Yugoslav Journal of Operations Research
28 (2018), Number 4, 539-566
DOI: https://doi.org/10.2298/YJOR150520023E

MODELING DYNAMIC COOPERATIVE ADVERTISING IN A DECENTRALIZED CHANNEL

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Received: May 2015 / Accepted: July 2018

Abstract: This work deals with cooperative advertising in a manufacturer-retailer supply channel using differential game theory. It considers the manufacturer as the Stackelberg leader and the retailer as the follower. It incorporates the manufacturer's advertising effort into Sethi's sales-advertising dynamics, and considers its effect on the retail advertising effort, the awareness share, the players' payoffs, and the channel payoff. These are achieved by considering two channel structures: a situation where retail advertising is subsidized, and a situation where it is not. In both situations, it obtains the Stackelberg equilibrium, which characterizes the effects of the manufacturer's advertising effort, including the relationships between the manufacturer's advertising effort and the retailer's advertising effort. The work shows that the direct involvement of the manufacturer in advertising is worthwhile.

Keywords: Cooperative Advertising, Supply Channel, Differential game Sethi's sales-advertising model.

MSC: 49N70, 91A23.

About the deceased professor Chukwuma R. Nwozo Chukwuma R. Nwozo was an Associate Professor at the Department of Mathematics, University of Ibadan, Nigeria. He was a scholar with a lot of local, national and international publications in highly rated journals. His areas of research were Operations Research, Optimization, and Financial Mathematics. He was due for the rank of a Professor which was yet to be announced at his passing on which took place on 4th December, 2017. His students and colleagues consider him a great mathematician. He is survived by a wife Sarah Nwozo (Associate Professor) three sons, and a daughter.

1. INTRODUCTION

Basically, companies use advertising to promote the sale of their products. Cooperative advertising may be of help to companies in a manufacturer-retailer supply chain. Cooperative advertising is an advertising design in which the manufacturer pays the retailer a certain percentage of the amount of money spent on retail advertising (Nagler [31]). While the retailer may engage in local advertising to stimulate “immediate” short term sales of the manufacturer’s product, the manufacturer may be involved in national advertising to build brand image name for his product. Since the retailer is closer to the consumers and has a good understanding of their behaviour, he uses local media at a lower cost to influence the consumers’ buying behaviour (Houk [17], Young and Greyser [39]). This work considers a manufacturer-retailer supply chain in dynamic setting and presents the obtained advertising strategies that optimize the players’ payoffs.

2. LITERATURE REVIEW

According to Jorgensen and Zaccour [21], cooperative advertising can be traced back to Lyon [29] as the first work to analyze cooperative advertising problems but without any mathematical model. It was followed by Hutchins [20], and Lockley [28]. Mathematical models on cooperative advertising can be categorised into static and dynamic. Berger [4] is probably the first paper to consider cooperative advertising using mathematical model, and was done on a static setting. It was followed by a number of static models which include Dant and Berger [9], Bergen and John [3], Karray and Zaccour [25], Yang et al. [38], He et al. [16].

Although Huang et al. [19] consider the use of static models as the appropriate in analyzing cooperative advertising the results from Chintagunta and Vilcassim [7], Fruchter and Kalish [13], and Naik et al. [33] suggest that it is more appropriate to employ dynamic models considering the carry-over and long-run effect of advertising.

In their review of dynamic advertising models Huang et al. [18] observed that, regarding the demand function involved, they can be classified into six groups, based on Nerlove-Arrow model (Nerlove and Arrow [30]), Vidale-Wolfe model (Vidale and Wolfe [36]), Lanchester model (Kimball [26], diffusion models, dynamic advertising competition models with other attributes, and empirical studies of dynamic advertising problems. In the course of their review Aust and Buscher [1] discovered that cooperative advertising models employ only the first three groups listed above.

Dynamic models on cooperative advertising are based on goodwill functions of Nerlove-Arrow model. This is related to the product brand image, influenced by national and local advertising effort. Jorgensen et al. [22] were the first to consider dynamic model on cooperative advertising using Nerlove-Arrow model. Other models in this category include Jorgensen et al. [23], Karray and Zaccour [24], De Giovanni [10], De Giovanni and Roseli [11].

Another group uses models which are based on Vidale-Wolfe model, extended in Sethi model (Sethi [35]). For models in this category, only the retailer is considered to be directly involved in advertising. The manufacturer participates only through subsidy to aid retail advertising. These models include Chutani and Sethi [8], He et al. [15].

The third category uses the Lanchester model (Kimball [26]), which is similar to the Vidale-Wolfe model. The Lanchester model typically models the dynamic shift in the market share between two competitors. Cooperative advertising models that are based on this model include He et al. [14]. For a comprehensive overview of the cooperative advertising literature, we refer readers to Jorgensen and Zaccour [21], and Aust and Buscher [1].

Considerations of cooperative advertising differential game models involving both the manufacturer and the retailer have only been carried out in the Nerlove-Arrow based models of goodwill. The direct involvement of both players in advertising has not been achieved in the Vidale-Wolfe based dynamics of differential games. In our work, we incorporate the manufacturer's advertising effort into the cooperative advertising literature using the Sethi advertising-sales dynamics, and by extension of the Vidale-Wolfe model. The players advertising effectiveness in this case are considered to be distinct. This is a more realistic consideration since different advertising efforts can influence the market awareness differently.

Further, none of these Vidale-Wolfe based models has been used to consider the effect of the manufacturer's advertising effort on the classical models, involving only retail advertising (that is without the manufacturer's advertising effort).

We use the resulting model to study the effect of the manufacturer's advertising effort on the retail advertising effort, i.e. the subsidy rate (manufacturer's participation rate); the manufacturer's payoff; the retailer's payoff; and the channel payoff. To see these effects, we will compare the results obtained with those of the cooperative advertising differential game (without the stochastic term) considered by He et al. [15].

3. MODEL FORMULATION

This work considers a situation where a manufacturer sells his product through the retailer to consumers. By using advertising spending and retail price, the players try to influence a fraction of the market towards buying the manufacturer's product.

It is important to note that some works in the cooperative advertising literature do not distinguish between the effects of both types of advertising on the payoffs (Berger [4], Little [27], He et al. [15], He et al. [14]). In this work we support the view that both types of advertising could influence payoffs differently, and as such, should be treated in their own rights (Jorgensen et al. [22], Huang et al. [19], Xie and Wei [37]).

The retailer decides the retail advertising effort, while the manufacturer decides the national advertising effort and advertising support scheme (subsidy) for retail advertising. Thus the manufacturer provides a certain fraction of the amount of money spent by the retailer on advertising. Specifically, the retailer decides the advertising effort

$a(t)$, while the manufacturer decides the advertising effort $A(t)$ and participation rate in the form of subsidy $\alpha(t)$.

We shall assume a quadratic cost function, a common assumption in the advertising literature. It implies diminishing marginal returns to advertising, (Deal [12], Chintagunta and Jain [5], Jorgensen et al. [22], Prasad and Sethi [34], He et al. [15], He et al. [14]). As such, the costs of advertising, quadratic in the manufacturer$A(t)$ and retailer's advertising efforts $a(t)$, are given by $\alpha a(t)^2 + A(t)^2$ and $(1-\alpha)a(t)^2$, respectively.

3.1. Dynamics of the Awareness Share

To model the dynamic effect of advertising on sales, we employ Sethi's advertising model (Sethi [35]), an improvement of the classical Vidale-Wolfe advertising model. It has been empirically validated by Chintagunta and Jain [6], and Naik et al. [32]. Using the above parameters, the sales dynamics is given by

$$x'(t) = \left(\beta_R a(t) + \beta_M A(t)\right)\sqrt{1-x(t)} - \delta x(t),$$
$$x(0) = x_0 \in [0,1], \ \ t \geq 0 \tag{1}$$

where $x(t)$ is the awareness share; it is a fraction of the total market at time t. It indicates the number of customers aware or informed of the product; x_0 is the initial condition, β_R and β_M measure the advertising effectiveness of the retailer and manufacturer respectively, and range between 0 and 1. They are known as the response constants; δ is the awareness decay parameter indicating the rate at which the potential consumers are lost due to background competition, forgetfulness, and product obsolesce.

3.2. The Leader-Follower Sequence of Events

We consider the channel members as playing a Stackelberg differential game. The decision process is modeled as a sequential Stackelberg differential gameover an infinite horizon with the manufacturer as the Stackelberg leader and the retailer as the follower. We will focus on feedback Stackelberg solutions where the optimal policy, in general, depends on the current state and time (Basar and Olsder [2], He et al. [15], He et al. [14]).

Now, the sequence of events of the game is as follows:

The manufacturer first declares the feedback national advertising effort rate $A(t)$ and the feedback participation rate $\alpha(t) \in [0,1]$ for local advertising.

In reaction to these decisions, announced by the manufacturer, the retailer decides the retail advertising effort rate $a(t)$. This is achieved by solving an optimal control problem to maximize the present value of his profit stream over the infinite horizon. This is given by

$$V^R(x) = \max_{a(t)\geq 0}\left\{\int_0^\infty e^{-\rho t}\{M_R x(t) - a(t)^2 + \alpha(t)a(t)^2\}\,dt\right\}, \tag{2}$$

subject to (1).

$V^R(x)$ is the retailer's value function; M_R is the manufacturer's margin; ρ is the discount rate.

In anticipation of the retailer's reactions, the manufacturer incorporates them (the retailers reactions) into his (manufacturer's) optimal control problem, and solves for his policies on national advertising effort $A(t)$ and participation rate $\alpha(t)$. Thus, we state his problem as

$$V^M(x) = \max_{\substack{A(t)\geq 0,\\ 0\leq\alpha(t)\leq 1}}$$

$$\left\{\int_0^\infty e^{-\rho t}\left\{M_M x(t) - \alpha(t)a\big(x(t)\big|A(t),\alpha(t)\big)^2 - A(t)^2\right\}dt\right\} \quad (3)$$

subject to

$$x'(t) = \Big(\beta_R a\big(x(t)\big|A(t),\alpha(t)\big) + \beta_M A(t)\Big)\sqrt{1-x(t)} - \delta x(t), \quad x(0) = x_0 \in [0,1], \quad (4)$$

Where $V^M(x)$ is the manufacturer's value function; M_M is the manufacturer's margin. We express the retailer's feedback advertising effort as $a(x) = a\big(x\big|A(t),\ \alpha(t)\big)$ since it is influenced by $A(t)$ and $\alpha(t)$.

At any given time $t \geq 0$, the state is denoted by $x(t)$. As such, the retailer's local advertising effort, the manufacturer's national advertising effort and the participation rate, denoted by $a(t)$, $A(t)$ and $\alpha(t)$, respectively, would be $a\big(x(t)\big)$, $A\big(x(t)\big)$ and $\alpha\big(x(t)\big)$, respectively. Thus, while we use $a(x)$, $A(x)$, and $\alpha(x)$ as feedback policies for a given awareness level x (that is the state), we use $a(t)$, $A(t)$, and $\alpha(t)$ as decision variables at time t. In a nutshell, we observe that the decision variables are functions of the state variable x, while x is a function of time t. This implies that all the decision variables are implicit functions of time.

4. THE PLAYERS' STRATEGIES AND VALUE FUNCTION

4.1. The Retailer's Advertising Effort and Value Function

In the next result, we obtain the retailer's advertising effort and value function, resulting from the manufacturer's announced policies. Although the advertising effort may appear too general as it does not specify the value or form of the subsidy provided by the manufacturer, it is a stepping stone to further results. The values and/or form of the rate of increase of the value function (payoff) and subsidy will be determined in subsequent results.

Proposition 4.1 *Let the manufacturer's advertising effort $A(x)$ be given, then, the retailer's advertising reaction policy is*

$$a(x|\,A,\alpha) = \frac{V_x^R\beta_R\sqrt{1-x}}{2\big(1-\alpha(x)\big)} \quad (5)$$

and his value function $V^R(x)$ satisfies

$$\rho V^R = M_R x + \frac{\left(V_x^R\right)^2 \beta_R^2 (1-x)}{4\left(1-\alpha(x)\right)} + V_x^R \beta_M A\sqrt{1-x} - V_x^R \delta x. \qquad (6)$$

Proof: *From (1) and (2), the Hamilton-Jacobi-Bellman (HJB) equation is*

$$\rho V^R = \max_{a(t)\geq 0}\left\{M_R x - \left(1-\alpha(x)\right)a(t)^2 + V_x^R\left[(\beta_R a + A)\sqrt{1-x} - \delta x\right]\right\}. \qquad (7)$$

The first order condition (FOC) for a maximum is

$$-2\left(1-\alpha(x)\right)a + V_x^R \beta_R \sqrt{1-x} = 0.$$

Thus

$$a(x|A,\alpha) = \frac{V_x^R \beta_R \sqrt{1-x}}{2\left(1-\alpha(x)\right)}. \qquad (8)$$

Now putting (8) in (7), we have

$$\rho V^R = M_R x - \left(1-\alpha(x)\right)\left(\frac{V_x^R \beta_R \sqrt{1-x}}{2\left(1-\alpha(x)\right)}\right)^2 + V_x^R\left[\left(\beta_R\left(\frac{V_x^R \beta_R \sqrt{1-x}}{2\left(1-\alpha(x)\right)}\right) + \beta_M A\right)\sqrt{1-x} - \delta x\right],$$

which gives the result. ■

We observe from (5) that setting $\alpha(x)$ equal to 1, that is, totally subsidising retail advertising will make the retailer's advertising effort and payoff in (6) to become unbounded. This does not make sense! Further, setting it very high would be to the detriment of the manufacturer since he would be bearing the burden of the retailer's local advertising in addition to his own national advertising.

We further note that the manufacturer's advertising effort acts on the unsold portion of the market to increase the retailer's payoff. Its effect on the retailer's payoff is high for very low market share, and as the market share increases, its effect reduces.

The retailer's margin plays a very important role in his payoff. Increasing it has to be done with caution, because it would be unnecessary if it leads to low market share, which would eventually cancel out the increase. In this situation, a wise retailer can use the manufacturer's advertising effort as a fallback position, knowing that it is very effort low awareness share.

4.2. The Manufacturer's Advertising Policy and Value Function

Proposition 4.2 *The manufacturer's feedback advertising policy is*

$$A(x) = \frac{V_x^M \beta_M \sqrt{1-x}}{2}, \qquad (9)$$

his subsidy rate to the retailer is

$$\alpha(x) = \begin{cases} \frac{2V_x^M - V_x^R}{2V_x^M + V_x^R}, & 2V_x^M > V_x^R \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

while his value function satisfies

$$\rho V^M = \max_{0\le\alpha(t)\le 1}\left\{M_M x - \frac{\alpha\left(V_x^R\right)^2\beta_R^2(1-x)}{4(1-\alpha)^2} + \frac{V_x^M V_x^R \beta_R^2(1-x)}{2(1-\alpha)} + \right.$$
$$VxM2\beta M21-x4-VxM\delta x, \tag{11}$$

Proof:*From (3) and (4), the HJB's equation is*

$$\rho V^M(x) = \max_{\substack{A(t)\ge 0,\\ 0\le\alpha(t)\le 1}} \quad . \tag{12}$$

$$\left\{M_M x - \alpha\left(\frac{V_x^R\beta_R\sqrt{1-x}}{2(1-\alpha(x))}\right)^2 - A^2 + V_x^M\left[\left(\beta_R\frac{V_x^R\beta_R\sqrt{1-x}}{2(1-\alpha(x))} + \beta_M A\right)\sqrt{1-x} - \delta x\right]\right\} \tag{12}$$

The FOC for maximum is

$$-2A + V_x^M\beta_M\sqrt{1-x} = 0$$

which implies that

$$A(x) = \frac{V_x^M\beta_M\sqrt{1-x}}{2}. \tag{13}$$

Putting (13) in (12), we have (11)
Now, maximizing (11) with respect to, we obtain

$$-\frac{\left(V_x^R\right)^2\beta_R^2(1-x)}{4}\left[\frac{(1-\alpha)^2+2\alpha(1-\alpha)}{(1-\alpha)^4}\right] + \frac{V_x^M V_x^R\beta_R^2(1-x)}{2(1-\alpha)^2}. \tag{14}$$

Recall that $\alpha \in [0,1]$. But from (14), $\alpha = 1$ is impossible. Thus, we are left with $\alpha \in [0,1)$ with $\alpha = 0$ corresponding to (14) being less than zero and $\alpha \in (0,1)$ corresponding to (14) being equal to zero.

Now suppose (14) is equal to zero, we have

$$\frac{(V_x^R)^2\beta_R^2(1-x)(1+\alpha)}{4(1-\alpha)} = \frac{V_x^M V_x^R\beta_R^2(1-x)}{2}$$

$$\alpha(x) = \begin{cases} \frac{2V_x^M - V_x^R}{2V_x^M + V_x^R}, & 2V_x^M > V_x^R \\ 0, & \text{otherwise} \end{cases} \quad . \tag{15}$$

Putting (13) into (12), we have (11). ■

From (9) we observe that as the awareness share increases, the manufacturer reduces his advertising effort. This is not out of place since there would be no need to advertise for patronage from those who are already patrons of the business, unless the purpose is to keep them as patrons. Observe that this effort is highest when the market share is zero. Further, if the advertising effectiveness and the rate of increase of his payoff are high, he will be motivated to advertise more.

4.3. Relationship between the Retail and Manufacturer's Advertising Efforts

Proposition 4.3. *For the differential games (1)-(2), and (3)-(4), the relationship between the manufacturer and retailer's advertising efforts for a given value of the awareness share is given by*

$$a(x|A,\alpha) = \frac{\beta_R V_x^R A(x)}{(1-\alpha)\beta_M V_x^M} \tag{16}$$

Proof: *From (5) and (9) , we have that for a given value of* x

$$\frac{a(x|A,\alpha)(1-\alpha)}{\beta_R V_x^R} = \frac{A(x)}{\beta_M V_x^M} = \frac{\sqrt{1-x}}{2},$$

which leads to (16). ■

From (16), we can also write

$$A(x) = \frac{(1-\alpha)\beta_M V_x^M a(x|A,\alpha)}{\beta_R V_x^R}. \tag{17}$$

From (17) we observe that as the subsidy increases, the manufacturer's advertising effort reduces, and from (16), we have that as the subsidy increases, the retail advertising effort increases. That is as the subsidy increases, the retail advertising effort increases, and the manufacturer's advertising effort reduces. In other words, as the manufacturer's advertising effort increases, the subsidy rate reduces, which subsequently leads to a reduction in the retail advertising effort. Thus, as the manufacturer gets directly involved in advertising and even increases his advertising effort, his subsidy to the retailer should reduce. This will eventually lead to the retailer reducing his advertising effort. Thus the manufacturer can decide to increase his advertising effort without bordering about the extra spending since he can reduce subsidy with his direct involvement, and vice versa. Further, total subsidy implies that he does not need to get involved in advertising.

5. MODELS WITHOUT THE MANUFACTURER'S ADVERTISING EFFORT (NON-STOCHASTIC VERSION OF HE ET AL. [15])

5.1. The Players' Optimal Control Problems

Before proceeding to consider the Stackelberg equilibrium (a, A), which characterizes non-provision of subsidy, let us first take a look at a dynamic (non-stochastic) version of the model considered by He et al. [15]. From their work, the retailer's optimal control problem is given by

$$V^R(x) = \max_{a(t)\geq 0} \int_0^\infty e^{-\rho t}\{M_R x(t) - a(t)^2 + \alpha(t)a(t)^2\}\, dt\,, t \geq 0 \quad (18)$$

subject to

$$x'(t) = \beta_R a(t)\sqrt{1 - x(t)} - \delta x(t), \quad x(0) = x_0 \in [0,1], \ t \geq 0, \quad (19)$$

where the parameters are as defined above.

The manufacturer's optimal control problem is given by

$$V^M(x) = \max_{0\leq\alpha(t)\leq 1} \int_0^\infty e^{-\rho t}\left\{M_M x(t) - \alpha(t)a\big(x(t)\big|\alpha(t)\big)^2\right\} dt \quad (20)$$

subject to

$$x'(t) = \beta_R a\big(x(t)\big|\alpha(t)\big)\sqrt{1 - x(t)} - \delta x(t), \quad x(0) = x_0 \in [0,1], t \geq 0, \quad (21)$$

where the parameters are as defined above.

In differential game models (18)-(19) and (20)-(21), the manufacturer is not directly involved in advertising. His involvement is through the provision of subsidy to the retailer.

5.2. The Player's Strategies when Subsidy is not Provided

From the models, it is shown that for a situation where the manufacturer does not provide subsidy, the retail advertising effort, the retailer's payoff, and the manufacturer's payoff are given by

$$a(x) = \frac{\beta_R B_R \sqrt{1-x}}{2}, \quad (22)$$

$$V^R(x) = C_R + B_R x$$

and

$$V^M(x) = C_M + B_M x$$

respectively; where

$$B_R = \frac{2M_R}{\sqrt{(\rho+\delta)^2 + \beta_R^2 M_R} + (\rho+\delta)},$$

$$B_M = \frac{2M_M}{2(\rho+\delta) + \beta_R^2 B_R},$$

$$C_R = \frac{\beta_R^2 B_R^2}{4\rho}$$

and

$$C_M = \frac{\beta_R^2 B_R B_M}{2\rho}$$

B_R, B_M are the slope (rate of increase) of the retailer's payoff function; the slope (rate of increase) of the manufacturer's payoff function; C_R is the intercept of the retailer's payoff function; and C_M is the intercept of the manufacturer's payoff function respectively.

5.3. The Players' Strategies and Payoffs for when Subsidy Is Provided

When the manufacturer participates in retail advertising, He et al. [15] showed that the retail advertising effort, the manufacturer's strategy, the retailer's payoff, and the manufacturer's payoff are given by

$$a(x) = \frac{\beta_R(B_R+2B_M)\sqrt{1-x}}{4}, \tag{23}$$

$$\alpha(x) = \frac{2B_M - B_R}{2B_M + B_R},$$

$$V^R(x) = C_R + B_R x$$

and

$$V^M(x) = C_M + B_M x$$

respectively, where

$$B_R = \frac{M_R}{\rho+\delta} - \frac{\beta_R^2(B_R + 2B_M)}{8(\rho+\delta)},$$

$$B_M = \frac{M_M}{\rho+\delta} - \frac{\beta_R^2(B_R + 2B_M)^2}{16(\rho+\delta)},$$

$$C_R = \frac{\beta_R^2(B_R + 2B_M)}{8\rho},$$

$$C_M = \frac{\beta_R^2(B_R + 2B_M)^2}{16\rho}.$$

6. STACKELBERG EQUILIBRIUM CHARACTERISING UNSUBSIDISED RETAIL ADVERTISING

We consider two types of equilibria. The first is the situation where the manufacturer does not provide any subsidy to aid retail advertising. In the second case, the manufacturer provides subsidy in support of retail advertising. We state these in Proposition 6.1 and Proposition 8.1, respectively.

Proposition 6.1. *For the given differential game (1)-(2), (3)-(4), the unique feedback Stackelberg equilibrium (a, A), characterizing the situation where the manufacturer does not support retail advertising effort, is given by*

$$A(x) = \frac{\beta_M B_M \sqrt{1-x}}{2}, \tag{24}$$

$$a(x) = \frac{\beta_R B_R A(x)}{\beta_M B_M} \tag{25}$$

and the associated value functions are

$$V^R(x) = C_R + B_R x, \tag{26}$$

$$V^M(x) = C_M + B_M x, \tag{27}$$

where

$$B_R = \frac{4M_R}{2\beta_M^2 B_M + \beta_R^2 B_R + 4(\rho+\delta)}, \tag{28}$$

$$B_M = \frac{4M_M}{2\beta_R^2 B_R + \beta_M^2 B_M + 4(\rho+\delta)}, \tag{29}$$

$$C_R = \frac{B_R}{4\rho}[\beta_R^2 B_R + 2\beta_M^2 B_M], \tag{30}$$

$$C_M = \frac{B_M}{4\rho}[2\beta_R^2 B_R + \beta_M^2 B_M]. \tag{31}$$

Proof*:* *Since there is no cooperative advertising, we have that $\alpha = 0$, and becomes*

$$a(x) = \frac{\beta_R V_x^R A(x)}{\beta_M V_x^M}. \tag{32}$$

Putting (13) and $\alpha = 0$ into (6) and (11), we respectively have

$$\rho V^R = M_R x + \frac{\left(V_x^R\right)^2 \beta_R^2 (1-x)}{4} + \frac{\beta_M^2 V_x^R V_x^M (1-x)}{2} - V_x^R \delta x. \tag{33}$$

and

$$\rho V^M = M_M x + \frac{\beta_R^2 V_x^R V_x^M (1-x)}{2} + \frac{\beta_M^2 \left(V_x^M\right)^2 (1-x)}{4} - V_x^M \delta x \tag{34}$$

respectively.

Because of the square root feature in the dynamics of our problem, we follow the approach of Sethi [35], He et al. [15], and He et al. [14] to obtain linear value functions which work for our model. Thus, let

$$V^R(x) = C_R + B_R x \tag{35}$$

and

$$V^M(x) = C_M + B_M x. \tag{36}$$

These imply that

$$V_x^M = B_M \text{and} \quad V_x^R = B_R \tag{37}$$

Using (37) in (9) and (32), we have (24) and (25) , respectively.
Putting (35) and (37) into (33), we have

$$\rho(C_R + B_R x) = M_R x + \frac{B_R^2 \beta_R^2 (1-x)}{4} + \frac{\beta_M^2 B_R B_M (1-x)}{2} - B_R \delta x. \tag{38}$$

Equating the coefficients of x and constants, we have (28) and (30), respectively.
Similarly, putting (36) and (37) into (34), we have

$$\rho(C_M + B_M x) = M_M x + \frac{\beta_R^2 B_R B_M (1-x)}{2} + \frac{\beta_M^2 B_M^2 (1-x)}{4} - B_M \delta x. \tag{39}$$

Equating the coefficients of x and constants, we have (29) and (31), respectively. ■

This result gives the strategies $a(x)$ and $A(x)$, and payoffs $V^R(x)$ and $V^M(x)$ for both players at equilibrium for a situation where no subsidy is provided. It allows us to see "at a glance" what both players are likely to invest (in this case their advertising efforts) and eventually gain through their value functions as payoffs.

A very important part of this result can be seen in (24) and (25) which give the unique feedback Stackelberg equilibrium when retail advertising is not subsidized. Particularly, it gives an explicit relationship between the manufacturer and retailer's advertising efforts for any given value of the awareness share.

From (25), we observe that the ratio $\frac{\beta_R B_R}{\beta_M B_M}$ is very important to the retailer. Obviously, high B_M, which implies a large $A(x)$ (from (24)), will imply a small $\frac{\beta_R B_R}{\beta_M B_M}$, and consequently, a small $a(x)$. Thus, with an effective direct involvement of the manufacturer in advertising, the retailer reduces his advertising effort.

7. EFFECT OF MANUFACTURER'S ADVERTISING EFFORT IN THE ABSENCE OF SUBSIDY

To clearly see the effect of the manufacturer's advertising effort on the retail advertising effort, awareness share, and the players' payoffs, we first determine the parameter values.

7.1. Choice of Parameter Values

In this work we are of the view that the retailer is closer to the consumer than the manufacturer. As such, his advertising effectiveness, β_R, is considered higher than the manufacturer's, β_M. Thus, we have that $\beta_R > \beta_M$. Further, we consider the effectiveness to be in percentage form (that is ratio). In particular, we take $\beta_R = 0.7$, and $\beta_M = 0.6$. Another important consideration is that we want the players to be foresighted. This is possible if ρ is set very low. Thus, we let $\rho = 0.05$. The decay rate cannot be higher than the advertising effectiveness else, it would be needless advertising. Also, it has to be small enough, reflecting that the rate of decay does not outwit the advertising effectiveness. Thus, we set it at $\delta = 0.2$. The manufacturer being the leader of the game has the first mover's advantage, and so his margin is assumed larger than that of the retailer. Thus, setting $M_M = 7$, we have that $M_R = 4$. Further, we assume that an initial awareness share of $x_0 = 0.1$. This is to create room for possible increase of the awareness share.

Note: We let the subscripts A^+ and A^- denote situations where the manufacturer is directly involved and where he is not directly involved in advertising, respectively. Also, let the subscripts $\alpha = 0$ and $\alpha > 0$ denote situations where the manufacturer does not subsidize and where he subsidizes retail advertising, respectively.

7.2. The Effect of the Manufacturer's Advertising Effort on the Retailer Advertising Effort (in the Absence of Subsidy)

We observe that with the manufacturer's direct involvement in advertising, the retail advertising effort improved from (22) to (25), to see this clearly consider Figure 1.

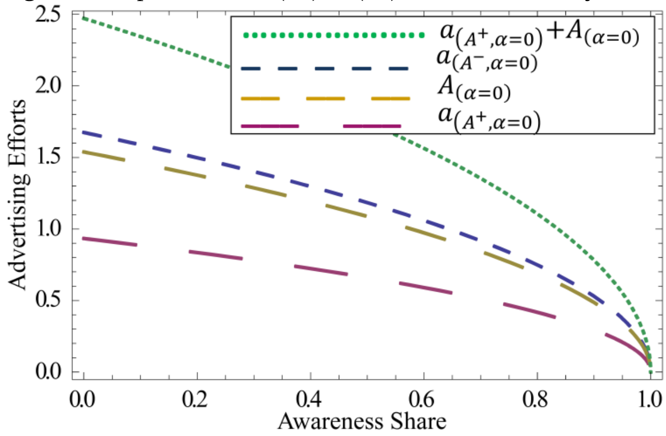

Figure 1: A comparison of the advertising efforts for a situation where the manufacturer is involved in advertising and where he is not involved (in the absence of subsidy) using the awareness share

It is obvious that with the manufacturer's direct involvement in advertising, the retailer is relieved of much of the advertising burden, which means, the reduction in his advertising effort. Also, we observe that with the manufacturer's involvement, the total advertising effort is larger compared to when he is not involved.

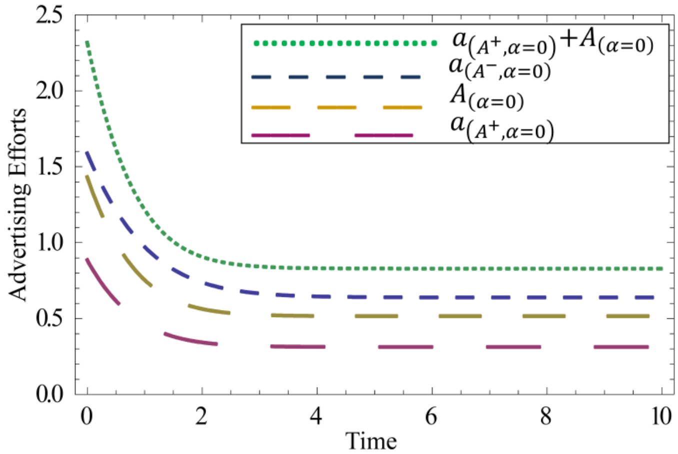

Figure 2: A comparison of the advertising efforts for a situation where the manufacturer is involved in advertising and when he is not involved (in the absence of subsidy) over time.

We can also illustrate the effect of the manufacturer's advertising involvement over time. To do this, we need explicit expressions of the awareness shares, using the dynamics in (19) and (1). This is achieved in (43) and (44), respectively and illustrated in Figure 2. Just like Figure 1, it shows that with the manufacturer's involvement in advertising, the retailer does not need to continue to spend the same amount on advertising. More specifically, the advertising effort reduced for all t. However, with the manufacturer's involvement, the total channel advertising effort increases.

7.3. The Awareness Shares in the Absence of Subsidy

7.3.1. Awareness Share without Manufacturer's Direct Involvement in Advertising (in the Absence of Subsidy)

From (22) and (19), we have that

$$x'(t) = \beta_R \frac{\beta_R B_R\sqrt{1-x(t)}}{2}\sqrt{1-x(t)} - \delta x(t)$$

$$= \frac{\beta_R^2 B_R}{2} - \frac{\beta_R^2 B_R + 2\delta}{2} x(t). \tag{40}$$

Using the integrating factor

$$I.F = \exp\left[\int\left(\frac{\beta_R^2 B_R+2\delta}{2}\right)dt\right] = \exp\left[\frac{\beta_R^2 B_R+2\delta}{2}t\right], \tag{41}$$

and multiplying (40) by (41), we have

$$\exp\left[\frac{\beta_R^2 B_R + 2\delta}{2}t\right]x' + \exp\left[\frac{\beta_R^2 B_R + 2\delta}{2}t\right]\left[\frac{\beta_R^2 B_R + 2\delta}{2}\right]x$$
$$= \exp\left[\frac{\beta_R^2 B_R + 2\delta}{2}t\right]\left[\frac{\beta_R^2 B_R}{2}\right].$$

Integrating and making x the subject, we have

$$x = \frac{\beta_R^2 B_R}{\beta_R^2 B_R + 2\delta} + \frac{C}{\exp\left[\frac{\beta_R^2 B_R + 2\delta}{2}t\right]}. \tag{42}$$

At $t = 0$, $x = x_0$. Thus, we have

$$C = x_0 - \frac{\beta_R^2 B_R}{\beta_R^2 B_R + 2\delta} .$$

Using C in (42), we have

$$x = \frac{\beta_R^2 B_R}{\beta_R^2 B_R + 2\delta} + \frac{(\beta_R^2 B_R + 2\delta)x_0 - \beta_R^2 B_R}{\beta_R^2 B_R + 2\delta}\exp\left[-\frac{\beta_R^2 B_R + 2\delta}{2}t\right]. \tag{43}$$

7.3.2. Awareness Share with Manufacturer's Direct Involvement in Advertising (in the Absence of Subsidy)

Using (24) and (25) in (1), we have

$$x'(t) = \left[\beta_R \frac{\beta_R B_{R(\alpha=0)}\sqrt{1-x}}{2} + \beta_M \frac{\beta_M B_{M(\alpha=0)}\sqrt{1-x}}{2}\right]\sqrt{1-x} - \delta x$$
$$= \left[\frac{\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)}}{2}\right] - \left[\frac{\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)} + 2\delta}{2}\right]x .$$

Using the integrating factor

$$\exp\left[\int\left(\frac{\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)} + 2\delta}{2}\right)dt\right] = \exp\left[\left(\frac{\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)} + 2\delta}{2}\right)t\right],$$

and proceeding by a similar argument as above, we have that

$$x = \frac{\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)}}{\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)} + 2\delta}$$
$$+ \frac{\left(\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)} + 2\delta\right)x_0 - \left(\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)}\right)}{\beta_R^2 B_{R(\alpha=0)} + \beta_M^2 B_{M(\alpha=0)} + 2\delta}$$

$$\times \exp\left[-\left(\frac{\beta_R^2 B_{R(\alpha=0)}+\beta_M^2 B_{M(\alpha=0)}+2\delta}{2}\right)t\right]. \quad (44)$$

7.4. The Effect of the Manufacturer's Advertising Effort on the Awareness Share (in the Absence of Subsidy)

Now let us consider the effect of the manufacturer's advertising effort on the awareness share, when there is no subsidy

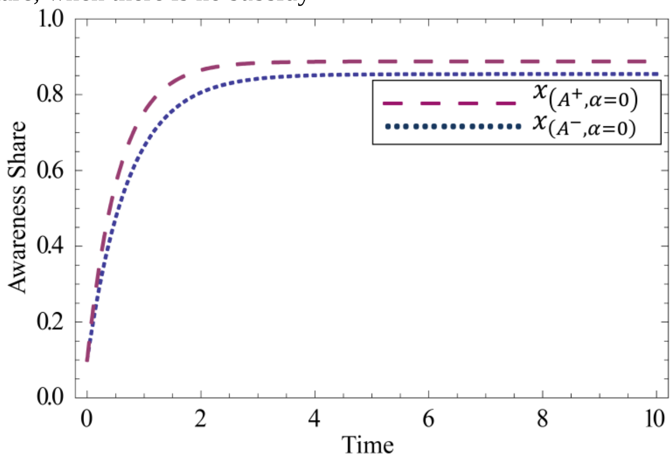

Figure 3: The awareness share for a situation where the manufacturer is involved in advertising and a situation where he is not involved (in the absence of subsidy).

From Figure 3, we observe that with the involvement of the manufacturer in advertising, the awareness increases. This implies that despite the fact that the manufacturer's involvement leads to a reduction in the retail advertising effort, the increase in the overall (channel) advertising effort leads to increase in the awareness share.

7.5. The Effect of the manufacturer's Advertising Effort on the Payoffs (in the Absence of Subsidy)

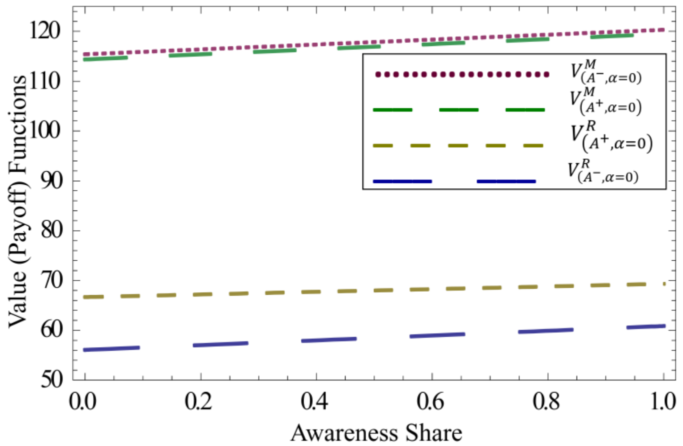

Figure 4: A comparison of the players' payoffs for a situation where the manufacturer is involved in advertising and where he is not involved (in the absence of subsidy).

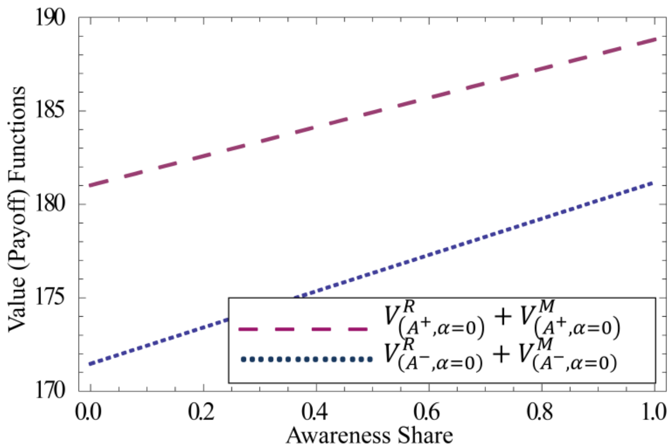

Figure 5: A comparison of the channel payoffs for a situation where the manufacturer is involved in advertising and where he is not involved (in the absence of subsidy).

Considering Figure 4, we observe that with the manufacturer's involvement in advertising in the absence of subsidy, his payoff reduces while the retailer's payoff increases. This (reduction) can be interpreted to be a result of the increase in advertising expenditure. However, a look at Figure 5 shows that this leads to increase in the total channel payoff. Thus, with a good profit sharing arrangement, the manufacturer will not be short changed.

8. EQUILIBRIUM CHARACTERIZING SUBSIDIZED RETAIL ADVERTISING

In the next result, we have the Stackelberg equilibrium characterizing a situation where retail advertising is subsidized. It gives the manufacturer and retailer's advertising efforts and the resulting payoffs for a situation where retail advertising is subsidized.

Proposition 8.1. *The Stackelberg equilibrium* (a, A, α) *characterizing the situation where the manufacturer participates in retail advertising is given by*

$$A(x) = \frac{\beta_M B_M \sqrt{1-x}}{2}, \tag{45}$$

$$a(x) = \frac{\beta_R (2B_M + B_R) A(x)}{4\beta_M B_M}, \tag{46}$$

$$\alpha = \frac{2B_M - B_R}{2B_M + B_R}, \tag{47}$$

and the condition is that

$$2B_M > B_R; \tag{48}$$

and the associated value functions are

$$V^R(x) = C_R + B_R x, \tag{49}$$

$$V^M(x) = C_M + B_M x, \tag{50}$$

where

$$B_R = \frac{8M_R}{2B_M(2\beta_M^2 + \beta_R^2) + \beta_R^2 B_R + 8(\rho+\delta)}, \tag{51}$$

$$B_M = \frac{16M_M - \beta_R^2 B_R^2}{4[\beta_R^2 B_R + (\beta_R^2 + \beta_M^2)B_M + 4(\rho+\delta)]}, \tag{52}$$

$$C_R = \frac{2\beta_R^2 B_R B_M + \beta_R^2 B_R^2 + 4\beta_M^2 B_R B_M}{8\rho}, \tag{53}$$

$$C_M = \frac{(8\beta_R^2 - 4\beta_R^2 + 4\beta_M^2)B_M^2 + \beta_R^2 B_R (B_R + 4B_M)}{16\rho}. \tag{54}$$

***Proof**: When subsidy is given by the manufacturer, we have that* $\alpha > 0$. *Now, from (15), we have that (16) becomes*

$$a(x) = \frac{\beta_R (2V_x^M + V_x^R) A(x)}{4\beta_M B_M} \tag{55}$$

Using (15) and (13) in (6) and (11), we have

$$\rho V^R = M_R x + \frac{\beta_R^2 V_x^R (1-x)(2V_x^M + V_x^R)}{8} + \frac{\beta_M^2 V_x^R V_x^M (1-x)}{2} - V_x^R \delta x \tag{56}$$

and

$$\rho V^M = M_M x - \frac{\beta_R^2((2V_x^M)^2 - (V_x^R)^2)(1-x)}{16} + \frac{\beta_R^2 V_x^M(2V_x^M + V_x^R)(1-x)}{4} + \frac{\beta_M^2 (V_x^M)^2 (1-x)}{4} - V_x^M \delta x \tag{57}$$

respectively.

Let

$$V^R(x) = C_R + B_R x \tag{58}$$

$$V^M(x) = C_M + B_M x \tag{59}$$

so that

$$V_x^R = B_R \quad \text{and} \quad V_x^M = B_M \tag{60}$$

Since subsidy is provided, using (60) in (9) and (55), we have (45) and (46), respectively. Now, putting (58) and (60) into (56), we have

$$\rho(C_R + B_R x) = M_R x + \frac{\beta_R^2 B_R(1-x)(2B_M + B_R)}{8} + \frac{\beta_M^2 B_R B_M(1-x)}{2} - B_R \delta x. \tag{61}$$

Equating the coefficients of x and constants, we have (51) and (53), respectively. Also putting (59) and (60) into (57), we have

$$\rho(C_R + B_R x) = M_M x - \frac{\beta_R^2(4B_M^2 - B_R^2)(1-x)}{16} + \frac{\beta_R^2 B_M(2B_M + B_R)(1-x)}{4} + \frac{\beta_M^2 B_M^2(1-x)}{4} - B_M \delta x. \tag{62}$$

Equating the coefficients of x and constants, we have (52) and (54), respectively. ■

Observe that (48) implies that

$$4B_M > 2B_M + B_R.$$

It follows from (31) that a large B_M implies a large A. Therefore, with subsidy, as the manufacturer's advertising effort increases, the retail advertising effort reduces. Using this result, we now consider the effect of the manufacturer's advertising effort on the retail advertising effort, the awareness shares, and the payoffs when subsidy is provided. This is the focus of the section.

9. THE EFFECT OF THE MANUFACTURER'S ADVERTISING EFFORT WHEN SUBSIDY IS PROVIDED

9.1. The Effect of the Manufacturer's Advertising Effort on the Retail Advertising Effort when Subsidy Is Provided

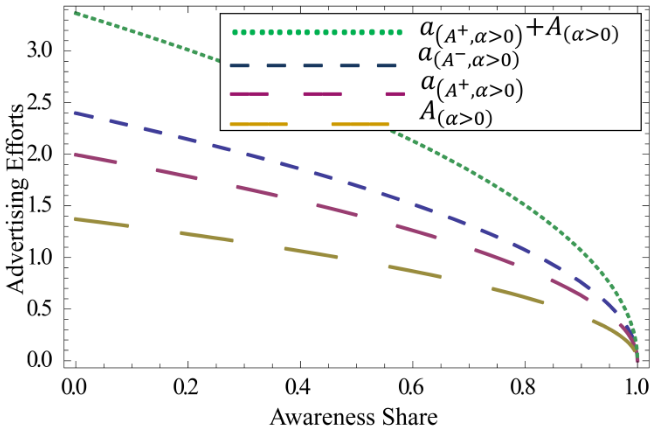

Figure 6: A comparison of the advertising efforts for a situation where the manufacturer is involved in advertising and a situation where he is not involved (in the presence of subsidy) using the awareness share.

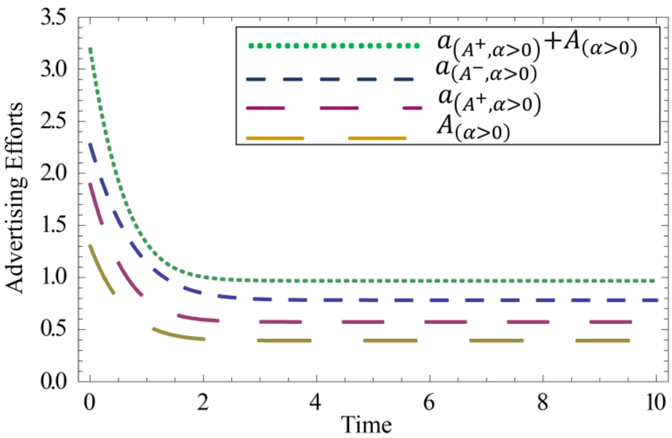

Figure 7: A comparison of the advertising efforts for a situation where the manufacturer is involved in advertising and where he is not involved (in the presence of subsidy) over time.

From Figure 6 and Figure 7 we observe that, just like the situation where there is no subsidy, the total channel advertising effort is larger with the manufacturer's involvement. This is further made clear in Figure 7, which shows that this improvement is consistent in the long-run.

9.2. *The Effect of the Manufacturer's Advertising Effort on the Awareness Share when Subsidy Is Provided*

To consider the effect of the manufacturer's advertising effort on the awareness share for a situation where subsidy is provided, we first obtain the awareness share for a situation where the manufacturer is directly involved and where he is not directly involved in advertising.

9.2.1. *Awareness Share in a Situation without the Manufacturer's Involvement in Advertising*

From (19) and (23), we have that

$$x'(t) = \beta_R \frac{\beta_R(B_R + 2B_M)\sqrt{1 - x(t)}}{4} \sqrt{1 - x(t)} - \delta x(t).$$

Proceeding as discussed in subsection 6.3, we have that

$$x(t) = \frac{\beta_R^2(B_R + 2B_M)}{\beta_R^2(B_R + 2B_M) + 4\delta} + \frac{[\beta_R^2(B_R + 2B_M) + 4\delta]x_0 - \beta_R^2(B_R + 2B_M)}{\beta_R^2(B_R + 2B_M) + 4\delta} e^{-\frac{\beta_R^2(B_R+2B_M)+4\delta}{2}t}.$$

9.2.2. *Awareness Share for a Situation where the Manufacturer Is Involved in Advertising*

Further, by using (45) and (46) in (1) and following similar argument above, we have that

$$x = \frac{\beta_R^2\left(2B_{M(\alpha>0)} + B_{R(S_R>0)}\right) + 2\beta_M^2 B_{M(\alpha>0)}}{\beta_R^2\left(2B_{M(\alpha>0)} + B_{R(\alpha>0)}\right) + 2\beta_M^2 B_{M(\alpha>0)} + 4\delta}$$

$$+ \frac{\left(\beta_R^2\left(2B_{M(\alpha>0)} + B_{R(\alpha>0)}\right) + 2\beta_M^2 B_{M(\alpha>0)} + 4\delta\right)x_0 - \left(\beta_R^2\left(2B_{M(\alpha>0)} + B_{R(\alpha>0)}\right) + 2\beta_M^2 B_{M(\alpha>0)}\right)}{\beta_R^2\left(2B_{M(\alpha>0)} + B_{R(\alpha>0)}\right) + 2\beta_M^2 B_{M(\alpha>0)} + 4\delta}$$

$$\times e^{-\left(\frac{\beta_R^2\left(2B_{M(\alpha>0)}+B_{R(\alpha>0)}\right)+2\beta_M^2 B_{M(\alpha>0)}+4\delta}{4}\right)t}.$$

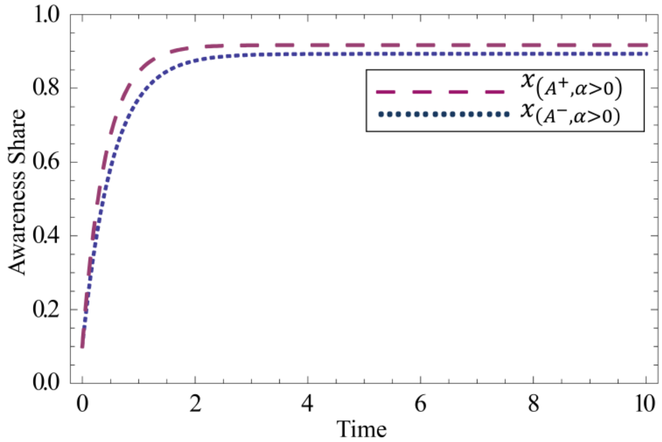

Figure 8: The awareness share for a situation where the manufacturer is involved in advertising and a situation where he is not involved (in the presence of subsidy).

We observe from Figure 8 that there is an increase in the awareness share, resulting from the manufacturer's involvement in advertising. Thus the reduction resulting from the manufacturer's involvement can be considered to be based on the confidence reposed by the retailer on the manufacturer's advertising effort. It therefore follows that this involvement can serve as additional support (in the presence of subsidy) for retail advertising.

9.3. The Effect of the Manufacturer's Advertising Effort on the Payoffs when Subsidy Is Provided

We observe that the improvement in advertising resulting from the manufacturer's involvement increased the awareness, which eventually led to increase in both the retailer and manufacturer's payoffs. This is clear from Figure 9. Further, Figure 10 illustrates the improvement of the channel payoff resulting from the manufacturer's involvement in advertising.

Now, considering Figure 4, we observe that in spite of the manufacturer's involvement in advertising, his payoff is lower in the absence of subsidy when compared to a situation where he is not involved in advertising. Figure 9 shows that with his involvement in advertising, his payoff is larger with subsidy. It is therefore clear that his direct involvement in advertising and indirect involvement through the provision of subsidy give him a better payoff.

Further, we observe that with subsidy and the manufacturer's direct involvement in advertising, both the retailer and the manufacturer's payoffs are better compared to a situation where the manufacturer is not directly involved in advertising, except through subsidy. Thus this aggressive advertising approach is justified.

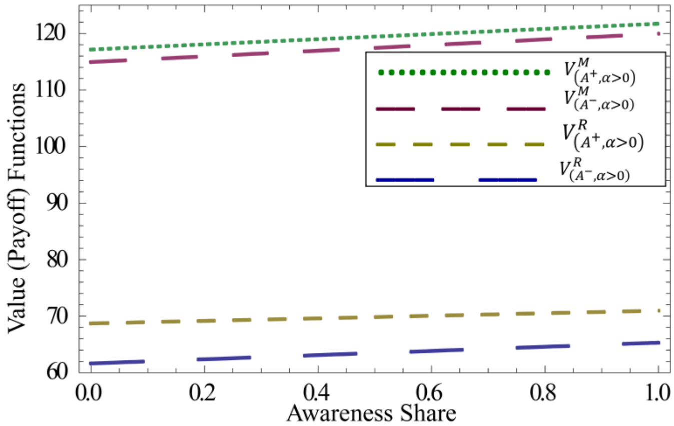

Figure 9: A comparison of the players' payoffs for a situation where the manufacturer is involved in advertising and where he is not involved (in the presence of subsidy).

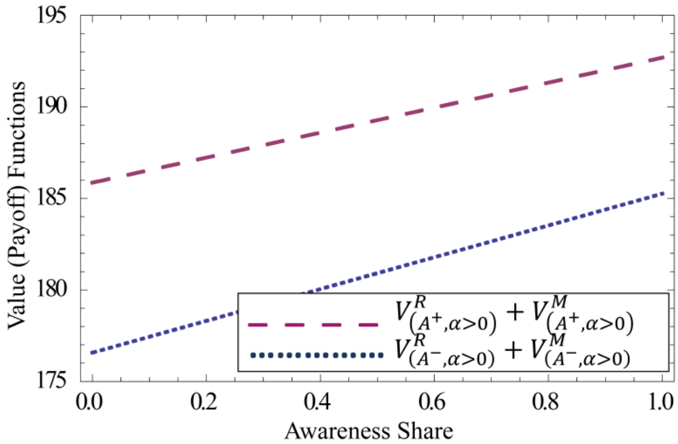

Figure 10: A comparison of the channel payoffs for a situation where the manufacturer is involved in advertising and where he is not involved (in the absence of subsidy).

10. EXISTENCE OF THE UNIQUE SOLUTION

Here we show that second order conditions are satisfied. To achieve this, it is sufficient to show that there exist unique solutions to the given differential games (1) to (4).

10.1. Uniqueness of Solution when Retail Advertising Is Unsubsidized

Now, observe from (38) and (39) that by equating the coefficients of x, we have (28) and (29), respectively. We show that (B_R, B_M) constitutes a unique solution to the coupled equation (28) and (29) (and by extension (38) and (39)) for a situation where retail advertising is unsubsidized.

Using (28) in (29), we have the equation

$$3\beta_R^4 B_R^4 + 16(\rho+\delta)\beta_R^2 B_R^3 + 8(2\beta_M^2 M_M - \beta_R^2 M_R + 2(\rho+\delta)^2)B_R^2 - 16M_R^2 = 0. \quad (63)$$

Now, let

$$a_3 = \frac{16(\rho+\delta)\beta_R^2}{3\beta_R^4}, \quad a_2 = \frac{8(2\beta_M^2 M_M - \beta_R^2 M_R + 2(\rho+\delta)^2)}{3\beta_R^4}, \quad a_1 = \frac{0}{3\beta_R^4}, \quad a_0 = \frac{16M_R^2}{3\beta_R^4}.$$

Thus (63) can be expressed as

$$F(B_R) = B_R^4 + a_3 B_R^3 + a_2 B_R^2 + a_1 B_R - a_0. \quad (64)$$

Now, from (64), we have that as $B_R \longrightarrow \pm\infty$, $F(B_R) \longrightarrow \infty$. Also, $F(B_R) < 0$ at $B_R = 0$. Further from (64), F is differentiable, which implies that it is continuous, passing through the B_R-axis at least twice.

Now,

- if all the four roots are real, then there will be three positive and one negative, or there will be three negative and one positive.
- if there are only two roots which are real, then while one will be positive the other will be negative.

1. F can be expressed as

$$F(B_R) = \big(B_R - B_R(1)\big)\big(B_R - B_R(2)\big)\big(B_R - B_R(3)\big)\big(B_R - B_R(4)\big).$$

Where $B_R(1),\ B_R(2),\ B_R(3),\ B_R(4)$ are the four roots with

$$B_R(4) > B_R(3) > B_R(2) > B_R(1), \quad B_R(2) > 0, \ B_R(1) < 0.$$

We observe that at $B_R(3)$, the slope is negative. That is

$$F'(B_R)|_{B_R=B_R(3)} = \big(B_R(3) - B_R(1)\big)\big(B_R(3) - B_R(2)\big)\big(B_R(3) - B_R(4)\big) < 0.$$

2. Now, differentiating, we have that

$$F'(B_R) = 12\beta_R^4 B_R^3 + 48(\rho+\delta)\beta_R^2 B_R^2 + 16(2\beta_M^2 M_M - \beta_R^2 M_R + 2(\rho+\delta)^2)B_R > 0 \ \forall\, B_R$$

since $M_M > M_R$.

Thus from 1 and 2 above, we infer that there is only one positive root which is unique.

10.2. Uniqueness of Solution when Retail Advertising Is Subsidized

By similar argument as the above, we have that (61) and (62) lead to (51) and (52).

Now, rearranging (51) and substituting into (52), we have

$$\frac{8M_R - \beta_R^2 B_R^2 - 8(\rho+\delta)B_R}{2(2\beta_M^2+\beta_R^2)B_R} - \frac{16M_M - \beta_R^2 B_R^2}{4\left(\beta_R^2 B_R + (\beta_R^2+\beta_M^2)\left(\frac{8M_R - \beta_R^2 B_R^2 - 8(\rho+\delta)B_R}{2(2\beta_M^2+\beta_R^2)B_R}\right) + 4(\rho+\delta)\right)} = 0$$

$$\Longrightarrow 4(\beta_M^2\beta_R^4 + 4\beta_R^2\beta_M^2)B_R^4 - 32(\rho+\delta)[4\beta_R^2\beta_M^2 + \beta_R^4]B_R^3$$
$$+64[(M_R - 4M_M)\beta_R^2\beta_M^2 - M_M(\beta_R^4 + 4\beta_M^4) - 4\beta_M^2(\rho+\delta)^2]B_R^2$$
$$-256(\rho\beta_M^2 + \delta\beta_R^2)M_R B_R + 256(\beta_R^2+\beta_M^2)M_R^2 = 0.$$

Let

$$a_0 = \frac{256(\beta_R^2+\beta_M^2)M_R^2}{4(\beta_M^2\beta_R^4 + 4\beta_R^2\beta_M^2)},$$
$$a_1 = -\frac{256(\rho\beta_M^2 + \delta\beta_R^2)M_R}{4(\beta_M^2\beta_R^4 + 4\beta_R^2\beta_M^2)},$$
$$a_2 = \frac{64[(M_R - 4M_M)\beta_R^2\beta_M^2 - M_M(\beta_R^4 + 4\beta_M^4) - 4\beta_M^2(\rho+\delta)^2]}{4(\beta_M^2\beta_R^4 + 4\beta_R^2\beta_M^2)},$$
$$a_3 = -\frac{32(\rho+\delta)[4\beta_R^2\beta_M^2 + \beta_R^4]}{4(\beta_M^2\beta_R^4 + 4\beta_R^2\beta_M^2)},$$

so that we can write

$$F(B_R) = B_R^4 + a_3 B_R^3 + a_2 B_R^2 + a_1 B_R + a_0.$$

Obviously, as $B_R \to \pm\infty$, $F(B_R) \to \infty$. Also, at $B_R = 0$, $F(B_R) > 0$. Clearly, F is continuous. It follows that its graph passes through the B_R-axis at least twice. As such, there will be

- four positive real roots, or
- two positive and two negative real roots.

Suppose that all four roots are positive and real, then, the slope at the largest must be positive. This means that if $B_R(1),\ B_R(2),\ B_R(3),\ B_R(4)$ are the roots such that $B_R(4) > B_R(3) > B_R(2) > B_R(1)$, then expressing F in terms of these roots, we have

$$F(B_R) = \big(B_R - B_R(1)\big)\big(B_R - B_R(2)\big)\big(B_R - B_R(3)\big)\big(B_R - B_R(4)\big),$$

and the slope at $B_R(4)$ is

$$F'(B_R)|_{B_R=B_R(4)} = \big(B_R(4) - B_R(1)\big)\big(B_R(4) - B_R(2)\big)\big(B_R(4) - B_R(3)\big) > 0.$$

But

$$\begin{aligned} F'(B_R) = {} & 16(\beta_M^2\beta_R^4 + 4\beta_R^2\beta_M^2)B_R^3 - 96(\rho+\delta)[4\beta_R^2\beta_M^2 + \beta_R^4]B_R^2 \\ & + 128[(M_R - 4M_M)\beta_R^2\beta_M^2 - M_M(\beta_R^4 + 4\beta_M^4) \\ & - 4\beta_M^2(\rho+\delta)^2]B_R - 256(\rho\beta_M^2 + \delta\beta_R^2)M_R \\ & < 0 \end{aligned}$$

since $M_M > M_R > 0$, $\beta_R, \beta_M \in [0,1]$ and $0 < \delta < \beta_R < \beta_M$. Hence, there exists a unique solution to the differential game.

11. CONCLUDING REMARKS

In this work we study the effect of the manufacturer's advertising involvement on the retail advertising effort, the awareness share, and subsequently the payoffs. To achieve this, the work considered a non-stochastic version of He et al. (2009) for a situation where retail advertising is subsidized and where it is not. We observe that with the manufacturer's direct involvement in advertising, the awareness share, the players' payoffs, and the channel payoffs are larger both for the subsidized and unsubsidized channels. However, the subsidized channel payoff is larger with the manufacturer's direct involvement in advertising.

This work has a few limitations and there are possible extensions. First, it considered a situation involving a single manufacturer and a single retailer. This can be extended to a situation where there is competition between a number of manufacturers and retailers. Secondly, instead of the manufacturer, we can consider the retailer as the Stackelberg leader since situations exist where the retailer is powerful enough to dictate terms to the manufacturer. Finally, recall that the involvement of the manufacturer in advertising increases the channel payoffs for the situation where retail advertising is subsidized and where it is not. Therefore, it is necessary to ensure that the manufacturer is not shortchanged in the process of providing subsidy and directly engaging in advertising. Thus a kind of agreement (a bargain) should be reached by the players. This can be incorporated into the work.

Acknowledgement: The first author would like to thank the reviewers, the editor(s), and management team for their useful comments, suggestions, patience and commitment towards this work.

REFERENCES

[1] Aust, G. and Buscher, U., "Cooperative advertising models in supply chain management: A review", *European Journal of Operational Research*, 234 (1) (2014) 1-14.

[2] Basar, T., and Olsder, G. J., *Dynamic noncooperative game theory*, SIAM, Philadelphia, PA, 1999.

[3] Bergen, M. and John, G., "Understanding cooperative advertising participation rates in conventional channels" *Journal Marketing Research*, 34 (3) (1997) 357-369.

[4] Berger, P. D., Vertical cooperative advertising ventures, *Journal of Marketing Research*, 9 (3) (1972) 309-312.

[5] Chintagunta, P. K. and Jain, D. C., "A dynamic model of channel member strategies for marketing expenditures", *Marketing Science*, 11 (2) (1992) 168-188.
[6] Chintagunta, P. K. and Jain, D. C., "Empirical analysis of a dynamic duopoly model of competition", *Journal of Economics and Management Strategies*, 4 (1) (1995) 109-131.
[7] Chintagunta P. K. and Vilcassim N. J., "Marketing investment decisions in a dynamic duopoly: a model and empirical analysis", *International Journal of Research in Marketing*, 11 (3) (1994) 287-306.
[8] Chutani, A. and Sethi, S. P., "Cooperative advertising in a dynamic retail market oligopoly", *Dynamic Games and Applications*, 2 (4) (2012) 347-375.
[9] Dant, R. and Berger, P., "Modelling cooperative advertising decisions in franchising", *Journal of the Operational Research Society*, 47 (9) (1996) 1120-1136.
[10] De Giovanni, P., "Quality improvement vs. advertising support: Which strategy works better for a manufacturer?" *European Journal of Operational Research*, 208 (2) (2011) 119-130.
[11] De Giovanni, P. and Roselli, M., "Overcoming the drawbacks of a revenue sharing contract through a support program", *Annals of Operations Research*, 196 (1) (2012) 201-222.
[12] Deal, K. R., "Optimizing advertising expenditures in a dynamic duopoly", *Operations Research*, 27 (4) (1979) 682-692.
[13] Fruchter, G. E. and Kalish, S., "Dynamic promotional budgeting and media allocation", *European Journal of Operational Research*, 111 (1) (1998) 15-27.
[14] He, X., Krishnamoorthy, A., Prasad, A. and Sethi, S. P., "Retail competition and cooperative advertising", *Operations Research Letters*, 39 (1) (2011) 11-16.
[15] He, X., Prasad, A. and Sethi, S. P., "Cooperative advertising and pricing in a dynamic stochastic supply chain: Feedback Stackelberg strategies", *Production and Operations Management*, 18 (1) (2009) 78-94.
[16] He, Y., Liu, Z. and Usman, K., "Coordination of cooperative advertising in a two-period fashion and textiles supply chain", *Mathematical Problems in Engineering*, http://dx.doi.org/10.1155/2014/356726, Article ID 356726 (2014) 10pp.
[17] Houk, B., *Co-op Advertising*, NTC Publishing, Lincolnwood, Illinois, 1995.
[18] Huang, J., Leng, M. and Liang, L., "Recent developments in dynamic advertising research", *European Journal of Operational Research*, 220 (3) (2012) 591-609.
[19] Huang, Z., Li, S. X. and Mahajan, V., "An analysis of manufacturer-retailer supply chain coordination in cooperative advertising", *Decision Science*, 33 (3) (2002) 469-494.
[20] Hutchins, M., *Cooperative Advertising - The way to make it pay*, Ronald Press, New York, 1953.
[21] Jørgensen, S. and Zaccour, G., "A survey of game-theoretic models of cooperative Advertising", *European Journal of Operational Research*, 237 (1) (2014) 1-14.
[22] Jørgensen, S., Sigue, S. P. and Zaccour, G., "Dynamic cooperative advertising in a channel" *Journal of Retailing*, 76(1) (2000) 71-92.
[23] Jørgensen, S., Taboubi, S. and Zaccour, G., "Retail promotions with negative brand image effects: Is cooperation possible?" *European Journal of Operational Research*, 150 (2) (2003) 395-405.
[24] Karray, S. and Zaccour, G., "A differential game of advertising for national and store brands", in: A. Haurie and G. Zaccour (eds.), *Dynamic games: Theory and applications*, Springer, Berlin, 2005, 213-229.
[25] Karray, S. and Zaccour, G., "Could co-op advertising be a manufacturer's counterstrategy to store brands?" *Journal of Business Research*, 59 (9) (2006) 1008-1015.
[26] Kimball, G. E., "Some industrial applications of military operations research methods", *Operations Research*, 5 (2) (1957) 201-204.
[27] Little, J. D. C., "Aggregate advertising models: The state of the art", *Operations Research* 27 (4) (1979) 629-667.
[28] Lockley, L. C., *Vertical cooperative advertising*, McGraw-Hill, New York, 1931.
[29] Lyon, L., *Advertising allowances*, The Brookings Institution, Washington, D. C., 1932.

[30] Nerlove, M. and Arrow, K. J., "Optimal advertising policy under dynamic conditions", *Economica*, 29(114) (1962) 129-142.
[31] Nagler, M.G., "An exploratory analysis of the determinants of cooperative advertising participation rates", *Marketing Letters*, 17(2) (2006) 91-102.
[32] Naik, P. A. Prasad, A and Sethi, S. P., "Building brand awareness in dynamic oligopoly markets", *Management Science*, 54(1) (2008) 129-138.
[33] Naik, P. A., Raman, K. and Winer, R. S., "Planning marketing-mix strategies in the presence of interactions", *Marketing Science*, 24(1) (2005) 25-34.
[34] Prasad, A. and Sethi, S. P., "Competitive advertising under uncertainty: A stochastic differential game approach", *Journal of Optimization Theory Application*, 123(1) (2004) 163-185.
[35] Sethi, S. P., "Deterministic and stochastic optimization of a dynamic advertising model", *Optimal Control Applications and Methods*, 4(2) (1983) 179-184.
[36] Vidale, M. L. and Wolfe, H. B., "An operations-research study of sales response to advertising", *Operations Research*, 5(3) (1957) 370-381.
[37] Xie, J. and Wei, J., "Coordination advertising and pricing in a manufacturer-retailer channel", *European Journal of Operational Research*, 197(2) (2009) 785-791.
[38] Yang, J., Xie, J., Deng, X. and Xiong, H., "Cooperative advertising in a distribution channel with fairness concerns", *European Journal of Operational Research*, 227(2) (2013) 401-407.
[39] Young, R. and Greyser, S. A., *Managing cooperative advertising: a strategic approach*, Lexington Books, Lanham, MD, (1983).