q_2) + \varepsilon_2\lambda_{\max}(Q_3)\right] \\ &+\delta\left[\varepsilon_3\lambda_{\max}(R_1) + \varepsilon_4\lambda_{\max}(R_2)\right] \end{aligned} \tag{40}$$

For LKL functional (32), the following inequality holds

$$V\left(x(t)\right) > x^T(t)Px(t) \tag{41}$$

Combining (39), (40) and (41) leads to

$$\begin{aligned} \lambda_{\min}(P)x^T(t)x(t) < e^{\gamma t}\alpha&\left[\lambda_{\max}(P) + \varepsilon_1\lambda_{\max}(Q_1) + \varepsilon_2\lambda_{\max}(Q_2) + \varepsilon_2\lambda_{\max}(Q_3)\right] \\ &+e^{\gamma t}\delta\left[\varepsilon_3\lambda_{\max}(R_1) + \varepsilon_4\lambda_{\max}(R_2)\right] \end{aligned}$$

If the following condition is satisfying

$$e^{\gamma t}\alpha\left[\lambda_{\max}(P)+\varepsilon_1\lambda_{\max}(Q_1)+\varepsilon_2\lambda_{\max}(Q_2)+\varepsilon_2\lambda_{\max}(Q_3)\right] \\ +e^{\gamma t}\delta\left[\varepsilon_3\lambda_{\max}(R_1)+\varepsilon_4\lambda_{\max}(R_2)\right]<\beta\lambda_{\min}(P) \quad (43)$$

then the system (1) is finite-time stable with respect to (α,β,T), i.e. $x^T(t)x(t)<\beta$, for all $t\in[0,T]$.

Let

$$\lambda_1<\lambda_{\min}(P),\ \lambda_{\max}(P)<\lambda_2,\ \lambda_{\max}(Q_1)<\lambda_3,\ \lambda_{\max}(Q_2)<\lambda_4, \\ \lambda_{\max}(Q_3)<\lambda_5,\ \lambda_{\max}(R_1)<\lambda_6,\ \lambda_{\max}(R_2)<\lambda_7 \quad (44)$$

Then the conditions (29) and (30) hold. The proof is completed. □

If the time delay is constant ($d(t)=d$), we can obtain the following corollary.

Corollary 1. [44] Nominal system (10) with constant time-delay, $d(t)=d$, is finite-time stable with respect to (α,β,T), $\alpha<\beta$, if there exist positive scalars γ, λ_i, $i=1,2,\cdots,4$, positive define matrices P, Q, R and matrix $N=\left[N_1^T\ N_2^T\right]^T$, such that the following conditions hold

$$\begin{bmatrix}\left[\Omega_{ij}\right]_{i,j=1,2} & cN\\ * & -cR\end{bmatrix}<0,\quad \Omega_{11}=A^TP+PA-\gamma P+Q+dA^TRA+N_1+N_1^T,$$
$$\Omega_{12}=PA_d+dA^TRA_d-N_1+N_2^T,\quad \Omega_{22}=-e^{\gamma d}Q+dA_d^TRA_d-N_2-N_2^T \quad (45)$$

$$\lambda_1 I<P<\lambda_2 I,\quad Q<\lambda_3 I,\quad R<\lambda_4 I \quad (46)$$

$$\alpha e^{\gamma T}\left(\lambda_2+\varepsilon_1\lambda_3\right)+\delta e^{\gamma T}\varepsilon_2\lambda_4-\beta\lambda_1<0 \quad (47)$$

where

$$\varepsilon_1=\left(e^{\gamma d}-1\right)/\gamma,\quad \varepsilon_2=\left(e^{\gamma d}-\gamma d-1\right)/\gamma^2,\quad c=\left(1-e^{-\gamma d}\right)/\gamma \quad (48)$$

3.2. FTS for time-delay systems with nonlinear perturbations.

Theorem 2. [44] The system (1) with nonlinear perturbations (4) is robust finite-time stable with respect to (α,β,T), $\alpha<\beta$, if there exist positive scalars γ, η, η_d, λ_i, $i=1,2,\cdots,7$, positive define matrices $P, Q_1, Q_2, Q_3, R_1, R_2$, matrices $M=\left[M_1^T M_2^T\ M_3^T M_4^T\ M_5^T\ M_6^T\right]^T$, $L=\left[L_1^T\ L_2^T\ L_3^T\ L_4^T\ L_5^T\ L_6^T\right]^T$ and $S=[S_1^T\ S_2^T\ S_3^T\ S_4^T\ S_5^T\ S_6^T]^T$, such that the following conditions hold

$$\Psi=\begin{bmatrix}\left[\Omega_{ij}\right]_{i,j=1,2\cdots,6} & c_1M & c_2L & c_3S\\ * & -c_1R_1 & 0 & 0\\ * & * & -c_2R_2 & 0\\ * & * & * & -c_3R_2\end{bmatrix}<0 \quad (49)$$

$$\Omega_{11}=A^TP+PA-\gamma P+Q_1+Q_2+Q_3+d_mA^TR_1A+d_MA^TR_2A \\ +M_1+M_1^T+S_1+S_1^T+\eta\varepsilon F^TF,$$
$$\Omega_{12}=PA_d+d_mA^TR_1A_d+d_MA^TR_2A_d+M_2^T+L_1-S_1+S_2^T,$$
$$\Omega_{13}=-M_1+M_3^T+S_3^T,\quad \Omega_{14}=M_4^T-L_1+S_4^T,$$

$$\Omega_{15} = P + d_m A^T R_1 + d_M A^T R_2 + M_5^T + S_5^T, \quad \Omega_{16} = P + d_m A^T R_1 + d_M A^T R_2 + M_6^T + S_6^T,$$
$$\Omega_{22} = -e^{\gamma d_m}(1-\rho)Q_3 + d_m A_d^T R_1 A_d + d_M A_d^T R_2 A_d + L_2 + L_2^T - S_2 - S_2^T + \eta_d \varepsilon_d F_d^T F_d,$$
$$\Omega_{23} = -M_2 + L_3^T - S_3^T, \quad \Omega_{24} = -L_2 + L_4^T - S_4^T,$$
$$\Omega_{25} = d_m A_d^T R_1 + d_M A_d^T R_2 + L_5^T - S_5^T, \quad \Omega_{26} = d_m A_d^T R_1 + d_M A_d^T R_2 + L_6^T - S_6^T,$$
$$\Omega_{33} = -e^{\gamma d_m} Q_1 - M_3 - M_3^T, \quad \Omega_{34} = -M_4^T - L_3, \quad \Omega_{35} = -M_5^T,$$
$$\Omega_{36} = -M_6^T, \quad \Omega_{44} = -e^{\gamma d_M} Q_2 - L_4 - L_4^T, \quad \Omega_{45} = -L_5^T,$$
$$\Omega_{46} = -L_6^T, \quad \Omega_{55} = d_m R_1 + d_M R_2 - \eta I, \quad \Omega_{56} = d_m R_1 + d_M R_2, \quad \Omega_{66} = d_m R_1 + d_M R_2 - \eta_d I$$

$$\lambda_1 I < P < \lambda_2 I, \quad Q_1 < \lambda_3 I, \quad Q_2 < \lambda_4 I, \quad Q_3 < \lambda_5 I, \quad R_1 < \lambda_6 I, \quad R_2 < \lambda_7 I \tag{50}$$

$$\alpha e^{\gamma T}\left(\lambda_2 + \varepsilon_1 \lambda_3 + \varepsilon_2 \lambda_4 + \varepsilon_2 \lambda_5\right) + \delta e^{\gamma T}\left(\varepsilon_3 \lambda_6 + \varepsilon_4 \lambda_7\right) - \beta \lambda_1 < 0 \tag{51}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, c_1, c_2$ and c_3 are defined by (31).

Proof. Let us consider the LKLF (32). Then, by using the following inequalities

$$\eta\left[\varepsilon x^T(t) F^T F x(t) - f^T(x(t),t) f(x(t),t)\right] \geq 0$$

$$\mu_d\left[\varepsilon_d x^T(t-d(t)) F_d^T F_d x(t-d(t)) - g^T(x(t-d(t)),t) g(x(t-d(t)),t)\right] \geq 0$$

we get

$$\dot{V}(x(t)) \leq \gamma V(x(t)) + \varsigma^T(t)\left[\Omega + c_1 M R_1^{-1} M^T + c_2 L R_2^{-1} L^T + c_3 S R_2^{-1} S^T\right]\varsigma(t) \tag{52}$$

where

$$\varsigma(t) = \left[x(t)^T \quad x(t-d(t))^T \quad x^T(t-d_m) \quad x^T(t-d_M) \quad f^T(x(t),t) \quad g^T(x(t-d(t)),t)\right]^T$$

If

$$\Omega + c_1 M R_1^{-1} M^T + c_2 L R_2^{-1} L^T + c_3 S R_2^{-1} S^T < 0 \tag{53}$$

then

$$\dot{V}\left(x(t)\right) < \gamma V\left(x(t)\right) \tag{54}$$

and (49) holds. The rest of the proof is similar to that of Theorem 1, thus omitted. □

If the time delay is constant, $d(t) = d$, we can obtain the following corollary.

Corollary 2. [44] The system (1) with nonlinear perturbations (4) and constant time-delay, $d(t) = d$, is robust finite-time stable with respect to (α, β, T), $\alpha < \beta$, if there exist positive scalars γ, η, η_d, λ_i, $i = 1, 2, \cdots, 4$, positive define matrices P, Q, R and matrix $N = \left[N_1^T N_2^T \; N_3^T N_4^T\right]^T$ such that the following conditions hold

$$\begin{bmatrix} \left[\Omega_{ij}\right]_{i,j=1,2,\cdots,4} & cN \\ * & -cR \end{bmatrix} < 0 \tag{55}$$

$$\Omega_{11} = A^T P + PA - \gamma P + Q + dA^T R A + \eta \varepsilon F^T F + N_1 + N_1^T,$$
$$\Omega_{12} = PA_d + dA^T R A_d - N_1 + N_2^T, \quad \Omega_{13} = P + dA^T R + N_3^T,$$
$$\Omega_{14} = P + dA^T R + N_4^T, \Omega_{22} = -e^{\gamma d} Q + dA_d^T R A_d + \eta_d \varepsilon_d F_d^T F_d - N_2 - N_2^T,$$
$$\Omega_{23} = dA_d^T R - N_3^T, \quad \Omega_{24} = dA_d^T R - N_4^T, \quad \Omega_{33} = dR - \eta I,$$

$$\Omega_{34} = dR, \quad \Omega_{44} = dR - \eta_d I$$

$$\lambda_1 I < P < \lambda_2 I, \quad Q < \lambda_3 I, \quad R < \lambda_4 I \tag{56}$$

$$\alpha e^{\gamma T}\left(\lambda_2 + \varepsilon_1\lambda_3\right) + \delta e^{\gamma T}\varepsilon_2\lambda_4 - \beta\lambda_1 < 0 \tag{57}$$

where $\varepsilon_1, \varepsilon_2$ and c are defined by (48).

3.3. FTS for time-delay systems with norm-bounded uncertainties. THEOREM 3. [44] The system (6) with norm-bounded time-varying structured uncertainties (7) is robust finite-time stable with respect to (α, β, T), $\alpha < \beta$, if there exist positive scalars γ, η, λ_i, $i = 1, 2, \cdots, 7$, positive define matrices $P, Q_1, Q_2, Q_3, R_1, R_2$, matrices $M = \left[M_1^T M_2^T \; M_3^T M_4^T \; M_5^T\right]^T$, $S = \left[S_1^T \; S_2^T \; S_3^T \; S_4^T \; S_5^T\right]^T$, and $S = \left[S_1^T \; S_2^T \; S_3^T \; S_4^T \; S_5^T\right]^T$, such that the following conditions hold

$$\Psi = \begin{bmatrix} \left[\Omega_{ij}\right]_{i,j=1,2\cdots,5} & c_1 M & c_2 L & c_3 S \\ * & -c_1 R_1 & 0 & 0 \\ * & * & -c_2 R_2 & 0 \\ * & * & * & -c_3 R_2 \end{bmatrix} < 0$$

$$\Omega_{11} = A^T P + PA - \gamma P + Q_1 + Q_2 + Q_3 + d_m A^T R_1 A + d_M A^T R_2 A$$
$$+ M_1 + M_1^T + S_1 + S_1^T + \eta H^T H,$$

$$\Omega_{12} = PA_d + d_m A^T R_1 A_d + d_M A^T R_2 A_d + M_2^T + L_1 - S_1 + S_2^T + \eta H^T H_d,$$

$$\Omega_{13} = -M_1 + M_3^T + S_3^T, \quad \Omega_{14} = M_4^T - L_1 + S_4^T,$$

$$\Omega_{15} = PG + d_m A^T R_1 G + d_M A^T R_2 G + M_5^T + S_5^T,$$

$$\Omega_{22} = -e^{\gamma d_m}(1-\rho)Q_3 + d_m A_d^T R_1 A_d + d_M A_d^T R_2 A_d + L_2 + L_2^T - S_2 - S_2^T + \eta H_d^T H_d,$$

$$\Omega_{23} = -M_2 + L_3^T - S_3^T, \quad \Omega_{24} = -L_2 + L_4^T - S_4^T,$$

$$\Omega_{25} = d_m A_d^T R_1 G + d_M A_d^T R_2 G + L_5^T - S_5^T,$$

$$\Omega_{33} = -e^{\gamma d_m} Q_1 - M_3 - M_3^T, \quad \Omega_{34} = -M_4^T - L_3,$$

$$\Omega_{35} = -M_5^T, \quad \Omega_{44} = -e^{\gamma d_M} Q_2 - L_4 - L_4^T,$$

$$\Omega_{45} = -L_5^T, \quad \Omega_{55} = d_m G^T R_1 G + d_M G^T R_2 G - \eta I,$$

$$\lambda_1 I < P < \lambda_2 I, \quad Q_1 < \lambda_3 I, \quad Q_2 < \lambda_4 I, \quad Q_3 < \lambda_5 I, \quad R_1 < \lambda_6 I, \quad R_2 < \lambda_7 I \tag{59}$$

$$\alpha e^{\gamma T}\left(\lambda_2 + \varepsilon_1\lambda_3 + \varepsilon_2\lambda_4 + \varepsilon_2\lambda_5\right) + \delta e^{\gamma T}\left(\varepsilon_3\lambda_6 + \varepsilon_4\lambda_7\right) - \beta\lambda_1 < 0 \tag{60}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, c_1, c_2$ and c_3 are defined by (31).

PROOF. Let us adopt the LKLF (32). Then, by using inequality

$$\eta\left[\left(Hx(t) + H_d x(t-d(t))\right)^T \left(Hx(t) + H_d x(t-d(t))\right) - z^T(t) z(t)\right] \geq 0$$

we get:

$$\dot{V}(x(t)) \le \gamma V(x(t)) + \sigma^T(t)\left[\Omega + c_1 M R_1^{-1} M^T + c_2 L R_2^{-1} L^T + c_3 S R_2^{-1} S^T\right]\sigma(t) \qquad (61)$$

where $\sigma(t) = \left[x^T(t) \quad x^T(t-d(t)) \quad x^T(t-d_m) \quad x^T(t-d_M) \quad z^T(t)\right]^T$. If

$$\Omega + c_1 M R_1^{-1} M^T + c_2 L R_2^{-1} L^T + c_3 S R_2^{-1} S^T < 0 \qquad (62)$$

then $\dot{V}(x(t)) < \gamma V(x(t))$. The rest of the proof is similar to that of Theorem 1, thus omitted. □

If the time delay is constant, $d(t) = d$, we will obtain the following criterion.

Corollary 3. [44] The system (6) with constant time-delay, $d(t) = d$, and norm-bounded time-varying structured uncertainties (7) is robust finite-time stable with respect to (α, β, T), $\alpha < \beta$, if there exist positive scalars γ, η, λ_i, $i = 1, 2, \cdots, 4$, positive define matrices P, Q, R and matrix $N = \left[N_1^T \ N_2^T \ N_3^T\right]^T$ such that the following conditions hold

$$\begin{bmatrix} \left[\Omega_{ij}\right]_{i,j=1,2,3} & cN \\ * & -cR \end{bmatrix} < 0$$

$$\Omega_{11} = A^T P + PA - \gamma P + Q + dA^T RA + \eta H^T H + N_1 + N_1^T$$

$$\Omega_{12} = PA_d + dA^T RA_d + \eta H^T H_d - N_1 + N_2^T, \quad \Omega_{13} = PG + dA^T RG + N_3^T$$

$$\Omega_{22} = -e^{\gamma d} Q + dA_d^T RA_d + \eta H_d^T H_d - N_2 - N_2^T, \quad \Omega_{23} = dA_d^T RG - N_3^T,$$

$$\Omega_{33} = dG^T RG - \eta I \qquad (63)$$

$$\lambda_1 I < P < \lambda_2 I, \quad Q < \lambda_3 I, \quad R < \lambda_4 I \qquad (64)$$

$$\alpha e^{\gamma T}\left(\lambda_2 + \varepsilon_1 \lambda_3\right) + \delta e^{\gamma T} \varepsilon_2 \lambda_4 - \beta \lambda_1 < 0 \qquad (65)$$

Remark 3. Observe that, the FTS analysis is based on solving two problems. The first problem is a determination of a less restrictive sufficient condition, such that the differential inequality $\dot{V}(x(t)) < \gamma V(x(t))$ is valid for $\forall t \in [0, T]$, $T > 0$. By solving this inequality, we get $V(x(t)) < e^{\gamma t} V(x(0))$. The second problem is estimation of an upper and a lower bound for $V(x(0))$ and $V(x(t))$, respectively. The precise estimation of these bounds provides that $V(x(t)) < e^{\gamma t} V(x(0))$ holds for lower values of the parameter β and higher values of the time-delay d. Note that the conservatism of derived stability criteria directly depends on solving accuracy of these two problems.

Remark 4. In the existing literature (see [38] and references therein), the approximation $e^{\gamma(t-s)} \le e^{\gamma d}$ is used for estimation of $V(x(0))$, and accordingly, conservative results are obtained. In this paper, this approximation is not used, such that the conservatism is reduced.

4. FTS for discrete-time systems

4.1. FTS for systems with non-linear perturbations.

Theorem 4. [47] The system (17) with $\Delta A(k) = \Delta A_d(k) = 0$ and nonlinear perturbations (20) is robust finite-time stable with respect to $\{\alpha, \beta, N\}$ if there exist a scalar $\gamma > 1$, positive scalars $\mu, \mu_d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8$, positive definite matrices P, Q_1, Q_2, R_1, R_2, matrices $L = [L_1^T\ L_2^T \cdots L_6^T]^T$, $S = [S_1^T\ S_2^T \cdots S_6^T]^T$ and $T = [T_1^T\ T_2^T \cdots T_6^T]^T$, such that the following inequalities hold:

$$\begin{bmatrix} [\Gamma_{ij}]_{i,j=1,2,\cdots,6} & \rho_1 L & \rho_1 S & \rho_2 T \\ * & -\rho_1 R_1 & 0 & 0 \\ * & * & -\rho_1 R_1 & 0 \\ * & * & * & -\rho_2 R_2 \end{bmatrix} < 0 \qquad (66)$$

$$\lambda_1 I < P < \lambda_2 I, \quad \lambda_3 I < Q_1 < \lambda_4 I, \quad \lambda_5 I < Q_2 < \lambda_6 I, \quad R_1 < \lambda_7 I, \quad R_2 < \lambda_8 I \qquad (67)$$

$$\gamma^N \left[\alpha\left(\lambda_2 + \delta_1\lambda_4 + \delta_2\lambda_6\right) + \delta\left(\delta_3\lambda_7 + \delta_4\lambda_8\right)\right] - \beta\left[\lambda_1 + \delta_1\lambda_3 + \delta_2\lambda_5\right] < 0 \qquad (68)$$

where

$$\begin{gathered} \Gamma_{11} = A^T PA - \gamma P + Q_1 + Q_2 + (A-I)^T R_{12}(A-I) + \mu F^T F + T_1 + T_1^T, \\ \Gamma_{12} = A^T PA_d + (A-I)^T R_{12} A_d + L_1 - S_1 + T_2^T, \quad \Gamma_{13} = S_1 - T_1 + T_3^T, \\ \Gamma_{14} = -L_1 + T_4^T, \quad \Gamma_{15} = A^T P + (A-I)^T R_{12} + T_5^T, \\ \Gamma_{16} = A^T P + (A-I)^T R_{12} + T_6^T, \\ \Gamma_{22} = A_d^T PA_d + A_d^T R_{12} A_d + \mu_d F_d^T F_d + L_2 + L_2^T - S_2 - S_2^T, \\ \Gamma_{23} = L_3^T + S_2 - S_3^T - T_2, \quad \Gamma_{24} = -L_2 + L_4^T - S_4^T, \quad \Gamma_{25} = A_d^T P + A_d^T R_{12} + L_5^T - S_5^T, \\ \Gamma_{26} = A_d^T P + A_d^T R_{12} + L_6^T - S_6^T, \quad \Gamma_{33} = -\gamma^{d_m} Q_2 + S_3 + S_3^T - T_3 - T_3^T, \\ \Gamma_{34} = -L_3 + S_4^T - T_4^T, \quad \Gamma_{35} = S_5^T - T_5^T, \quad \Gamma_{36} = S_6^T - T_6^T, \\ \Gamma_{44} = -\gamma^{d_M} Q_1 - L_4 - L_4^T, \quad \Gamma_{45} = -L_5^T, \quad \Gamma_{46} = -L_6^T, \quad \Gamma_{55} = P + R_{12} - \mu I, \\ \Gamma_{56} = P + R_{12}, \quad \Gamma_{66} = P + R_{12} - \mu_d I, \quad R_{12} = (d_M - d_m)R_1 + d_m R_2 \end{gathered} \qquad (69)$$

$$\rho_1 = \begin{cases} d_M - d_m, & \gamma = 1 \\ (\gamma^{-d_m} - \gamma^{-d_M})/(\gamma - 1), & \gamma \neq 1 \end{cases}, \quad \rho_2 = \begin{cases} d_m, & \gamma = 1 \\ (1 - \gamma^{-d_m})/(\gamma - 1), & \gamma \neq 1 \end{cases} \qquad (70)$$

$$\begin{gathered} \delta_1 = \begin{cases} d_M, & \gamma = 1 \\ (\gamma^{d_M} - 1)/(\gamma - 1), & \gamma \neq 1 \end{cases}, \quad \delta_2 = \begin{cases} d_m, & \gamma = 1 \\ (\gamma^{d_m} - 1)/(\gamma - 1), & \gamma \neq 1 \end{cases}, \\ \delta_3 = \begin{cases} \dfrac{d_M(d_M+1)}{2} - \dfrac{d_m(d_m+1)}{2}, & \gamma = 1 \\ (\gamma^{d_M+1} - \gamma^{d_m+1} - (\gamma-1)(d_M - d_m))/(\gamma-1)^2, & \gamma \neq 1 \end{cases}, \\ \delta_4 = \begin{cases} \dfrac{d_m(d_m+1)}{2}, & \gamma = 1 \\ (\gamma(\gamma^{d_m} - 1) - (\gamma-1)d_m)/(\gamma-1)^2, & \gamma \neq 1 \end{cases} \end{gathered} \qquad (71)$$

Proof. Choose the following LKLF:

$$V(k) = V_1(k) + V_2(k) + V_3(k) \qquad (72)$$

where

$$V_1(k) = x^T(k)Px(k),$$

$$V_2(k) = \sum_{j=k-d_M}^{k-1} \gamma^{k-1-j} x^T(j)Q_1 x(j) + \sum_{j=k-d_m}^{k-1} \gamma^{k-1-j} x^T(j)Q_2 x(j)$$

$$V_3(k) = \sum_{i=-d_M}^{-d_m-1} \sum_{j=k+i}^{k-1} \gamma^{k-1-j} y^T(j)R_1 y(j) + \sum_{i=-d_m}^{-1} \sum_{j=k+i}^{k-1} \gamma^{k-1-j} y^T(j)R_2 y(j)$$

$$y(k) = x(k+1) - x(k) \tag{73}$$

The forward difference of $V_i(k)$ ($i = 1,2,3$) along the trajectories of the system (17) yields

$$\begin{aligned}
\Delta V(k) = {} & (\gamma-1)V(k) + x^T(k)\left(A^T PA - \gamma P\right)x(k) + 2x^T(k)A^T PA_d x(k-d(k)) \\
& +2x^T(k)A^T Pf(\cdot,\cdot) + 2x^T(k)A^T Pg(\cdot,\cdot) + x^T(k-d(k))A_d^T PA_d x(k-d(k)) \\
& +2x^T(k-d(k))A_d^T Pf(\cdot,\cdot) + 2x^T(k-d(k))A_d^T Pg(\cdot,\cdot) + f^T(\cdot,\cdot)Pf(\cdot,\cdot) \\
& +2f^T(\cdot,\cdot)Pg(\cdot,\cdot) + g^T(\cdot,\cdot)Pg(\cdot,\cdot) + x^T(k)\left(Q_1 + Q_2\right)x(k) \\
& -\gamma^{d_M} x^T(k-d_M)Q_1 x(k-d_M) - \gamma^{d_m} x^T(k-d_m)Q_2 x(k-d_m) \\
& +y^T(k)\left(d_M - d_m)R_1 + d_m R_2\right)y(k) - \sum_{j=k-d_M}^{k-d_m-1} \gamma^{k-j} y^T(j)R_1 y(j) \\
& - \sum_{j=k-d_m}^{k-1} \gamma^{k-j} y^T(j)R_2 y(j)
\end{aligned} \tag{74}$$

The upper bounds of the last two terms in Eq. (74) can be obtained by using Lemma 2.

$$-\sum_{j=k-d_M}^{k-d_m-1} \gamma^{k-j} y^T(j)R_1 y(j) \le \xi^T(k)\left(\Sigma_1 + \rho_1 LR_1^{-1}L^T + \rho_1 SR_1^{-1}S^T\right)\xi(k) \tag{75}$$

$$-\sum_{j=k-d_m}^{k-1} \gamma^{k-j} y^T(j)R_2 y(j) \le \xi^T(k)\left(\Sigma_2 + \rho_2 TR_2^{-1}T^T\right)\xi(k) \tag{76}$$

where

$$\xi(k) = [x^T(k) \quad x^T(k-d(k)) \quad x^T(k-d_m) \quad x^T(k-d_M) \quad f^T\left(x(k),k\right) \quad g^T\left(x(k-d(k)),k\right)]^T$$

$$\Sigma_1 = \begin{bmatrix} 0 & L_1 - S_1 & S_1 & -L_1 & 0 & 0 \\ * & L_2 + L_2^T - S_2 - S_2^T & L_3^T + S_2 - S_3^T & -L_2 + L_4^T - S_4^T & L_5^T - S_5^T & L_6^T - S_6^T \\ * & * & S_3 + S_3^T & -L_3 + S_4^T & S_5^T & S_6^T \\ * & * & * & -L_4 - L_4^T & -L_5^T & -L_6^T \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \tag{77}$$

$$\Sigma_2 = \begin{bmatrix} T_1 + T_1^T & T_2^T & -T_1 + T_3^T & T_4^T & T_5^T & T_6^T \\ * & 0 & -T_2 & 0 & 0 & 0 \\ * & * & -T_3 - T_3^T & -T_4^T & -T_5^T & -T_6^T \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \tag{78}$$

$$\rho_1' = (\gamma^{-d(k)} - \gamma^{-d_M})/(\gamma - 1) \le (\gamma^{-d_m} - \gamma^{-d_M})/(\gamma - 1) = \rho_1$$
$$\rho_1'' = (\gamma^{-d_m} - \gamma^{-d(k)})/(\gamma - 1) \le (\gamma^{-d_m} - \gamma^{-d_M})/(\gamma - 1) = \rho_1 \tag{79}$$

By combining (74)-(76) and the following perturbation conditions

$$\mu x^T(k) F^T F x(k) - \mu f^T\left(x(k), k\right) f\left(x(k), k\right) \ge 0$$
$$\mu_d x^T(k - d(k)) F_d^T F_d x(k - d(k)) - \mu_d g^T\left(x(k - d(k)), k\right) g\left(x(k - d(k)), k\right) \ge 0 \tag{80}$$

$\Delta V(k)$ can be finally written as

$$\Delta V(k) \le (\gamma - 1)V(k) + \xi^T(k)\left(\Gamma + \rho_1 L R_1^{-1} L^T + \rho_1 S R_1^{-1} S^T + \rho_2 T R_2^{-1} T^T\right)\xi(k) \tag{81}$$

If

$$\Gamma + \rho_1 L R_1^{-ü} L^T + \rho_1 S R_1^{-} S^T + \rho_2 T R_2^{-} T^T < 0 \tag{82}$$

then

$$\Delta V(k) - (\gamma - 1)V(k) < 0 \tag{83}$$

Note that the condition (83) can be rewritten as

$$V(k) < \gamma^k V(0), \quad k = 1, 2, 3, \cdots \tag{84}$$

By applying Schur complement, the inequality (82) is equivalent to (66).
An upper bound of the initial value of LKLF can be written as

$$V(0) \le \alpha\left(\lambda_{\max}(P) + \delta_1 \lambda_{\max}(Q_1) + \delta_2 \lambda_{\max}(Q_2)\right) + \delta\left(\delta_3 \lambda_{\max}(R_1) + \delta_4 \lambda_{\max}(R_2)\right) \tag{85}$$

In addition, a lower bound of the LKLF can be written as

$$V(k) > \lambda_{\min}(P) x^T(k) x(k) + \lambda_{\min}(Q_1) \sum_{j=k-d_M}^{k-1} \gamma^{k-1-j} x^T(j) x(j)$$
$$+ \lambda_{\min}(Q_2) \sum_{j=k-d_m}^{k-1} \gamma^{k-1-j} x^T(j) x(j) \tag{86}$$

If the following condition is valid

$$\gamma^N\left[\alpha\left(\lambda_{\max}(P) + \delta_1 \lambda_{\max}(Q_1) + \delta_2 \lambda_{\max}(Q_2)\right) + \delta\left(\delta_3 \lambda_{\max}(R_1) + \delta_4 \lambda_{\max}(R_2)\right)\right]$$
$$< \beta\left[\lambda_{\min}(P) + \delta_1 \lambda_{\min}(Q_1) + \delta_2 \lambda_{\min}(Q_2)\right] \tag{87}$$

then

$$\begin{aligned} &\lambda_{\min}(P) x^T(k) x(k) + \lambda_{\min}(Q_1) \sum_{j=k-d_M}^{k-1} \gamma^{k-1-j} x^T(j) x(j) \\ &\qquad + \lambda_{\min}(Q_2) \sum_{j=k-d_m}^{k-1} \gamma^{k-1-j} x^T(j) x(j) \\ &< V(k) \quad < \gamma^N \alpha\left(\lambda_{\max}(P) + \delta_1 \lambda_{\max}(Q_1) + \delta_2 \lambda_{\max}(Q_2)\right) \\ &\qquad + \gamma^N \delta\left(\delta_3 \lambda_{\max}(R_1) + \delta_4 \lambda_{\max}(R_2)\right) \\ &< \beta\left[\lambda_{\min}(P) + \delta_1 \lambda_{\min}(Q_1) + \delta_2 \lambda_{\min}(Q_2)\right] \end{aligned} \tag{88}$$

which implies $x^T(k)x(k) < \beta$ for $k = 1, 2, \cdots, N$. From this, it can be concluded that the system (17) is finite-time stable. From (67) and (88) follows (68). This completes the proof. □

REMARK 5. The conservatism of FTS criteria generally depends on a restrictiveness of the inequalities (84), (85) and (86), i.e. $V(k) < \gamma^k V(0)$, $V(0) < \Theta_1$ and $V(k) > \Theta_2$, where Θ_1 and Θ_2 are the estimations of the upper bound of $V(0)$ and the lower bound of $V(k)$, respectively. This estimations depend from the parameters $\alpha, \beta, N, \delta, d_m, d_M$ and γ. In this paper, four innovations are proposed in order to reduce the conservativeness of the estimations. The two innovations are related to inequality (84), and the rest of innovations deal with the inequalities (85) and (86).

In above analysis, it is assumed that the delay is time-varying. If the time-delay is constant $(d(k) = d)$, the following corollary can be obtained.

COROLLARY 4. [47] The system (17) with $\Delta A(k) = \Delta A_d(k) = 0$, nonlinear perturbations (20) and constant time-delay is robust finite-time stable with respect to $\{\alpha, \beta, N\}$ if there exist a scalar $\gamma > 1$, positive scalars $\mu, \mu_d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, positive definite matrices $P, Q, R,$ matrices $L = [L_1^T \ L_2^T L_3^T \ L_4^T]^T$, $S = [S_1^T \ S_2^T \ S_3^T \ S_4^T]^T$ and $T = [T_1^T \ T_2^T \ T_3^T \ T_4^T]^T$, such that the following inequalities hold:

$$\begin{bmatrix} [\tilde{\Gamma}_{ij}]_{i,j=1,2,3,4} & \rho L \\ * & -\rho R \end{bmatrix} < 0 \tag{89}$$

$$\lambda_1 I < P < \lambda_2 I, \quad \lambda_3 I < Q < \lambda_4 I, \quad R < \lambda_5 I \tag{90}$$

$$\gamma^N \left[\alpha \left(\lambda_2 + \delta_1 \lambda_4 \right) + \delta \delta_2 \lambda_5 \right] - \beta \left(\lambda_1 + \delta_1 \lambda_3 \right) < 0 \tag{91}$$

where

$$\begin{gathered} \tilde{\Gamma}_{11} = A^T P A - \gamma P + Q + d(A-I)^T R (A-I) + L_1 + L_1^T + \mu F^T F, \\ \tilde{\Gamma}_{12} = A^T P A_d + d(A-I)^T R A_d - L_1 + L_2^T, \\ \tilde{\Gamma}_{13} = A^T P + d(A-I)^T R + L_3^T, \quad \tilde{\Gamma}_{14} = A^T P + d(A-I)^T R + L_4^T, \\ \tilde{\Gamma}_{22} = A_d^T P A_d - \gamma^d Q + d A_d^T R A_d - L_2 - L_2^T + \mu_d F_d^T F_d, \quad \tilde{\Gamma}_{23} = A_d^T P + d A_d^T R - L_3^T, \\ \tilde{\Gamma}_{24} = A_d^T P + d A_d^T R - L_4^T, \\ \tilde{\Gamma}_{33} = P + R - \mu I, \quad \tilde{\Gamma}_{34} = P + R, \quad \tilde{\Gamma}_{44} = P + R - \mu_d I \end{gathered} \tag{92}$$

$$\rho = \begin{cases} d, & \gamma = 1 \\ (1-\gamma^{-d})/(\gamma-1), & \gamma \neq 1 \end{cases}, \quad \delta_1 = \begin{cases} d, & \gamma = 1 \\ (\gamma^d - 1)/(\gamma-1), & \gamma \neq 1 \end{cases},$$

$$\delta_2 = \begin{cases} d(d+1)/2, & \gamma = 1 \\ (\gamma(\gamma^d - 1) - (\gamma-1)d)/(\gamma-1)^2, & \gamma \neq 1 \end{cases} \tag{93}$$

4.2. FTS for uncertain systems with norm-bounded uncertainties. In this section, two new sufficient FTS conditions are derived for the uncertain system (17) with $f\left(x(k),k\right)=g\left(x(k-d(k)),k\right)=0$, norm-bounded uncertainties (7)-(8) and time-varying or constant delay.

Theorem 5. [47] The uncertain system (17) with $f\left(x(k),k\right)=g\left(x(k-d(k)),k\right)=0$, norm-bounded uncertainties satisfying (7)-(8) and time-varying delay is robust finite-time stable with respect to $\{\alpha,\beta,N\}$ if there exist a scalar $\gamma>1$, positive scalars $\mu,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8$, positive definite symmetric matrices P,Q_1,Q_2,R_1,R_2, matrices $L=[L_1^T\ L_2^T\cdots L_5^T]^T$, $S=[S_1^T\ S_2^T\cdots S_5^T]^T$ and $T=[T_1^T\ T_2^T\cdots T_5^T]^T$, such that the following inequalities hold:

$$\begin{bmatrix} [\hat{\Gamma}_{i,j}]_{i,j=1,2,\cdots,5} & \rho_1 L & \rho_1 S & \rho_2 T \\ * & -\rho_1 R_1 & 0 & 0 \\ * & * & -\rho_1 R_1 & 0 \\ * & * & * & -\rho_2 R_2 \end{bmatrix} < 0$$

$$\lambda_1 I < P < \lambda_2 I,\quad \lambda_3 I < Q_1 < \lambda_4 I,\quad \lambda_5 I < Q_2 < \lambda_6 I,\quad R_1 < \lambda_7 I,\quad R_2 < \lambda_8 I \tag{95}$$

$$\gamma^N\left[\alpha\left(\lambda_2+\delta_1\lambda_4+\delta_2\lambda_6\right)+\delta\left(\delta_3\lambda_7+\delta_4\lambda_8\right)\right]-\beta\left[\lambda_1+\delta_1\lambda_3+\delta_2\lambda_5\right]<0 \tag{96}$$

where

$$\begin{aligned}
&\hat{\Gamma}_{11}=A^TPA-\gamma P+Q_1+Q_2+(A-I)^TR_{12}(A-I)+\mu H^TH+T_1+T_1^T,\\
&\hat{\Gamma}_{12}=A^TPA_d+(A-I)^TR_{12}A_d+\mu H^TH_d+L_1-S_1+T_2^T,\\
&\hat{\Gamma}_{13}=S_1-T_1+T_3^T,\quad \hat{\Gamma}_{14}=-L_1+T_4^T,\quad \hat{\Gamma}_{15}=A^TPG+(A-I)^TR_{12}G+T_5^T,\\
&\hat{\Gamma}_{22}=A_d^TPA_d+A_d^TR_{12}A_d+\mu H_d^TH_d+L_2+L_2^T-S_2-S_2^T,\\
&\hat{\Gamma}_{23}=L_3^T+S_2-S_3^T-T_2,\quad \hat{\Gamma}_{24}=-L_2+L_4^T-S_4^T,\\
&\hat{\Gamma}_{25}=A_d^TPG+A_d^TR_{12}G+L_5^T-S_5^T,\\
&\hat{\Gamma}_{33}=-\gamma^{d_m}Q_2+S_3+S_3^T-T_3-T_3^T,\quad \hat{\Gamma}_{34}=-L_3+S_4^T-T_4^T,\quad \hat{\Gamma}_{35}=S_5^T-T_5^T,\\
&\hat{\Gamma}_{44}=-\gamma^{d_M}Q_1-L_4-L_4^T,\quad \hat{\Gamma}_{45}=-L_5^T,\\
&\hat{\Gamma}_{55}=G^TPG+G^TR_{12}G-\mu I,\quad R_{12}=(d_M-d_m)R_1+d_mR_2
\end{aligned} \tag{97}$$

and the constants ρ_1, ρ_2, δ_1, δ_2, δ_3 and δ_4 are defined by (70) and (71).

Proof. Let us adopt (73) for LKLF. Then, the forward difference of $\Delta V(k)$ along the trajectories of the uncertain system (17) with $f\left(x(k),k\right)=g\left(x(k-d(k)),k\right)=0$, amounts:

$$\begin{aligned}
\Delta V(k)\le{}&(\gamma-1)V(k)+x^T(k)\left(A^TPA-\gamma P\right)x(k)+2x^T(k)A^TPA_dx(k-d(k))\\
&+2x^T(k)A^TPGz(k)+x^T(k-d(k))A_d^TPA_dx(k-d(k))\\
&+2x^T(k-d(k))A_d^TPGz(k)+z^T(k)G^TPGz(k)+x^T(k)\left(Q_1+Q_2\right)x(k)\\
&-\gamma^{d_M}x^T(k-d_M)Q_1x(k-d_M)-\gamma^{d_m}x^T(k-d_m)Q_2x(k-d_m)\\
&+y^T(k)R_{12}y(k)+\hat{\xi}^T(k)\left(\hat{\Sigma}+\rho_1LR_1^{-1}L^T+\rho_1SR_1^{-1}S^T+\rho_2TR_2^{-1}T^T\right)\hat{\xi}(k)
\end{aligned} \tag{98}$$

where $\hat{\xi}(k) = \begin{bmatrix} x^T(k) & x^T(k-d(k)) & x^T(k-d_m) & x^T(k-d_M) & z^T(k) \end{bmatrix}^T$ and

$$\hat{\Sigma} = \begin{bmatrix} T_1 + T_1^T & L_1 - S_1 + T_2^T & S_1 - T_1 + T_3^T & -L_1 + T_4^T & T_5^T \\ * & L_2 + L_2^T - S_2 - S_2^T & L_3^T + S_2 - S_3^T - T_2 & -L_2 + L_4^T - S_4^T & L_5^T - S_5^T \\ * & * & S_3 + S_3^T - T_3 - T_3^T & -L_3 + S_4^T - T_4^T & S_5^T - T_5^T \\ * & * & * & -L_4 - L_4^T & -L_5^T \\ * & * & * & * & 0 \end{bmatrix} \quad (99)$$

By combining (98) and the following perturbation condition

$$\mu x^T(k)H^T Hx(k) + 2\mu x^T(k)H^T H_d x(k-d(k)) + \mu x^T(k-d(k))H_d^T H_d x(k-d(k)) - \mu z^T(k)z(k) \geq 0 \quad (100)$$

we get

$$\Delta V(k) \leq (\gamma-1)V\left(x(k)\right) + \hat{\xi}^T(k)\left(\hat{\Gamma} + \rho_1 LR_1^{-1}L^T + \rho_1 SR_1^{-1}S^T + \rho_2 TR_2^{-1}T^T\right)\hat{\xi}(k) \quad (101)$$

If the following inequality is satisfied:

$$\hat{\Gamma} + \rho_1 LR_1^{-1}L^T + \rho_1 SR_1^{-1}S^T + \rho_2 TR_2^{-1}T^T < 0 \quad (102)$$

then the condition (84) holds. From (62), by using Schur complement, we get the condition (94). The rest of the proof is similar to the proof of Theorem 4. □

If the time-delay is constant ($d(k) = d$), the following corollary can be obtained.

COROLLARY 5. [47] The uncertain system (17) with $f\left(x(k),k\right) = g\left(x(k-d(k)),k\right) = 0$, norm-bounded uncertainties satisfying (7)-(8) and constant time-delay is robust finite-time stable with respect to $\{\alpha, \beta, N\}$ if there exist a scalar $\gamma > 1$, positive scalars $\mu, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, positive definite matrices P, Q, R, matrices $L = [L_1^T \; L_2^T L_3^T]^T$, $S = [S_1^T \; S_2^T \; S_3^T]^T$ and $T = [T_1^T \; T_2^T \; T_3^T]^T$, such that the following inequalities hold:

$$\begin{bmatrix} [\hat{\Gamma}_{ij}]_{i,j=1,2,3} & \rho L \\ * & -\rho R \end{bmatrix} < 0 \quad (103)$$

$$\lambda_1 I < P < \lambda_2 I, \quad \lambda_3 I < Q < \lambda_4 I, \quad R < \lambda_5 I \quad (104)$$

$$\gamma^N \left[\alpha\left(\lambda_2 + \delta_1\lambda_4\right) + \delta\delta_2\lambda_5\right] - \beta\left(\lambda_1 + \delta_1\lambda_3\right) < 0 \quad (105)$$

where

$$\begin{aligned} \hat{\Gamma}_{11} &= A^T PA - \gamma P + Q + d(A-I)^T R(A-I) + \mu H^T H + L_1 + L_1^T, \\ \hat{\Gamma}_{12} &= A^T PA_d + d(A-I)^T RA_d + \mu H^T H_d - L_1 + L_2^T, \\ \hat{\Gamma}_{13} &= A^T PG + d(A-I)^T RG + L_3^T, \\ \hat{\Gamma}_{22} &= A_d^T PA_d - \gamma^d Q + dA_d^T RA_d + \mu H_d^T H_d - L_2 - L_2^T, \\ \hat{\Gamma}_{23} &= A_d^T PG + dA_d^T RG - L_3^T, \quad \hat{\Gamma}_{33} = G^T PG + dG^T RG - \mu I \end{aligned} \quad (106)$$

and the constants ρ, δ_1 and δ_2 are defined by (93).

4.3. FTS for uncertain systems with non-linear perturbations and norm-bounded uncertainties. In this section, we consider the uncertain system (17) with norm-bounded uncertainties and nonlinear perturbations that satisfy (7)-(8) and (20), respectively, and give a new FTS criterion.

THEOREM 6. [47] The uncertain system (17) with nonlinear perturbations (20), norm-bounded uncertainties satisfying (7)-(8) and time-varying delay is robust finite-time stable with respect to $\{\alpha,\beta,N\}$ if there exist a scalar $\gamma>1$, positive scalars $\varepsilon,\varepsilon_d,\mu,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8$, positive definite symmetric matrices P,Q_1,Q_2,R_1,R_2, matrices $L=[L_1^T\ L_2^T\cdots L_7^T]^T$, $S=[S_1^T\ S_2^T\cdots S_7^T]^T$ and $T=[T_1^T\ T_2^T\cdots T_7^T]^T$, such that the following inequalities hold:

$$\bar{\Psi}=\begin{bmatrix}[\underline{\Gamma}_{i,j}]_{i,j=1,2,\cdots,7} & \rho_1 L & \rho_1 S & \rho_2 T\\ * & -\rho_1 R_1 & 0 & 0\\ * & * & -\rho_1 R_1 & 0\\ * & * & * & -\rho_2 R_2\end{bmatrix}<0 \tag{107}$$

$$\lambda_1 I<P<\lambda_2 I,\quad \lambda_3 I<Q_1<\lambda_4 I,\quad \lambda_5 I<Q_2<\lambda_6 I,\quad R_1<\lambda_7 I,\quad R_2<\lambda_8 I \tag{108}$$

$$\gamma^N\left[\alpha\left(\lambda_2+\delta_1\lambda_4+\delta_2\lambda_6\right)+\delta\left(\delta_3\lambda_7+\delta_4\lambda_8\right)\right]-\beta\left[\lambda_1+\delta_1\lambda_3+\delta_2\lambda_5\right]<0 \tag{109}$$

where

$$\begin{gathered}\underline{\Gamma}_{11}=A^TPA-\gamma P+Q_1+Q_2+(A-I)^TR_{12}(A-I)+\varepsilon F^TF+\mu H^TH+T_1+T_1^T,\\ \underline{\Gamma}_{12}=A^TPA_d+(A-I)^TR_{12}A_d+\mu H^TH_d+L_1-S_1+T_2^T,\quad \underline{\Gamma}_{13}=S_1-T_1+T_3^T,\\ \underline{\Gamma}_{14}=-L_1+T_4^T,\quad \underline{\Gamma}_{15}=A^TPG+(A-I)^TR_{12}G+T_5^T,\\ \underline{\Gamma}_{16}=A^TP+(A-I)^TR_{12}+T_6^T,\quad \underline{\Gamma}_{17}=A^TP+(A-I)^TR_{12}+T_7^T,\\ \underline{\Gamma}_{22}=A_d^TPA_d+A_d^TR_{12}A_d+\varepsilon_dF_d^TF_d+\mu H_d^TH_d+L_2+L_2^T-S_2-S_2^T,\\ \underline{\Gamma}_{23}=L_3^T+S_2-S_3^T-T_2,\quad \underline{\Gamma}_{24}=-L_2+L_4^T-S_4^T,\\ \underline{\Gamma}_{25}=A_d^TPG+A_d^TR_{12}G+L_5^T-S_5^T,\\ \underline{\Gamma}_{26}=A_d^TP+A_d^TR_{12}+L_6^T-S_6^T,\quad \underline{\Gamma}_{27}=A_d^TP+A_d^TR_{12}+L_7^T-S_7^T,\\ \underline{\Gamma}_{33}=-\gamma^{d_m}Q_2+S_3+S_3^T-T_3-T_3^T,\quad \underline{\Gamma}_{34}=-L_3+S_4^T-T_4^T,\quad \underline{\Gamma}_{35}=S_5^T-T_5^T,\\ \underline{\Gamma}_{36}=S_6^T-T_6^T,\quad \underline{\Gamma}_{37}=S_7^T-T_7^T,\quad \underline{\Gamma}_{44}=-\gamma^{d_M}Q_1-L_4-L_4^T,\quad \underline{\Gamma}_{45}=-L_5^T,\\ \underline{\Gamma}_{46}=-L_6^T,\quad \underline{\Gamma}_{47}=-L_7^T,\quad \underline{\Gamma}_{55}=G^TPG+G^TR_{12}G-\mu I,\quad \underline{\Gamma}_{56}=G^TP+G^TR_{12},\\ \underline{\Gamma}_{57}=G^TP+G^TR_{12},\quad \underline{\Gamma}_{66}=P+R_{12}-\varepsilon I,\quad \underline{\Gamma}_{67}=P+R_{12},\\ \underline{\Gamma}_{77}=P+R_{12}-\varepsilon_dI,\quad R_{12}=(d_M-d_m)R_1+d_mR_2\end{gathered} \tag{110}$$

and the constants ρ_2, ρ_2, δ_1, δ_2, δ_3 and δ_4 are defined by (70) and (71).

PROOF. The proof is similar to the proof of Theorem 4 and 5, thus omitted. □

4.4. FTS for nominal systems. For the nominal system (10), the following FTS criterion can be obtained.

COROLLARY 6. [47] The nominal system (10) with time-varying delay is finite-time stable with respect to $\{\alpha,\beta,N\}$ if there exist a scalar $\gamma>1$, positive scalars

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8$, positive definite matrices P, Q_1, Q_2, R_1, R_2, matrices $L=[L_1^T\ L_2^T\ L_4^T\ L_4^T]^T$, $S=[\mathrm{S}_1^T\ S_2^T\ S_4^T\ S_4^T]^T$ and $T=[\mathrm{T}_1^T\ T_2^T\ T_4^T\ T_4^T]^T$, such that the following inequalities hold:

$$\begin{bmatrix} [\underset{\sim}{\Gamma}_{ij}]_{i,j=1,2,3,4} & \rho_1 L & \rho_1 S & \rho_2 T \\ * & -\rho_1 R_1 & 0 & 0 \\ * & * & -\rho_1 R_1 & 0 \\ * & * & * & -\rho_2 R_2 \end{bmatrix} < 0 \tag{111}$$

$$\lambda_1 I < P < \lambda_2 I, \quad \lambda_3 I < Q_1 < \lambda_4 I, \quad \lambda_5 I < Q_2 < \lambda_6 I, \quad R_1 < \lambda_7 I, \quad R_2 < \lambda_8 I \tag{112}$$

$$\gamma^N \left[\alpha\left(\lambda_2 + \delta_1\lambda_4 + \delta_2\lambda_6\right) + \delta\left(\delta_3\lambda_7 + \delta_4\lambda_8\right)\right] - \beta\left[\lambda_1 + \delta_1\lambda_3 + \delta_2\lambda_5\right] < 0 \tag{113}$$

where

$$\begin{aligned} \underset{\sim}{\Gamma}_{11} &= A^T PA - \gamma P + Q_1 + Q_2 + (A-I)^T R_{12}(A-I) + T_1 + T_1^T, \\ \underset{\sim}{\Gamma}_{12} &= A^T PA_d + (A-I)^T R_{12} A_d + L_1 - S_1 + T_2^T, \\ \underset{\sim}{\Gamma}_{13} &= S_1 - T_1 + T_3^T, \quad \underset{\sim}{\Gamma}_{14} = -L_1 + T_4^T, \\ \underset{\sim}{\Gamma}_{22} &= A_d^T PA_d + A_d^T R_{12} A_d + L_2 + L_2^T - S_2 - S_2^T, \\ \underset{\sim}{\Gamma}_{23} &= L_3^T + S_2 - S_3^T - T_2, \quad \underset{\sim}{\Gamma}_{24} = -L_2 + L_4^T - S_4^T, \\ \underset{\sim}{\Gamma}_{33} &= -\gamma^{d_m} Q_2 + S_3 + S_3^T - T_3 - T_3^T, \quad \underset{\sim}{\Gamma}_{34} = -L_3 + S_4^T - T_4^T, \\ \underset{\sim}{\Gamma}_{44} &= -\gamma^{d_M} Q_1 - L_4 - L_4^T \end{aligned} \tag{114}$$

and the constants ρ_1, ρ_2, δ_1, δ_2, δ_3 and δ_4 are defined by (70) and (71).

In special case, when the time-delay is constant ($d(k)=d$), the following corollary is obtained.

Corollary 7. [47] The nominal system (10) with constant time-delay is finite-time stable with respect to $\{\alpha, \beta, N\}$ if there exist a scalar $\gamma > 1$, positive scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, positive definite symmetric matrices Q, R, matrices $L=[L_1^T\ L_2^T]^T$, $S=[S_1^T\ S_2^T]^T$ and $T=[T_1^T\ T_2^T]^T$, such that the following inequalities hold:

$$\begin{bmatrix} \underset{\frown}{\Gamma} & \rho L \\ * & -\rho R \end{bmatrix} < 0 \tag{115}$$

$$\lambda_1 I < P < \lambda_2 I, \quad \lambda_3 I < Q < \lambda_4 I, \quad R < \lambda_5 I \tag{116}$$

$$\gamma^N \left[\alpha\left(\lambda_2 + \delta_1\lambda_4\right) + \delta\delta_2\lambda_5 \right] - \beta\left(\lambda_1 + \delta_1\lambda_3\right) < 0 \tag{117}$$

where

$$\begin{aligned} \underset{\frown}{\Gamma}_{11} &= A^T PA - \gamma P + Q + d(A-I)^T R(A-I) + L_1 + L_1^T, \\ \underset{\frown}{\Gamma}_{12} &= A^T PA_d + d(A-I)^T RA_d - L_1 + L_2^T, \\ \underset{\frown}{\Gamma}_{22} &= A_d^T PA_d - \gamma^d Q + dA_d^T RA_d - L_2 - L_2^T \end{aligned} \tag{118}$$

and the constants ρ, δ_1 and δ_2 are defined by(93) .

5. Illustrative examples and simulations

In order to compare our results with existing ones, the following two criteria are adopted: a) a minimum allowable lower bound (MALB) of the parameter β, $\beta_{\min}$, such that the concerned system is FTS for any parameter β greater than the MALB, b) a maximum allowable upper bound (MAUB) of the time-delay d, $d_{\max}$, such that the concerned system is FTS for any delay size less than the MAUB. Note that a criterion that gives a lower value of MALB or a higher value of MAUB is less conservative with respect to other criteria.

5.1. Continuous-time case.

Example 1. [44] Consider the following nominal continuous-time system with time-varying delay:

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t))$$

$$A = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.02 \\ 0.02 & 0.01 & 0.02 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \end{bmatrix} \tag{119}$$

Let $d_m = 2 \le d(t) \le d_M = 5$ and $\rho = 0.1$. Base on [19] and Theorem 1, the MALB $\beta_{\min}$ is calculated for $\alpha = 0.5$, $\delta = 0.5$ and $T \in (10,\ 20,\ 30,\ 40,\ 50)$, and results are listed in Table 1. From the table, it can be seen that Theorem 1 provides much less conservative results than [19]. Using the second criterion for comparison, the MAUB $d_{\max}$ is computed for $\alpha = 0.5$, $\beta = 600$, $\delta = 0.5$ and $T \in (10,\ 20,\ 30,\ 40,\ 50)$, and results are shown in Table 2. From the table, it is clearly seen that our method is more effective than the recently reported one. Moreover, [19] is not feasible (NF) for $T \ge 30$.

Table 1. The MALB $\beta_{\min}$ for the system with $2 \le d(t) \le 5$, $\delta = 0.5$ and $\alpha = 0.5$

T	10	20	30	40	50
[19]	1483.9	44463.3	1332301.7	$39922 \cdot 10^3$	$1197 \cdot 10^6$
Theorem 1	2.4	9.4	38.0	154.1	624.6

Table 2. The MAUB $d_{\max}$ for the system with $d_m = 2$ and $\beta = 600$.

T	10	20	30	40	50
[19]	4.3	2.8	NF	NF	NF
Theorem 1	54.9	44.7	34.4	23.2	3.7

Figures 1 and 2 show simulations (the state variable and the norm of the state vector) of the above system (119) with the time delay $d(t) = 3|\sin(0.03t)| + 2$ and the initial condition $\phi(\theta) = [0.1\theta + 0.2 \quad 0.1\theta + 0.2 \quad 0.1\theta + 0.2]^T$, $\theta \in [-5, 0]$, which satisfied $\sup_{\theta \in [-5,0]} \phi^T(\theta)\phi(\theta) = 0.27 < 0.5 = \alpha$ and $\sup_{\theta \in [-5,0]} \dot{\phi}^T(\theta)\dot{\phi}(\theta) = 0.03 \le 0.5 = \delta$. Obviously, $2 \le d(t) \le 5$, $\max \dot{d}(t) = 0.09 \le 0.1 = \rho$ and the considered system is not asymptotically stable, but it is FTS. Further, the norm of the state vector with Figure 2 does not exceed the MALB $\beta_{\min}$ from Table 1, which confirms the correctness of the proposed result.

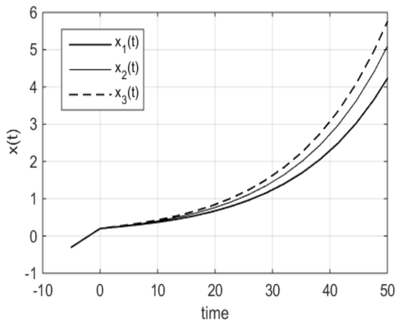

FIGURE 1. The state variables of the system

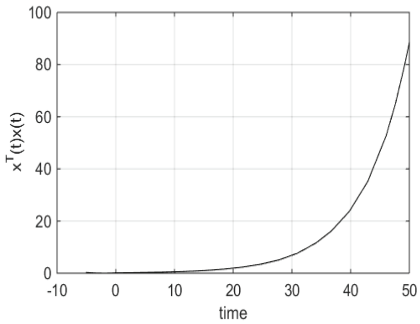

FIGURE 2. The norm of the state vector of the system

EXAMPLE 2. [44] Consider the following system with time-varying delay and nonlinear perturbations (4)

$$\begin{aligned}&\dot{x}(t) = Ax(t) + A_d x(t-d(t)) + f(x(t),t) + g(x(t-d(t)),t)\\ &d_m = 2, \quad \delta = 0.5, \quad \rho = 0.1, \quad F = F_d = 0.05 I_{3\times 3}, \ \varepsilon = \varepsilon_d = 0.05\end{aligned} \tag{120}$$

where the matrices A and A_d are defined in (119). By using Theorem 2, the MALB $\beta_{\min}$ is calculated for $d_M = 5$, $\alpha = 0.5$, $T \in (10,\ 20,\ 30,\ 40,\ 50)$ and results are listed in Table 3. Table 4 lists the MAUB values $d_{\max}$ which are obtained by Theorem 2 for $\beta = 600$, $\alpha = 0.5$ and $T \in (10,\ 20,\ 30,\ 40,\ 50)$.

TABLE 3. The MALB $\beta_{\min}$ for the system (120) and $2 \le d(t) \le 5$

T	10	20	30	40	50
Theorem 2	3.7	22.3	134.5	813.2	4919.5

TABLE 4. The MAUB $d_{\max}$ for the system and $\beta = 600$

T	10	20	30	40	50
Theorem 2	40.0	29.9	19	NF	NF

EXAMPLE 3. [44] Consider the following time-varying delay system (120) with parametric uncertainties (7):

$$\begin{aligned}&\dot{x}(t) = \left(A + \Delta A(t)\right)x(t) + \left(A_d + \Delta A_d(t)\right)x(t-d(t))\\ &d_m = 2, \quad \delta = 0.5, \quad \rho = 0.1, \quad G = H = H_d = 0.05 I_{3\times 3}\end{aligned} \tag{121}$$

where the matrices A_d and A_d are defined in (119). By using Theorem 1 in [19] and Theorem 3 (this paper), the MALB $\beta_{\min}$ is calculated for $d_M = 5$, $\alpha = 0.5$, $T \in (10,\ 20,\ 30,\ 40,\ 50)$, and results are listed in Table 5. From this table, we can see that $\beta_{\min}$ in this paper is significantly smaller than those in [19].

Based on Theorem 1 [19], and Theorem 3 (this paper), using the second criterion for comparison, the MAUB $d_{\max}$ is computed for $\alpha = 0.5$, $\beta = 600$ and $T \in (10,\ 20,\ 30,\ 40,\ 50)$, and results are shown in Table 6. From the table, we can see that the values of MAUB $d_{\max}$ in this paper are significantly larger than those in [21]. Moreover, Theorem 1 [19] is not feasible for $T \geq 30$.

Table 5. The MALB $\beta_{\min}$ for the system (121) and $2 \leq d(t) \leq 5$.

T	10	20	30	40	50
[19], $\varepsilon = 0.1$	1484.1	44469.0	1332473	$39926 \cdot 10^3$	$1197 \cdot 10^6$
Theorem 3	2.7	13.3	65.8	325.6	1612.6

Table 6. The MAUB $d_{\max}$ for the system (121) and $\beta = 600$.

T	10	20	30	40	50
[19], $\varepsilon = 0.1$	4.3	2.6	NF	NF	NF
Theorem 3	51.2	41.1	30.6	17.2	NF

5.2. Discrete-time case. In this section, we give two numerical examples to show the effectiveness of the proposed results for discrete-time systems and their improvement over the existing literature.

Example 4. [47] Consider the discrete-time system and nominal system [33], [34] and [53] with the corresponding parameters:

$$A = \begin{bmatrix} 0.60 & 0.00 \\ 0.35 & 0.70 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.20 & 0.25 \\ 0.25 & 0.15 \end{bmatrix}, \quad d_m = 2, \quad d_M = 5, \quad \delta = 1.1 \tag{122}$$

$$F = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad F_d = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} \tag{123}$$

$$G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad H_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \tag{124}$$

We adopt the MALB criterion to illustrate the applicability of our results and compare them with existing ones.

(i) First we analyse the system (17) with $\Delta A(k) = \Delta A_d(k) = 0$ and parameters (122) and (123). By using Theorem 4, the MALB $\beta_{\min}$ is calculated for $\alpha = 3$ and $N \in (5,\ 10,\ 20,\ 40)$. In special case, when the time-delay is constant $(d(k) = 5)$, $\alpha = 3$ and $N \in (5,\ 10,\ 20,\ 40)$, the MALB $\beta_{\min}$ is computed by using Corollary 4. The obtained results are listed in Table 7.

(ii) Now we consider the system (17) with $f\left(x(k),k\right) = g\left(x(k-d(k)),k\right) = 0$ and the parameters (122) and (124). By using Theorem 5, the MALB $\beta_{\min}$ is calculated for $\alpha = 3$ and $N \in (5,\ 10,\ 20,\ 40)$. For the constant time-delay $d(k) = d = 5$ and the same values of the parameters α and N, we computed the MALB $\beta_{\min}$ by using Corollary 5. The obtained results are listed in Table 7.

(iii) In addition, we consider the system (17) with nonlinear perturbations, parameter uncertainties, interval time-varying delay and the parameters (122), (123) and (124). By applying Theorem 6 and solving corresponding matrix inequalities, the

MALB $\beta_{\min}$ is computed for $\alpha = 3$ and $N \in (5,\ 10,\ 20,\ 40)$. The corresponding results are also shown in Table 7.

(iv) In order to compare our results with existing ones given in [33, 34, 53], we consider the nominal system (10) with time-varying delay $2 \le d(k) \le 5$ and the parameters (122). The MALB $\beta_{\min}$ is computed by using [33, Theorem 1], [34, Theorem 1], [53, Theorem 6] and Corollary 6 for $\alpha = 3$ and $N \in (5,\ 10,\ 20,\ 40)$. The obtained results are listed in Table 8. From this table it can be seen that: a) our results are less conservative than those in [33] and [34]; b) [34, Theorem 1] is not feasible (NF) for $N \ge 40$ and c) 53, Theorem 6] represents a special case of Theorems 4-6 for nominal system.

TABLE 7. The MALB $\beta_{\min}$ for $\alpha = 3$ and the system (17) with $2 \le d(k) \le 5$ and the parameters (122), (123) and (124).

N	5	10	20	40
Theorem 4 $(\Delta A(k) = \Delta A_d(k) = 0)$, $2 \le d(k) \le 5$	48	352	$1.53 \cdot 10^4$	$2.57 \cdot 10^7$
Corollary 4 $(\Delta A(k) = \Delta A_d(k) = 0)$, $d(k) = 5$	11	35	346	$3.00 \cdot 10^4$
Theorem 5 $(f(\cdot) = g(\cdot) = 0)$, $2 \le d(k) \le 5$	35	180	3674	$1.32 \cdot 10^6$
Corollary 5 $(f(\cdot) = g(\cdot) = 0)$, $d(k) = 5$	9	22	121	3409
Theorem 6, $2 \le d(k) \le 5$	52	412	$2.11 \cdot 10^4$	$4.79 \cdot 10^7$

TABLE 8. The MALB $\beta_{\min}$ for $\alpha = 3$ and the nominal system (10) with $2 \le d(k) \le 5$ and the parameters (122).

N	5	10	20	40
[34]	$9.85 \cdot 10^3$	$1.53 \cdot 10^6$	$3.63 \cdot 10^{10}$	NF
[33]	168	741	$1.04 \cdot 10^4$	$1.70 \cdot 10^6$
[53]	33	154	2701	$7.09 \cdot 10^5$
Corollary 6	33	154	2701	$7.09 \cdot 10^5$

In order to verify previous results, we chose the following initial values $\phi(j) = \begin{bmatrix} 0.1j + 0.2 & 0.1j + 0.2 \end{bmatrix}$, $j \in \{-5, -4, \cdots, -1, 0\}$ and compute the state response of the system (10) with $d(k) = \lfloor 3|\sin(k/15)| \rfloor + 2$, $k = 1, 2, \cdots, N$, where $\lfloor \cdot \rfloor$ denotes rounding to the nearest integer. Obviously, it can be seen that the time-delay and initial values satisfy the following conditions:

$$2 \le d(k) \le 5 \text{ and } \sup_{j \in \{-5, -5, \cdots, -1\}} \left(\phi(j+1) - \phi(j)\right)^T \left(\phi(j+1) - \phi(j)\right) = 0.02 \le 1.1 .$$

Figures 3 and 4 show the state response $x(k)$ of the nominal system (10) and the evolution of the norm $x^T(k)x(k)$. Based on the figure, we can see that the considered system is not asymptotic stable, but the norm $x^T(k)x(k)$ does not exceed the MALB $\beta_{\min}$ from Table 10, which means that the above system is FTS with respect to $(3, \beta_{\min}, N)$, $N \in (5,\ 10,\ 20,\ 40)$.

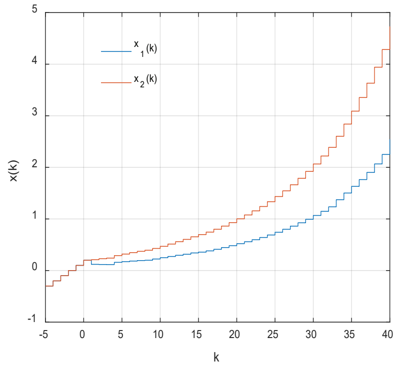

FIGURE 3. The state variable of the system with the parameters and $2 \le d(k) \le 5$.

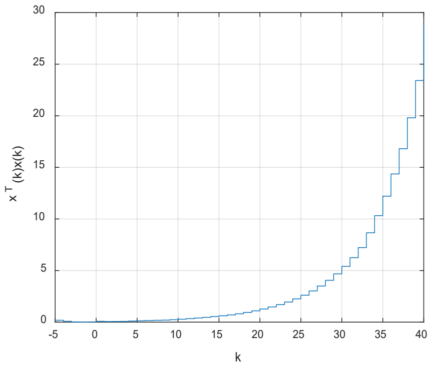

FIGURE 4. The norm of the state vector of the system with the parameters and $2 \le d(k) \le 5$.

TABLE 9. The MALB $\beta_{\min}$ for $\alpha = 3$ and the nominal system (10) with $d(k) = d = 5$ and the parameters (122).

N	5	10	20	40
[34]	104	791	$4.02 \cdot 10^4$	$1.03 \cdot 10^8$
[33]	29	74	362	$7.06 \cdot 10^3$
[54]	8	19	93	1941
Corollary 7	8	19	93	1941

(v) Finally, we consider the nominal system (10) with constant time-delay $d(k) = 5$ and the parameters (122). For $\alpha = 3$ and $N \in (5,\ 10,\ 20,\ 40)$, the MALB $\beta_{\min}$ is computed by using [29, Corollary 1], [32, Corollary 1], [33, Theorem 7], Corollary 7, and the obtained results are listed in Table 9. From this table, it can be seen that Corollary 7 and [33, Theorem 7] provide less conservative results than those in [29] and [32], and [54, Theorem 7] represents a special case of Theorems 4-6 (for nominal system with constant time-delay).

6. Conclusion

In this paper, finite-time stability for classes of uncertain time-varying delay continuous and discrete systems with nonlinear perturbations and parametric uncertainties have been investigated. New integral and finite sum inequalities with the exponential and power function are derived in order to estimate the upper limit of a quadratic form. By using these inequalities and Lyapunov-Krasovskii-like functional with exponential and power functions, some sufficient conditions of finite-time stability are obtained in form of linear matrix inequalities. Finally, Numerical examples have been presented to verify the effectiveness of the proposed results and some improvement over some existing work in the literature.

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