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ON THE BRACHISTOCHRONIC MOTION OF A NON-CONSERVATIVE DYNAMIC SYSTEM

Đ. S. Djukić

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Abstract

The brachistochronic motion of mechanical system with dissipative generalized forces is considered. The generalized workless forces, which must be added to the system to realize brachistochronic motion, are obtained. As an application of the theory two particular problems are studied.

1. Introduction

The optimization of mechanical system during the motion is a very old problem but contemporary current in recent years. John Bernoulli (1696) first formulated and solved a problem of this kind. It was the brachistochrone problem.

Let us consider a dynamic system which is moving from a given configuration A to another given configuration B. It is required to find the control forces which will carry the system from A to B in a minimal-stationary time. The motion of the system is called brachistochronic, while the corresponding trajectory of the system is known as a brachistochrone. Following a concept of classical mechanics, we will suppose that the total work of the control forces is equal to zero, and we will say that these forces are workless. Namely, in the classical mechanics these control forces are reactions of the constraints imposed on the system during the motion. We will consider the control forces as the forces which must be added to the system to realize the brachistochronic motion.

Bernoulli considered the brachistochronic motion of a particle under the influence of gravity. The brachistochronic motion of a particle in a central force field have been solved by Kleinschmidt and Schulze [4]. The brachistochronic motion of a conservative mechanical system of n degrees of freedom was studied by Pennachietti [1] and McConnel [2] for the holonomic system, and by Djukić [3] for the nonholonomic case. In [9]—[11] the brachistochrone of a particle with Coulomb friction has been treated. The brachistochrone of a particle in a resisting medium was solved, in the sense of reduction to quadratures, by Euler (see [6] p. 241), and recently by Drummond and Downes [7]. Stojanovitch [8] considered the brachistochronic motion of a scleronomic holonomic dynamical system in the field ot non-conservative forces that are independent of time and velocities of the system.

We will consider the brachistochronic motion of scleronomic holonomic non-conservative mechanical system, where the nonconservative generalized forces are dependent on time, generalized coordinates and generalized velocities. As an application of the derived theory, two particular problems will be analyzed.

In this paper the notation of tensor calculus is used throughout and the summation convention will be observed. Small italic indices imply a range of values from 1 to n.

2. The equations of brachistochronic motion

Let us consider a holonomic scleronomic nonconservative mechanical system with n degrees of freedom, where the q^i are mutually independent generalized coordinates and t is the time. The mechanical system is characterized by the kinetic energy

$$T=\frac{1}{2}\,a_{ij}(q^r)\,\dot{q}^i\,\dot{q}^j, \tag{1}$$

potential enegy π, nonconservative generalized forces Q_i and by the generalized control forces u_i. The potential energy is function of the generalized coordinates and the nonconservative forces are dependent on the generalized coordinates, generalized velocities and time. Here $\dot{q}^i$ are the derivative of q^i with respect to time and a_{ij} are the functions of the $q^{'s}$ only. The corresponding differential equations of motion of the mechanical system are

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial \dot{q}^i}-\frac{\partial T}{\partial q^i}=-\frac{\partial \pi}{\partial q^i}+Q_i+u_i. \tag{2}$$

Let us suppose that the mechanical system is moving in a Riemannian configuration space V_n with the metric form

$$(ds)^2=2T\,(dt)^2=a_{ij}\,dq^i\,dq^j. \tag{3}$$

From this equation we immediately have that the time needed by the system to pass from a given initial configuration

$$q^i(0)=q^{i0} \tag{4}$$

to another known terminal configuration

$$q^i(\tau)=q^{i1}, \tag{5}$$

along a curve in the V_n, is

$$I=\int\limits_0^\tau \frac{\psi^{1/2}}{(2T)^{1/2}}\,dt, \tag{6}$$

where

$$\psi(q^i,\,\dot{q}^i)=a_{ij}\,\dot{q}^i\,\dot{q}^j. \tag{7}$$

According to our assumption total work* of the control forces u_i is equal to zero, i.e.

$$u_i \dot{q}^i = 0. \tag{8}$$

Hence, between the change of mechanical energy and the work of nonconservative forces done on the elementary displacement of the system there exists the relation

$$\frac{d}{dt}(T+\pi) = Q_i \dot{q}^i. \tag{9}$$

Using (3), above equation takes the form

$$\dot{T} + \frac{\partial \pi}{\partial q^i}\dot{q}^i - \Phi \psi^{1/2} = 0, \tag{10}$$

where

$$\Phi = \frac{1}{\sqrt{2T}} Q_i(t, q^r, \dot{q}^r)\dot{q}^i. \tag{11}$$

Further, we will restrict our analysis on the problems where the function Φ may be always expressed as a function of time, generalized coordinates and kinetic energy, that is,

$$\Phi = \Phi(t, q^i, T). \tag{12}$$

In the brachistochronic motion of the mechanical system the functional (6), where the terminal time τ is not specified, must be minimal subject to the equation (10) as a constraint. The variational problem is equivalent (see [6] p.222) to the minimization of a new functional

$$I^* = \int_0^\tau \left[\psi^{1/2}(q^i, \dot{q}^i)\, H(t, q^i, \lambda, T) + \lambda\left(\dot{T} + \frac{\partial \pi}{\partial q^i}\dot{q}^i\right)\right] dt, \tag{13}$$

where λ is Languange's multiplier and

$$H(t, q^i, \lambda, T) = \frac{1}{\sqrt{2T}} - \lambda\,\Phi(t, q^i, T). \tag{14}$$

Assuming that the variation operator δ and the differentiating operator d are commutative, the first variation of (13), where the τ is not specified, is given by

$$\begin{aligned}\delta I^* = \int_0^\tau \Big\{\delta q^i\Big[\frac{1}{2}\psi^{-1/2}\frac{\partial \psi}{\partial q^i}H + \psi^{1/2}\frac{\partial H}{\partial q^i} - \frac{d}{dt}\Big(\frac{1}{2}\psi^{-1/2}\frac{\partial \psi}{\partial \dot{q}^i}H\Big) -\\ - \dot{\lambda}\frac{\partial \pi}{\partial q^i}\Big] + \delta T\Big(\psi^{1/2}\frac{\partial H}{\partial T} - \dot{\lambda}\Big)\Big\} dt + \lambda\,\delta T\Big|_0^\tau + \Big(\frac{1}{2}\psi^{-1/2}\frac{\partial \psi}{\partial \dot{q}^i}H +\\ + \lambda\frac{\partial \pi}{\partial q^i}\Big)\delta q^i\Big|_0^\tau + H\psi^{1/2}\Big(1 - \frac{1}{2\psi}\frac{\partial \psi}{\partial \dot{q}^i}\dot{q}^i\Big)\delta t\Big|_{t=\tau}.\end{aligned} \tag{15}$$

* indeed total power

Following standard procedure of variational calculus, using (1), (7) and the facts that $\delta q^i(0)=\delta q^i(\tau)=0$, the condition that I^* is stationary, i.e. $\delta I^*=0$, yields

$$\frac{d}{dt}\left(\frac{H}{\sqrt{2T}}\frac{\partial \psi}{\partial \dot{q}^i}\right)-\frac{H}{\sqrt{2T}}\frac{\partial \psi}{\partial q^i}=2\left(\sqrt{2T}\frac{\partial H}{\partial q^i}-\dot{\lambda}\frac{\partial \pi}{\partial q^i}\right), \tag{16}$$

$$\dot{\lambda}=\sqrt{2T}\frac{\partial H}{\partial T}, \tag{17}$$

$$\lambda\,\delta T\,\big|_0^\tau=0. \tag{18}$$

Combining (1), (2), (7), (16), we have the optimal control forces

$$u_i=\frac{\partial \pi}{\partial q^i}-Q_i+\frac{2T}{H}\frac{\partial H}{\partial q^i}-\frac{\sqrt{2T}}{H}\left[\dot{\lambda}\frac{\partial \pi}{\partial q^i}+\frac{\partial T}{\partial \dot{q}^i}\frac{d}{dt}\left(\frac{H}{\sqrt{2T}}\right)\right]. \tag{19}$$

Hence, it follows:

Theorem: The brachistochronic motion of a holonomic scleronomic nonconservative mechanical system with kinetic energy (1) in the field of forces of potential π and nonconservative forces Q_i is described by the differential equations (16), (17) and (10) and realized with the help of the control forces (19). The transversality condition (18) will be satisfied if

$$T(0)=T_0 \quad \text{or} \quad \lambda(0)=0, \tag{20}$$

and

$$T(\tau)=T_1 \quad \text{or} \quad \lambda(\tau)=0, \tag{21}$$

where T_0 and T_1 are initial and terminal values of the kinetic energy. Combining (1) and (7) we obtain the following equation

$$\psi=2T \quad \text{for} \quad t\in[0,\tau]. \tag{22}$$

After integration of n second order differential equations (16) and two first order differential equations (17) and (10) we have the q^i, T and λ as functions of time and $2n+2$ constants of integration. These $2n+2$ constants and the terminal time τ of the motion may be found from the $2n+3$ algebraic equations (4), (5) and (20)–(22).

3. A brachistochronic stabilisation

Let us consider the perturbed motion about a circular orbit of a material point in a Newtonian central force field. The corresponding differential equations for the disturbances x and y are (for more details consult [12] p. 617):

$$\ddot{x}-2\dot{y}=0; \quad \ddot{y}+2\dot{x}-3y=0. \tag{23}$$

Adding to the system a constant force -k in the y direction, we wish to transfer the initial disturbances

(24) $$x(0)=x_0;\quad y(0)=y_0;\quad T(0)=T_0$$

into the zeros, i.e. to the state

(25) $$x(\tau)=0;\quad y(\tau)=0;\quad T(\tau)=0$$

for a minimal time. Hence, we may call the problem under consideration "the brachistochronic stabilization problem". In this case, the characteristic function of the problem are

(26) $$\psi=\dot{x}^2+\dot{y}^2;\quad 2T=\dot{x}^2+\dot{y}^2;\quad \pi=ky-\frac{3}{2}y^2;\quad \Phi=0;\quad H=\frac{1}{\sqrt{2T}}.$$

Combining (10), (16), (17) and (26) we obtain the differential equations of the brachistochronic motion

(27) $$\dot{T}+(k-3y)\dot{y}=0.$$

(28) $$\frac{d}{dt}\left(\frac{\dot{x}}{2T}\right)=0;\quad \frac{d}{dt}\left(\frac{\dot{y}}{2T}\right)=-\dot{\lambda}(k-3y);$$

(29) $$\dot{\lambda}=-\frac{1}{2T}.$$

From (24), (25) and (27) we have

(30) $$2T=3y^2-2ky;\quad k=\frac{3}{2}y_0-\frac{T_0}{y_0}.$$

Solving the equations (18) and (29) and using (22) (26) and (30) we obtain following solution of the problem

(31) $$x=b-\frac{k}{3}\left[\frac{1}{a}F\left(\alpha,\frac{\sqrt{a^2-1}}{a}\right)-aE\left(\alpha,\frac{\sqrt{a^2-1}}{a}\right)\right],$$

(32) $$t=c+\frac{\sqrt{a^2-1}}{a\sqrt{3}}F\left(\alpha,\frac{\sqrt{a^2-1}}{a}\right),$$

where

(33) $$\alpha=\arcsin\left\{\frac{a}{\sqrt{a^2-1}}\cos\left[\arcsin\frac{1}{a}\left(\frac{3}{k}y-1\right)\right]\right\},$$

a, b and c are integration constants, and where $F(\psi, k)$ and $E(\psi, k)$ are the elliptic integrals of the first and second order respectively.

Substituting (31) — (33) into the equations (24) and (25) we obtain the following system of algebraic equations for finding the constants a, b and c and the minimal time τ

$$b=\frac{k}{3}\left[\frac{1}{a}F(\alpha_1, e)-aE(\alpha_1, e)\right]; \quad e=\frac{\sqrt{a^2-1}}{a};$$

$$c=-\frac{\sqrt{a^2-1}}{a\sqrt{3}}F(\alpha_0, e); \quad \tau=\frac{e}{\sqrt{3}}[F(\alpha_1, e)-F(\alpha_2, e)]; \tag{34}$$

$$\frac{3x_0}{k}=\frac{1}{k}[F(\alpha_1, e)-F(\alpha_0, e)]-a[E(\alpha_1, e)-E(\alpha_0, e)],$$

where $\alpha_0(\alpha)$ and $\alpha_1(a)$ are obtained from (33) for $y=y_0$ and $y=0$ respectively.

4. Brachistochronic motion of a material point in a resting medium

A material point of unit mass with prescribed kinetic energy T_0 is moving, under the influence of gravity along a curve in space from a point A to another point B. When the kinetic energy is T the motion is resisted by a force $R(T)$ per unit mass, but there is no other frictional force. The problem is to find the form of the curve so that the point reaches B in the shortest possible time. The same problem of motion, but in vertical plane, is treated in [6] p. 241. The characteristic functions of the problem are

$$\pi=-gz; \quad 2T=\dot{x}^2+\dot{y}^2+\dot{z}^2; \quad \psi=\dot{x}^2+\dot{y}^2+\dot{z}^2; \quad \Phi=-R(T),$$

$$H=\frac{1}{\sqrt{2T}}+\lambda R(T) \tag{35}$$

where x, y and z are the Cartesian coordinates of the material point, and g is the acceleration due to gravity.

Substituting (35) into (10), (16) and (17) we have the following equations of brachistochronic motion

$$\frac{d}{dt}\left(\frac{H}{\sqrt{2T}}\dot{x}\right)=0; \tag{36}$$

$$\frac{d}{dt}\left(\frac{H}{\sqrt{2T}}\dot{y}\right)=0; \tag{37}$$

$$\frac{d}{dt}\left(\frac{H}{\sqrt{2T}}\dot{z}\right)=\dot{\lambda}g; \tag{38}$$

$$\dot{\lambda}=\sqrt{2T}\frac{\partial H}{\partial T}, \tag{39}$$

$$\dot{T}-g\dot{z}+R(T)\sqrt{2T}=0. \tag{40}$$

From (35)—(38) we have

$$H^2 = C_1^2 + C_2^2 + (\lambda g + C_3)^2, \tag{41}$$

where C_1, C_2 and C_3 are integration constants. Combination (41) and (35) yields the Lagrange's multiplier λ as a function of the kinetic energy

$$\lambda(T) = \frac{-\left(g C_3 - \frac{R}{\sqrt{2T}}\right) \pm \sqrt{f(T)}}{g^2 - R^2}, \tag{42}$$

where

$$f(T) = \left(g C_3 - \frac{R}{\sqrt{2T}}\right)^2 - (g^2 - R^2)\left(C_1^2 + C_2^2 + C_3^2 - \frac{1}{2T}\right). \tag{43}$$

Now, from (35) and (41) it follows that

$$H \frac{\partial H}{\partial T} dT = \pm \sqrt{f(T)}\, d\lambda, \tag{44}$$

and the equations (36)—(40) become

$$C_2 x - C_1 y = C_4, \qquad C_4 = \text{const.}, \tag{45}$$

$$dy = C_2 \frac{dT}{\sqrt{f(T)}}, \tag{46}$$

$$dz = \frac{[C_3 + \lambda(T) g]\, dT}{\sqrt{f(T)}}, \tag{47}$$

$$dt = \frac{1}{R\sqrt{2T}} \left\{ \frac{g[C_3 + \lambda(T) g]}{\sqrt{f(T)}} - 1 \right\} dT, \tag{48}$$

where the positive square root of the function $f(T)$ is taken. For any given dependence $R(T)$, the problem can therefore be solved by quadratures.

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Faculty of Technical Sciences,
University of Novi Sad, 21000 Novi Sad,
Yugoslavia.