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# ON THE APPROXIMATIVE ROOTS OF MAXIMAL MONOTONE MAPPINGS

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**Abstract**. For unsolvable generalized equations with monotone mappings on the left side the notion of approximative roots is introduced. The (existence) theorems are proved and numerical methods for finding approximative solutions are proposed. The application areas are the analysis of contradictory models of optimization, game theory, economic equilibrium, etc.

**Keywords:** Variational inequality, monotone mappings, quasi roots, contradictory systems of quasi equations, improper mathematical programming problems.

## 1. INTRODUCTION

Monotone mappings play an important role in the optimization theory because many problems that involve convexity can be re-formulated in terms of such mappings, e.g. convex mathematical programming problems as well as saddle-point problems, complementarity problems, and problems of the game theory of the theory of economic equilibrium [1-2]. In this paper the recent problematics of improper (in other terms ill-posed or unresolvable) mathematical programming problems [3-4] is transferred to generalized equations with monotone mappings on the left side.

Let $E$ be a finite-dimensional Euclidean space, $G$ - some subset of the direct product $E \times E$, $\langle \cdot , \cdot \rangle$ - scalar product of two vectors. Set $G$ is called monotone, if for any two points $(z', y') \in G$, $(z'', y'') \in G$ the following inequality holds

$$\langle z' - z'', y' - y'' \rangle \geq 0,$$

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and maximal monotone, if it is not properly contained in any other monotone set.

Set $G$ generates the pair of point-to-set mappings $T: E \to 2^E$ and $T^{-1}: E \to 2^E$, namely

$$Tz = \{y:(z,y) \in G\}, \quad T^{-1}y = \{z:(z,y) \in G\}$$

If $G$ is monotone (maximal monotone), both of these mappings are called monotone (maximal monotone).

The problem we are interested in is concerned with the roots of $T$, i.e. such vectors $z \in E$, that

$$Tz \ni 0. \tag{1}$$

We shall study the case when this generalized equation has no solution and therefore must be corrected. Our analysis is based on the approach developed in [4] for improper mathematical programming problems.

The main framework is as follows. Let

$$T(\lambda) z \ni 0 \tag{2}$$

be a parametrical imbedding of the original equation (1) into some family of equations depending on the vector parameter $\lambda \in E_\lambda$ (of finite dimension) so that under some values $\lambda_i = \lambda_i'$ we have $T(\lambda') \equiv T$ and the set

$$W = \{\lambda \in E_\lambda : \text{ equation } T(\lambda) z \ni 0 \text{ has at least one root}\}$$

is non-empty.

Any choice of the parameter $\lambda \in W$ determines the concrete variant of correction of unsolvable equation (1). Assume that this choice is restricted by some set $R$, $R \cap W \neq \varnothing$, and that there is a point-to-set-mapping $S: E_\lambda \to 2^{E_\lambda}$ evaluating this choice (e.g. subdifferential mapping of some convex scalar function evaluating the difference $\lambda - \lambda'$). According to general ideas of [4-6], let us consider the problem of finding $S$-optimal correction parameter value $\lambda_0$ in the following form

find the vector $\lambda_0 \in W_R \overset{\Delta}{=} cl(R \cap W)$ such that for some $s_0 \in S\lambda_0$

and for any $\lambda \in W_R$ the following inequality holds

$$\langle s_0, \lambda_0 - \lambda \rangle \leq 0. \tag{3}$$

Note that this problem belongs to the class of so-called variational inequalities, for which the quality theory and numeric techniques have been developed [1-2, 7 and others]. The solution to equation (2), corresponding to $\lambda_0$ (or its $\varepsilon$-approximation, if $\lambda_0 \notin W$), will be called to the $S$-approximative solution of equation (1).

A closure operation in the definition of the set $W_R$ is introduced for some regularization of the problem.

If $S\lambda = \lambda - \tilde{\lambda}$ and the set $cl\ W$ is convex then the S-approximation is equivalent to finding the element of $cl\ W$ closest to $\tilde{\lambda}$.

## 2. MAIN RESULTS

In the sequel we assume that some monotone mapping $F: E \times E_\lambda \to 2^{E \times E_\lambda}$ of arguments $z$, $\lambda$ exists with the property

$$T(\lambda)\, z = \{y \in E : \exists h \in E_\lambda \text{ such that } (y, h) \in F(z, \lambda)\}. \tag{4}$$

Since $F$ is monotone, the mappings $T(\lambda)$ are monotone too for any $\lambda$.

**Theorem 1**. Let $F$ be monotone and condition (4) holds.
Then mapping $f: E_\lambda \to 2^{E_\lambda}$ defined by the formula

$$f\lambda = \{y \in E_\lambda : \exists\, z \text{ such that } (0, y) \in F(z, \lambda)\}, \tag{5}$$

is monotone too. Moreover

$$W = \{\lambda \in E_\lambda : T(\lambda) z \ni 0 \text{ solvable}\} = dom\, f,$$

where $dom\ f$ denotes the effective domain of the mapping $f$ - the set of all points with non-empty images.

**Proof**. If $y' \in f\lambda'$, $y'' \in f\lambda''$ are taken arbitrarily then by definition of $f$ the points $z', z'' \in E$ exist such that $h' = (z', \lambda') \in dom\, F$, $h'' = (z'', \lambda'') \in dom\, F$, $w' = (0, y') \in F(h')$, $w'' = (0, y'') \in F(h'')$. Consequently (due to monotonocity of $F$) we have

$$\langle y' - y'', \lambda' - \lambda'' \rangle = \langle y' - y'', \lambda' - \lambda'' \rangle + \langle 0 - 0, z' - z'' \rangle =$$
$$= \langle w' - w'', h' - h'' \rangle \geq 0.$$

To show that $W = dom\, f$ let us take any $\lambda \in W$. For some $z \in E$ we have $0 \in T(\lambda) z = \{y: \exists h \in E_\lambda (y, h) \in F(z, \lambda)\}$. Hence there exists $h \in E_\lambda$ such that $(y, h) \in F(z, \lambda)$, i.e. $h \in f\lambda$, and therefore $f\lambda \neq \varnothing$. Conversely, if we take any $\lambda \in dom\, f$, then points $y \in E_\lambda$ and $z \in E$ exist such that $(0, y) \in F(z, \lambda)$. Due to (4) this means the inclusion $T(\lambda) z \ni 0$.

A set $M$ is called almost convex if $ri\,(conv\, M) \subset M$. From the known theorem [7] a about the structure of the effective domain of maximal monotone mappings there follows.

**Corollary 1**. If mapping $f$ in relation (5) is maximal monotone then set $W$ in problem (3) of finding the $S$ - optimal correction parameter value of equation (1) is almost convex.

Let us present the conditions that will be used below in different combinations.

(a) Mappings $F: E \times E_\lambda \to 2^{E \times E_\lambda}$ and $f: E_\lambda \to 2^{E_\lambda}$ from formulae (4)-(5) are maximal monotone.

(b) Set $R$ is closed and convex and $ri\, R \cap ri\,(dom\, f) \neq \varnothing$.

(c) Mapping $S: E_\lambda \to 2^{E_\lambda}$ is monotone, upper semi-continuous and its values are convex compact sets over $W_R = cl\,(dom\, f \cap R)$.

(d) The point $h_0 \in ri\, W_R$ and a compact set $D$ surrounding it exist with the property: for any $\lambda \in W_R \cap D$ there exists $s \in S\lambda$ such that $\langle s, \lambda - h_0 \rangle \geq 0$.

Set $D$ is said to surround a point $h_0$ if for any $l \in E$ there exists $\varepsilon = \varepsilon(l) > 0$ such that $h_0 + \varepsilon\, l \in D$ (e.g. the sphere $\{\lambda : \| \lambda - h_0 \| = r\}$ with the center at $h_0$ surrounds it).

Assumptions (a) - (d) guarantee that the problem (3) has solutions and seen the weakest among the conditions of such type for monotone variational inequalities [1-2, 8-9].

Define auxiliary mappings $H^\alpha: E \times E_\lambda \to 2^{E \times E_\lambda}$, $f^\alpha: E_\lambda \to 2^{E_\lambda}$ by the rules

$$H^\alpha(z,\lambda) = \begin{cases} \alpha F(z,\lambda) + \{0\} \times S_R\lambda & \text{for } (z,\lambda) \in dom\, F \cap L, \\ \varnothing & \text{otherwise;} \end{cases}$$

$$f^\alpha(\lambda) = \begin{cases} \alpha f\,\lambda + S_R\lambda & \text{for } \lambda \in dom\, S_R \cap dom\, f, \\ \varnothing & \text{otherwise.} \end{cases}$$

Here $S_R = S + N_R$, mapping $N_R: E_\lambda \to 2^{E_\lambda}$ is generated by normal cones to set $R$ by rule $N_R\lambda = \{l : \langle l, \lambda - h \rangle \geq 0 \text{ for any } h \in R\}$, $dom\, S_R = R \cap dom\, S$, $L = \{(z,\lambda) : z \in E,\ \lambda \in dom\, S_R\}$, $\alpha > 0$.

Under the above assumptions mapping $f^\alpha$ will be maximal monotone. The mapping $H^\alpha$ will be maximal monotone if condition (c) holds for the whole $R$.

**Theorem 2**. Generalized equations $f^\alpha\,\lambda \ni 0$, $H^\alpha(z,\lambda) \ni 0$ are solvable or not simultaneously.

This follows from the fact that if $(z,\lambda)$ is any root of mapping $H^\alpha$, then $\lambda$ is a root of mapping $f^\alpha$, and conversely, if $\lambda$ is any root of $f^\alpha$, then there exists $z \in E$ such that $(z,\lambda)$ is a root of $H^\alpha$.

**Theorem 3**. Let $F$ be monotone, the conditions (b), (c) hold and the sequence $\{\lambda_n\}$ consists of the roots of the mappings $f^{\alpha_n}$, where $\alpha_n \to +0$. Then all cluster points (if any exist) satisfy variational inequality (3).

**P r o o f**. Let $\lambda_0 \in W_R$ be an arbitrary cluster point of the sequence $\{\lambda_n\}$. We may suppose $\lambda_0 = \lim_{(n)} \lambda_n$. Theorem 2 implies that with sequence $\{\lambda_n\}$ there may be associated some sequences $\{z_n\}$, $\{s_n\}$, $\{u_n\}$ and $\{l_n\}$ such that $s_n \in S\lambda_n, l_n \quad N_R\lambda_n$, $(0, u_n) \in F(z_n, \lambda_n)$ and

$$s_n = -l_n - \alpha_n u_n. \tag{6}$$

Since sequence $\{\lambda_n\}$ is bounded then, due to the declared properties of mapping $S$, sequence $\{s_n\}$ associated with it is bounded too. All its cluster points belong to $S\lambda_0$.

Let $s_0$ be any one of them (one can think $s_0 = \lim_{(n)} s_n$). Let us show that $\langle s_0, \lambda_0 - \lambda\rangle \le 0$ for all $\lambda \in R \cap dom\, f$ (therefore for all $\lambda \in W_R$).

Indeed, since $f$ is monotone (see Theorem 1) and $u_n \in f\lambda_n$, we have

$$\langle u_n, \lambda - \lambda_n \rangle \le \langle u, \lambda - \lambda_n \rangle.$$

From the definition of normal cones it follows that

$$\langle l_n, \lambda - \lambda_n \rangle \le 0.$$

Hence for a fixed $\lambda \in dom\, f$ and any $n$ we have

$$\langle s_n, \lambda_n - \lambda \rangle = \langle l_n, \lambda - \lambda_n \rangle + \alpha_n \langle u_n, \lambda - \lambda_n \rangle \le \alpha_n \langle u, \lambda - \lambda_n \rangle \tag{7}$$

Letting $n$ to infinity, we shall get the desired inequality.

**Corollary 2**. Let all assumptions of Theorem 3 and condition (a) be valid, and let the set $W_R^0$ of the solutions of inequality (3) be not empty and lie in $W$. If $\lambda_0$ is any cluster point of the sequence $\{\lambda_n\}$ and $f\lambda_0$ is bounded then for some $u_0 \in f\lambda_0$ the inequality

$$\langle u_0, \lambda_0 - \lambda \rangle \le 0$$

holds for any $\lambda \in W_R^0$.

**P r o o f**. Maximal monotone mappings are upper semi-continuous. Then under the above assumptions there exists a cluster point $u_0$ of the sequence $\{u_n\}$ (one can think $u_0 = \lim_{(n)} u_n \in f\lambda_0$). Take any $\lambda \in W_R^0$. From the definition of set $W_R^0$ as well as the properties of normal cones and the monotonicity of $S$ there exists $s \in S\lambda$ such that (see (6))

$$\langle u_n, \lambda_n - \lambda\rangle = 1/\alpha_n \langle -s_n - l_n, \lambda_n - \lambda\rangle \le 1/\alpha_n \langle s, \lambda - \lambda_n \rangle \le 0.$$

Letting $n$ to infinity, we shall get the inequality we wanted.

Theorem 3 shows how to find the $S$-optimal vector of correction $\lambda_0$ and the approximative root of equation (1) corresponding to it: we must choose a small $\alpha > 0$

and find the root $(z_{\alpha,} \lambda_\alpha)$ of the mapping a $H_\alpha$.The vector $\lambda_0$ will be good $\alpha$ aaproximation to $\lambda_0$ , as close to it as $\alpha$ is small.

We shall now give some conditions which guarantee the solvability of equations $f^\alpha\lambda \ni 0,\ 0<\alpha<\alpha_0$, and the boundedness of their solution sets in coupling.

**Theorem 4**. Let take that the assumptions (a)-(c) hold and the mapping $S$ are strongly monotone on $W_R$ with module $\gamma>0$, i.e. for all $\lambda', \lambda'' \in W_R,\ s' \in S\lambda',\ s'' \in S\lambda''$ the following inequality holds

$$\langle s'-s'',\ \lambda'-\lambda''\rangle \geq \gamma\| \lambda'-\lambda''\|^2.$$

Then the mapping $f^\alpha$ is strongly monotone too it has the unique root $\lambda_\alpha$ and the set $\Lambda_0 = \bigcup_{0<\alpha<\alpha_0} \lambda_\alpha$ is bounded for any $\alpha_0>0$.

**P r o o f**. Take any $u' \in f^\alpha\lambda',\ u'' \in f^\alpha\lambda''$. We can present them as the sums $u'=s'+l'+\alpha t',\ u''=s''+l''+\alpha t''$, where $s' \in S\lambda',\ s'' \in S\lambda'',\ l' \in N_R\lambda',\ l'' \in N_R\lambda'',\ t' \in f\lambda'$, $t'' \in f\lambda''$. Since $f$ and $N_R$ are monotone and $S$ is strongly monotone, we have

$$\langle u'-u'', \lambda'-\lambda''\rangle = \langle s'-s'',\ \lambda'-\lambda''\rangle +$$
$$+\langle l'-l'',\ \lambda'-\lambda''\rangle + \alpha\langle t'-t'',\ \lambda'-\lambda''\rangle \geq \gamma\| \lambda'-\lambda''\|^2,$$

i.e. the mapping $f^\alpha$ also satisfies the condition of strong monotonocity. Being maximal monotone it has the unique root $\lambda_\alpha$ [1] .

Next, let us show the set $\Lambda_0$ bounded.

Take any $\lambda_0 \in ri\, W_R = ri\, R \cap ri\,(dom\, f),\ s_0 \in S\lambda_0,\ l_0 \in N_R\lambda_0,\ t_0 \in f\lambda_0$. Since $f^\alpha$ is strongly monotone with the module $\gamma>0$ and $0 \in f^\alpha\lambda_\alpha$ we have

$$\gamma\|\ \lambda_\alpha-\lambda_0\ \|^2 \leq -\langle\ s_0+l_0+\alpha t_0,\ \lambda_\alpha-\lambda_0\ \rangle$$

Then

$$\gamma\|\ \lambda_\alpha-\lambda_0\ \|^2 \leq (\|\ s_0\ \|+\|\ l_0\ \|+\alpha\|\ t_0\ \|)\ \|\|\ \lambda_\alpha-\lambda_0\ \|,$$

i.e.

$$\|\ \lambda_\alpha-\lambda_0\ \| \leq \gamma^{-1}(\|\ s_0\ \|+\|\ l_0\ \|+\alpha_0\|\ t_0\ \|)$$

for all $\alpha<\alpha_0$ .

Note that the condition of strong monotonicity of $S$ over $W_R$ implies that over condition (d) is true.

**Corollary 3**. Suppose that all assumptions of Theorem 4 hold and $\lambda_0$ *dom f* is the solution to problem (3). Then

$$\|\cdot\lambda_\alpha-\lambda_0\ \| \leq \alpha\gamma^{-1}\|\ z_0\ \|$$

for any $z_0 \in f\lambda_0$.

**P r o o f**. The mapping of $S$ is strongly monotone. Therefore for all $s_\alpha \in S\lambda_\alpha$, $s_0 \in S\lambda_0$ the following inequality holds

$$\langle s_\alpha - s_0, \lambda_\alpha - \lambda_0 \rangle \geq \gamma \| \lambda_\alpha - \lambda_0 \|^2.$$

Using inequalities (3), (7) we have

$$\gamma \| \lambda_\alpha - \lambda_0 \|^2 \leq \langle s_\alpha, \lambda_\alpha - \lambda_0 \rangle + \langle s_0, \lambda_0 - \lambda_\alpha \rangle \leq \langle s_\alpha, \lambda_\alpha - \lambda_0 \rangle \leq$$
$$\leq -\alpha \langle z_0, \lambda_\alpha - \lambda_0 \rangle \leq \alpha \| z_0 \| \, \| \lambda_\alpha - \lambda_0 \|.$$

which proves the Corollary.

The condition of strong monotonocity of $S$ may be omitted if we use a more restrictive variant of condition (d). Let $V^\alpha$ denote the set of roots of mapping $f^\alpha$.

**Theorem 5**. Let assumptions (a)-(c) be valid as well as the following variant of condition (d): there exists a point $h_0 \in ri\, W_R \subset ri\,(dom\, f)$ and a compact set $D$ surrounding it with diameter $d$ such that for any $\lambda \in dom\, f^\alpha \cap D$ there exists $s \in S\lambda$ for which $\langle s, \lambda - h_0 \rangle \geq \delta$, where $dom\, f^\alpha = R \cap dom\, f$, $\delta > 0$ is fixed.. If there exists $0 \neq t_0 \in f\, h_0$ then the sets $V^\alpha$, $0 < \alpha < \alpha_0 = \delta / (d \| t_0 \|)$, are non-empty and bounded in coupling (namely they all lie in $D$). If in addition $f h_0 \ni 0$, then $V^\alpha \subset D$ for all $\alpha > 0$.

**P r o o f**. The mappings of $f^\alpha$ are maximal monotone. We will show that for $0 < \alpha < \alpha_0$ all of them satisfy the condition of type (d) relative to set $D$. Indeed, let $\lambda \in D \cap R \cap dom\, f$, $l \in N_R \lambda$, $t \in f\lambda$ and $s \in S\lambda$ be such that $\langle s, \lambda - h_0 \rangle \geq \delta$. Then it is obvious that $s_\alpha = s + l + \alpha t \in f^\alpha \lambda$ and for all $0 < \alpha < \alpha_0$ we have inequality

$$\langle s_\alpha, \lambda - h_0 \rangle = \langle s, \lambda - h_0 \rangle + \langle l, \lambda - h_0 \rangle + \alpha \langle t, \lambda - h_0 \rangle \geq$$
$$\geq \delta + \alpha \langle t_0, \lambda - h_0 \rangle \geq \delta - \alpha_0 \| t_0 \| d = 0.$$

The reference to the theorem on the existence of the roots of monotone mappings [see 9, p.228] ends the proof.

## 3. THE CASE OF LINEAR PARAMETERIZATION

Assume that

$$T(\lambda) z = Tz - \lambda, \quad \lambda \in E = E_\lambda.$$

We shall call it the case of linear correction (shifting) of equation (1).

Obviously, one can include the linear case into the framework of Section 2: it is enough to take $F'(z, \lambda) = (Tz - \lambda) \times \{ z \}$ as the mapping $F$. This implies that $f = T^{-1}$ and equation

$$Tz - \lambda \ni 0$$

has a solution if and only if $\lambda \in dom\, T^{-1}$.

The variants of conditions (a)-(d) corresponding to this case are enumerated below.

(a') Mapping $T$ is maximal monotone (the main consequence of this is that $T$ will be upper semi-continuous over $ri(dom T)$ and the sets $dom\, T$, $dom\, T^{-1}$ will be almost convex [7].

(b') Set $R$ is convex, closed and $ri\, R \cap ri\,(dom\, T^{-1}) \neq \varnothing$.

(c') Mapping $S$ is monotone, upper semi-continuous and its values are convex compact sets over $W_R$.

(d') There exists a point $h_0 \in ri\, W_R$ and a compact set $D$ surrounding it with the property: for any $\lambda \in ri\, W_R \cap D$ there exists $s \in S\lambda$ such that $\langle\ s,\ \lambda - h_0\ \rangle \geq 0$.

Let us again construct the auxiliary mappings $T^\alpha : E \to 2^E$, $S^\alpha : E \to 2^E$, $H^\alpha$ : $E \times E \to 2^{E \times E}$ as follows

$$T^\alpha z = \begin{cases} Tz - S_R^{-1}(-\alpha z) & \text{for } z \in dom\, T \cap (-1/\alpha dom\, S_R^{-1}), \\ \varnothing & \text{otherwise;} \end{cases}$$

$$S^\alpha \lambda = \begin{cases} S_R \lambda + \alpha T^{-1} \lambda & \text{for } \lambda \in dom S_R \cap dom T^{-1} \\ \varnothing & \text{otherwise;} \end{cases},$$

$$H^\alpha(z,\lambda) = \begin{cases} \{\alpha T z - \alpha\lambda\} \times \{\ S_R \lambda + \alpha z\ \} & \text{for } (z,\lambda) \in dom\, H^\alpha \\ \varnothing & \text{otherwise;} \end{cases}$$

Here $S_R = S + N_R$, $N_R : E \to 2^E$ is monotone mapping generated by normal cones to the set $R$, $dom\, S_R = R \cap dom\, S$, $dom\, H^\alpha = \{\ (z,\lambda) : z \in dom\, T,\ \lambda \in dom\, S_R\ \}$, $\alpha > 0$.

Conditions (a')- (c') guarantee the maximal monotonocity of mapping $S^\alpha$ only. The mapping $H^\alpha$ will be maximal monotone if condition (c') holds not only on $W_R$ but on the whole set $R$. The maximal monotonocity of $T^\alpha$ may be obtained by assumption $ri\,(dom\, T) \cap ri\,(-1/\alpha dom\, S_R^{-1}) \neq \varnothing$, which in turn may be obtained (for small $\alpha > 0$) from inclusion $0 \in \mathrm{int}\,(dom\, S_R^{-1})$. All these facts are consequences of the well-known theorem on maximal monotonocity of the sum of maximal monotone mappings (see e.g. [9], p.234).

Let us present (without proofs) the analogouses of Theorems 2 - 4 for the linear case.

**Theorem 6.** The equations

$$S^\alpha \lambda \ni 0, \quad T^\alpha z \ni 0, \quad H^\alpha(z,\lambda) \ni 0$$

are solvable or not solvable at the same time.

Let $Z^\alpha$ be the set of the roots of the mapping $T^\alpha$ and $W^\alpha$ be the set of the roots of the mapping $S^\alpha$.

**Theorem 7**. Assume the mapping $T$ is monotone, conditions (b'), (c') hold and the sequence $\{ \lambda_n \}$ consists of the roots of the mappings $S^{\alpha_n}$, where $\alpha_n \to +0$. If this sequence has cluster points then all of them are solutions to the variational inequality (3).

Theorem 7 shows how to find the $S$-optimal shifting vector $\lambda_0$ and the corresponding approximative solution of equation (1): we must choose a small $\alpha > 0$ and calculate the root $(z_\alpha, \lambda_\alpha)$ of the mapping $H_\alpha$ (or the root $z_\alpha$ of the mapping $T^\alpha$ and associated with it the vector $\lambda_\alpha \in W(z_\alpha)$, or the root $\lambda_\alpha$ of the mapping $S^\alpha$ and associated with it the vector $z_\alpha \in Z(\lambda_\alpha)$). The vector $\lambda_\alpha$ will approximate $\lambda_0$ we are interested in, more precisely if $\alpha$ is small. Moreover, $Tz_\alpha - \lambda_\alpha \ni 0$, i.e. $Tz_\alpha - \lambda_\alpha \ni 0$, i.e. $z_\alpha$ is the $S$-approximative solution of the original equation.

**Theorem 8**: Let conditions (a')-(c') be valid and the mapping $S$ be strongly monotone on $W_R$ with module $\gamma > 0$. Then a) any mapping $S^\alpha$ is strongly monotone too and has the unique root $\lambda_\alpha$; b) the set $\Lambda_0 = \bigcup_{0<\alpha<\alpha_0} \lambda_\alpha$ is bounded for all $\alpha_0 > 0$.

**Corollary 4**. Let all assumptions of Theorem 3 be valid and $\lambda_0 \in dom\, T^{-1}$ be the solution to problem (3). Then

$$\| \lambda_\alpha - \lambda_0 \| \le \alpha \gamma^{-1} \| z_0 \|,$$

where $z_0 \in T^{-1} \lambda_0$ is taken arbitrarily.

Strong monotonocity condition on $S$ may be weakened if we use some more restrictive variant of condition (d').

**Theorem 9**: Let conditions (a')-(c') be valid as well as the following variant of condition (d'): a point $h_0 \in ri\, W_R \subset ri\,(dom\, T^{-1})$ and a compact set $D$ surrounding it (with diameter $d$) exist such that for any $\lambda \in dom\, S^\alpha \cap D$ and some $s \in S\lambda$ one has $\langle s, \lambda - h_0 \rangle \ge \delta$, where $dom\, S^\alpha = R \cap dom\, T^{-1}$, $\delta > 0$ is fixed. If there exists $0 \ne t_0 \in T^{-1} h_0$ then the sets $W^\alpha, 0 < \alpha < \alpha_0 = \delta / (d \| t_0 \|)$, are non-empty and bounded in the coupling (they all lie in $D$). More over, if $T^{-1} h_0 \ni 0$ then $W^\alpha$ are non-empty and lie in $D$ for all $\alpha_0 > 0$.

Remark: Theorem 6 provides the opportunity to obtain the boundedness and non-emptiness of set $W^\alpha$ from some properties of the inverse mapping $S_0^{-1}$.

**Theorem 10**. Let the mapping $T$ be maximal monotone, $R$=$E$ and the mapping $S_0^{-1} \equiv S^{-1}$ be upper semi-continuous, compact-valued and strongly monotone on $E$ with module $\gamma > 0$. Then for any $\alpha_0 > 0$ the sets $W^\alpha$, $0 < \alpha < \alpha_0$, are not empty and bounded in the coupling.

**P r o o f**: Under the above assumptions the mappings $T^\alpha$ are maximal and strongly monotone. Indeed, take any $t'_\alpha \in T^\alpha z'$, $t''_\alpha \in T^\alpha z''$, $t'_\alpha + s'$, $t''_\alpha = t'' + s''$, where $t' \in Tz'$, $t'' \in Tz''$, $s' \in -S^{-1}(-\alpha z')$, $s'' \in -S^{-1}(-\alpha z'')$. Then $\langle t'_\alpha - t''_\alpha,\ z' - z'' \rangle = \langle t' - -t'',\ z' - z'' \rangle + \langle s' - s'',\ z' - z'' \rangle \ge \alpha\gamma \| z' - z'' \|^2$.

It implies that each of them has one and only one root $z_\alpha$ and for any $z_0$ $dom\,T$ and $\lambda_0 \in Tz_0$, $s^0_\alpha \in -S^{-1}(-\alpha z_0)$ we have successively

$$\alpha\gamma \| z_\alpha - z_0 \|^2 \le -\langle \lambda_0 + s^0_\alpha,\ z_\alpha - z_0 \rangle \le$$
$$\le \| z_\alpha - z_0 \| \,(\langle \lambda_0 \rangle + \max_{0<\alpha<\alpha_0} \ \max_{s \in S^{-1}(-\alpha z_0)} \|s\|) \le$$
$$\le \| z_\alpha - z_0 \| \,(\| \lambda_0 \| + \max_{\|z\| \le r} \ \max_{s \in S^{-1}(z)} \|s\|),$$

where $r = \alpha_0 \| z_0 \|$. Therefore

$$\alpha \| z_\alpha \| \le \alpha_0 \| z_0 \| + \gamma^{-1} (\| \lambda_0 \| + \max_{\|z\| \le r} \ \max_{s \in S^{-1}(z)} \|s\|)$$

provided that $0 < \alpha < \alpha_0$.

Consequently, the set $\bigcup_{0<\alpha<\alpha_0} \alpha z_\alpha$ is bounded for any $\alpha_0$. Hence, due to the inclusion $\lambda_\alpha \in W(z_\alpha) \subset S^{-1}(-\alpha z_\alpha)$ and assumptions about properties of $R$ and $S^{-1}$, the set $\bigcup_{0<\alpha<\alpha_0} W^\alpha$ is bounded too.

We end this section by describing the structure of the set of all approximative roots of maximal monotone mappings.

**Theorem 11**. Let $R = E$, $\lambda_0$ be the $S$-optimal shifting vector for the maximal monotone mapping $T$ and $Z_0 \overset{\Delta}{=} T^{-1}\lambda_0 \ne \varnothing$. Then there exists a vector $s_0 \in S\lambda_0$ such that $-s_0$ is the recessive heading of the set $Z_0$ (i.e. $Z_0 - s_0 \subset Z_0$).

**P r o o f**. The set $Z_0$ is closed and convex [1-2]. Let $z_0 \in Z_0$ and $s_0$ be the vector in $S\lambda_0$ for which for all $\lambda \in W_R$ $(= cl\,dom\,T^{-1})$ inequality (3) is valid:

$$\langle s_0,\ \lambda_0 - \lambda \rangle \le 0.$$

Since $T$ is monotone, the inequality

$$\langle z_0 - s_0 - z,\ \lambda_0 - \lambda \rangle = \langle z_0 - z,\ \lambda_0 - \lambda \rangle - \langle s_0,\ \lambda_0 - \lambda \rangle \ge 0$$

is valid for any $(z, \lambda) \in G$. It implies (due to the maximal monotonocity of $T$) the inclusion $z_0 - s_0 \in T^{-1}\lambda_0 = Z_0$.

## 4. EXAMPLES

Let us consider the dual pair of linear programming (LP) problems

$$\max\{\ \langle\ c,x\ \rangle : Ax \le b,\ x \ge 0\ \}, \tag{8}$$

$$\min\{\ \langle\ b,y\ \rangle : A^{\mathrm{T}}y \ge c,\ y \ge 0\ \}, \tag{9}$$

where $x \in R^n, c \in R^n, y \in R^m, b \in R^m$, $A$ is an $(m \times n)$-matrix, the superscript T denotes transposition.

The problems (8)-(9) may be improper. Let $A = [\ A_0^{\mathrm{T}}, A_1^{\mathrm{T}}\ ]^{\mathrm{T}} = [\ B_0, B_1\ ]$, $c^{\mathrm{T}} = (c_0^{\mathrm{T}}, c_1^{\mathrm{T}})$, $b^{\mathrm{T}} = (b_0^{\mathrm{T}}, b_1^{\mathrm{T}})$, $x^{\mathrm{T}} = (x_0^{\mathrm{T}}, x_1^{\mathrm{T}})$, $y^{\mathrm{T}} = (y_0^{\mathrm{T}}, y_1^{\mathrm{T}})$ be a partition of the original problems. Let us define the sets

$$M(v) = \{\ x : A_0 x \le b_0,\ \ A_1 x \le b_1 + v,\ x \ge 0\ \},$$

$$M^*(u) = \{\ y : B_0^{\mathrm{T}} y \ge c_0,\ \ B_1^{\mathrm{T}} y \ge c_1 - u,\ y \ge 0\ \},$$

$$W_1 = \{\ u : M^*(u) \ne \varnothing\ \},\quad W_2 = \{\ v : M(v) \ne \varnothing\ \},\quad W = W_1 \times W_2.$$

All these sets are polyhedrons. Let us assume that $W \ne \varnothing$. Our task is to find the elements $w = [\ u_0^{\mathrm{T}}, v_0^{\mathrm{T}}\ ] \in W$ which are minimal relative to some vector monotone norm $d(u,v) = d_1(u) + d_2(v)$.

The sense of this problem is to obtain consistent systems of constraints in (8) - (9) by means of the minimal correction of some groups of its right-hand sides $b_1$ and $c_1$.

It is easy to reduce this problem to that of finding the $S$ - optimal shifting vector for the monotone mapping $T$, as regarded in Section 3, if we denote

$$z = \begin{bmatrix} x_0 \\ x_1 \\ y_0 \\ y_1 \end{bmatrix},\quad T^0 z = \begin{bmatrix} B_0^{\mathrm{T}} y - c_0 \\ B_1^{\mathrm{T}} y - c_1 \\ b_0 - A_0 x \\ b_1 - A_1 x \end{bmatrix},\quad Tz = \begin{cases} T^0 z + N^+ z, & \text{for } x \ge 0,\ y \ge 0; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Here $N^+ : R^{n+m} \to 2^{R^{n+m}}$ is the mapping generated by normal cones to the non-negative ortant of $R^{n+m}$,

$$\lambda = \begin{bmatrix} p \\ u \\ q \\ v \end{bmatrix},\quad R = \{\ \lambda = \begin{bmatrix} p \\ u \\ q \\ v \end{bmatrix} : p = 0,\ q = 0\ \},\quad S\lambda = \begin{bmatrix} 0 \\ \partial d_1(u) \\ 0 \\ \partial d_2(v) \end{bmatrix}.$$

Using the well-known inversability of subdifferential mappings of conjugate convex functons [8] we can reformulate the auxiliary problem of finding the root of the mapping $T^{\alpha}$ as the following dual pair of LP problems

$$\max\{ \langle c, x \rangle - rd_2((A_1x - b_1)^+) : A_0x \le b_0,\ d_1^*(x_1) \le r, \quad x^{\mathrm{T}} = [\ x_0^{\mathrm{T}}, x_1^{\mathrm{T}}\ ] \ge 0 \}, \tag{10}$$

$$\min\{ \langle b, y \rangle + rd_1((c_1 - B_1^{\mathrm{T}}y)^+) : B_0^{\mathrm{T}}y \ge c_0,\ d_2^*(y_1) \le r, \quad y^{\mathrm{T}} = [\ y_0^{\mathrm{T}}, y_1^{\mathrm{T}}\ ] \ge 0 \}, \tag{11}$$

where $\lambda^+ = \max(0, \lambda)$ for the number $\lambda$, $p^+ = [\ p_1^+, \dots, p_l^+$ for the vector $p = [\ p_1, \dots, p_l\ ]$; $r \quad 1/\alpha > 0$, $d_1^*$ and $d_2^*$ are conjugate norms to $d_1$ and $d_2$, respectively. Note that problems (10)-(11) was considered in [4] where the dual relations for improper LP problems were established.

Let $M_r$ be the optimal set for the problem (10) and $M_r^*$ be the optimal set for problem (11). In [4] there it was established that if $\{ A_0x \le b_0, d_1^*(x_1) < r, x \ge 0 \ne \varnothing, M_r \ne \varnothing$, or $\{ B_0^{\mathrm{T}}y \ge c_0, d_2^*(y_1) < r, y \ge 0 \} \ne \varnothing, M_r^* \ne \varnothing$, then problems (10)-(11) are solvable and their optimal values are equal to each other. Moreover, if $x_r \in M_r$, $y_r \in M_r^*$, then $x_r, y_r$ are optimal vectors of the LP problems.

$$L_0(u_r, v_r) = \max\{ \langle c_0, x_0 \rangle + \langle c_1 - u_r, x_1 \rangle : A_0x \le b_0,\ A_1x \le b_1 + v_r, \quad x^{\mathrm{T}} = [\ x_0^{\mathrm{T}}, x_1^{\mathrm{T}}\ ] \ge 0 \},$$

$$L^0(u_r, v_r) = \min\{ \langle b_0, y_0 \rangle + \langle b_1 - v_r, y_1 \rangle : B_0^{\mathrm{T}}y \ge c_0,\ b_1^{\mathrm{T}}y \ge c_1 - u_r, \quad y^{\mathrm{T}} = [\ y_0^{\mathrm{T}}, y_1^{\mathrm{T}}\ ] \ge 0 \},$$

where $u_r = (c_1 - B_1^{\mathrm{T}}y_r)^+ \in W_1$, $v_r = (A_1x_r - b_1)^+ \in W_2$.

Note that for the maximal monotonocity of $T^{\alpha}$ in this case it is sufficient to demand (instead of conditions (b), (b')) the relation $W \ne \varnothing$. Then due to the results of Sections 2-3 we have

**Theorem 12.** If $W = W_1 \times W_2 \ne \varnothing$ then

$$\lim_{r \to \infty} \max_{x \in M_r} \min_{v \in W_2^0} \| (A_1x - b_1)^+ - v \| = 0,$$

$$\lim_{r \to \infty} \max_{y \in M_r^*} \min_{u \in W_1^0} \| (c_1 - B_1^{\mathrm{T}}y)^+ - u \| = 0,$$

where $W_1^0 \times W_2^0$ is the set of saddle points for the equivalent class of convex-concave closed functions $L_0(u,v)$, $L^0(u,v)$ relative to the region $Arg \min_{u \in W_1} d_1(u) \times Arg \min_{v \in W_2} d_2(v)$.

Let us now consider the convex programming (CP) problem in the form

$$\sup\{ f_0(x) : f_j(x) \le 0, \quad j = 1,\dots,m, \quad x \in R^n \}, \tag{12}$$

and the dual one

$$\inf_{y \le 0} \sup_x L(x, y), \tag{13}$$

where the functions $-f_0(x), f_1(x),\dots,f_m(x)$ are assumed to be convex and differentiable everywhere, $L(x,y) = f_0(x) + \langle y, f(x) \rangle$, $y = (y_1,\dots,y_m)^{\mathrm{T}}$, $f(x) = (f_1(x),\dots,f_m(x))^{\mathrm{T}}$.

If the CP problem is improper we are interested in finding

$$(u_0, v_0) = \arg \min_{(u,v) \in cl\, W'} \|(u,v)\|,$$

where $W' = \{ (u,v) \in R^n \times R^m :$ functions $L(x,y) - < u, x > - \langle v, y \rangle$ has a saddle point in the region $R^n \times R^m_- \}$, $R^m_- = \{ y \in R^m : y \le 0 \}$.

In the general scheme of Section 3 set $z = (x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$, $\lambda = (u^{\mathrm{T}}, v^{\mathrm{T}})^{\mathrm{T}}$, $Tz = T_0 z + Nz$, $R = R^{n+m}$, $S\lambda = (u, v)$, where $N : R^{n+m} \to 2^{R^{n+m}}$ is the mapping generated by normal cones to the product $R^n \times R^m_-$,

$$T_0 z = \begin{cases} (-\nabla_x L(x,y), \ \nabla_y L(x,y)^{\mathrm{T}})^{\mathrm{T}} & \text{if } y \le 0, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Then the problem of optimal correction of the CP problems (12)-(13) may be reduced to that of finding the $S$-optimal shifting vector for the monotone mapping $T$. The auxiliary problem of finding the root of the mapping $T^{\alpha}$ may be reformulated as follows: find

$$\max\{ f_0(x) - (\alpha/2)\|x\|^2 - 1/(2\alpha) \sum_{j=1}^{m} [\, f_j(x)^+ \,]^2 : x \in R^n \}, \tag{14}$$

$$\min\{ f_0(x) + \langle y, f(x) \rangle - (\alpha/2)(\|x\|^2 - \|y\|^2) : \alpha x = \\ = \nabla f_0(x) + \langle y, \nabla f(x) \rangle, \ y \in R^m_- \}. \tag{15}$$

By $M_\alpha$ and $M^*_\alpha$ let us denote the optimal set of problem (14) and the projection of the optimal set of problem (15) on the subspace of the variables $y$, respectively. From the general results of sections 2-3 it follows

**Theorem 13**. Under the above assumptions the optimal values of problems (14), (15) are the same and the sets $M_\alpha$, $M^*_\alpha$ are non-empty. If $x_\alpha \in M_\alpha$, $y_\alpha \in M^*_\alpha$, then $(\alpha x_\alpha^{\mathrm{T}}, f^+(x_\alpha)^{\mathrm{T}})^{\mathrm{T}} \in W'$, $x_\alpha$ is the optimal vector of the problem

$$\sup\{ f_0(x) - \langle \alpha x_\alpha, x \rangle : f_j(x) \le f_j^+(x_\alpha), \quad j = 1,\dots,m, \quad x \in R^n \}$$

$y_\alpha$ is the optimal vector of the dual problem

$$\inf_{y\le 0}\sup_{x}\,(L(x,y)-\langle\, \alpha x_\alpha, x \,\rangle-\langle\, f^{+}(x_\alpha), y \,\rangle),$$

and $\lim_{\alpha\to+0}(\alpha x_\alpha, f^{+}(x_\alpha))=(u_0, v_0)$.

Problem (14), as the one replacing (12), was regarded in [10-11] where it was regular (therefore proper). Dual relations between problems (14), (15) were considered in [12]. Relations between problems (12), (13) and (14), (15), in the case when they are improper LP problems of the first kind (i.e. only the primal system of constraints is inconsistent, not the dual), have also been established earlier.

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