Visualization of fractional calculus

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Abstract

The visualizations of definite integral for area problems are considered. The functions defined by definite integral, the convolution and the composite of integrals are visualized and analyzed. The definitions of fractional integral and derivative are introduced and visualized by using GeoGebra Packages. The presented visualizations were used in teaching calculus for the students, mayor physics, at the Faculty of Science, University of Novi Sad. The students were tested after the presentations and they show good results.

Key words: fractional calculus, visualization, integral.

MSC: 26A33, 44A10, 34A08, 97U70.

1. Introduction

Nowadays many books, surveys, journals and papers address problems involving fractional calculus, i.e., fractional derivatives and integrals with appropriate initial or boundary conditions. The reason for this lies in the fact that the fractional calculus got wide use in numerous physical and other applications, as viscoelastic materials, fluid flow, diffusive transport, electrical networks, electromagnetic theory, probability and others.

Fractional calculus can be considered as the generalization (extension) of the well known differential and integral calculus of noninteger order. The origins of the fractional calculus go back all the way to the end of the 17th century. L'Hospital asked in Leibniz about the sense of the notation:

$$\frac{D^n}{Dx^n} \quad \text{if} \quad n = 1/2$$

i.e., the fractional derivative of order $1/2.$ Then Leibniz's answer was: "An apparent paradox, from which one day useful consequences will be drawn."

In the continuation of the paper we shall visually introduced fractional integral, $J^a,$ of order $a > 0,$ in the sense Riemann-Liouville and fractional derivative, $D^a,$ of order $a > 0,$ in Caputo sense (rather than the Riemann-Liouville one), because it is more suitable for application to the problems with initial and boundary conditions.

2. Visualization of definite integral

The application of definite integral for determining the corresponding area between the graph of the considered function and x axes is well known. Interactive presentation of the area is presented on Figure 1. Click on Figure 1 and you will get the interactive file. By changing the values of lower and upper limits, with the slider, the values of integral $I := \int_a^b f(x)dx,$ for $f(x) = -x(x-4),$ and the values of corresponding area P are changing.

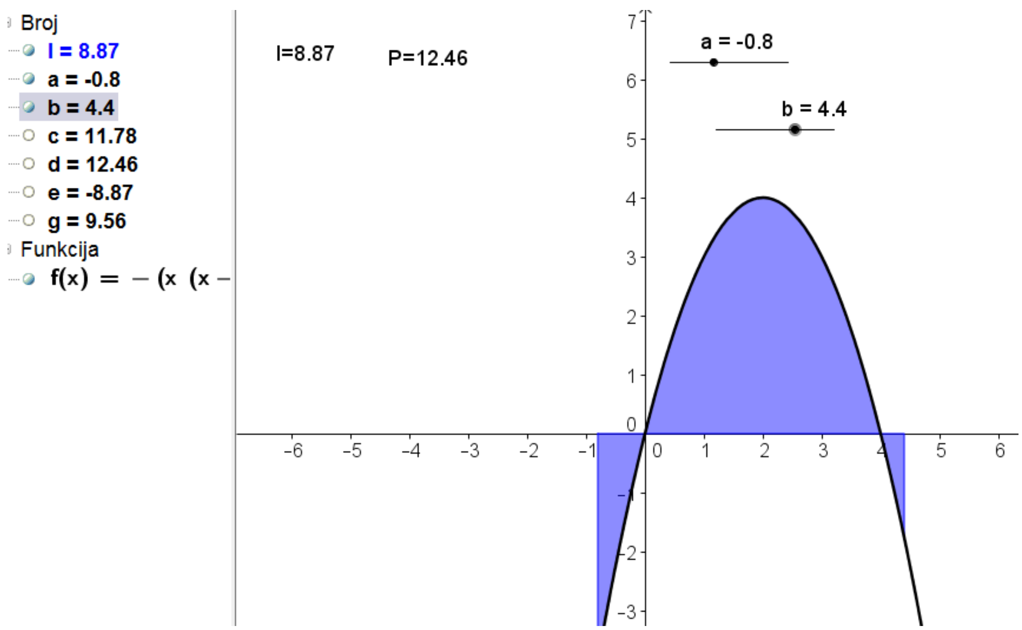

Figure 1.

3. Visualization of the function given by integral

The function F, defined by definite integral:

$$F(t) = \int_a^t f(x)dx\,,$$

can be visualized and analyzed in the sense of previous considerations of definite integral and its applications. The properties of the function F is conditioned by the function f, because it is its primitive function. As an example, we consider the function $f(x) = \frac{1}{x}, \quad x > 0,$ and the function

$$F(t)=\int_1^t \frac{1}{x}dx,$$

for $t>1$, and $t<1$. By the click on Figure 2 the interactive applications can be obtained. The function $\ln t$ is defined for $t>1$, and $t<1$, and the points A (on the left hand side, for $t>1$,) and C (on the right hand side $t<1$) belong to their graphs, but their $y-$coordinates are the values of the function F. Coordinates of these points can be followed on Figure 2, by using sliders t, ad t_1, respectively.

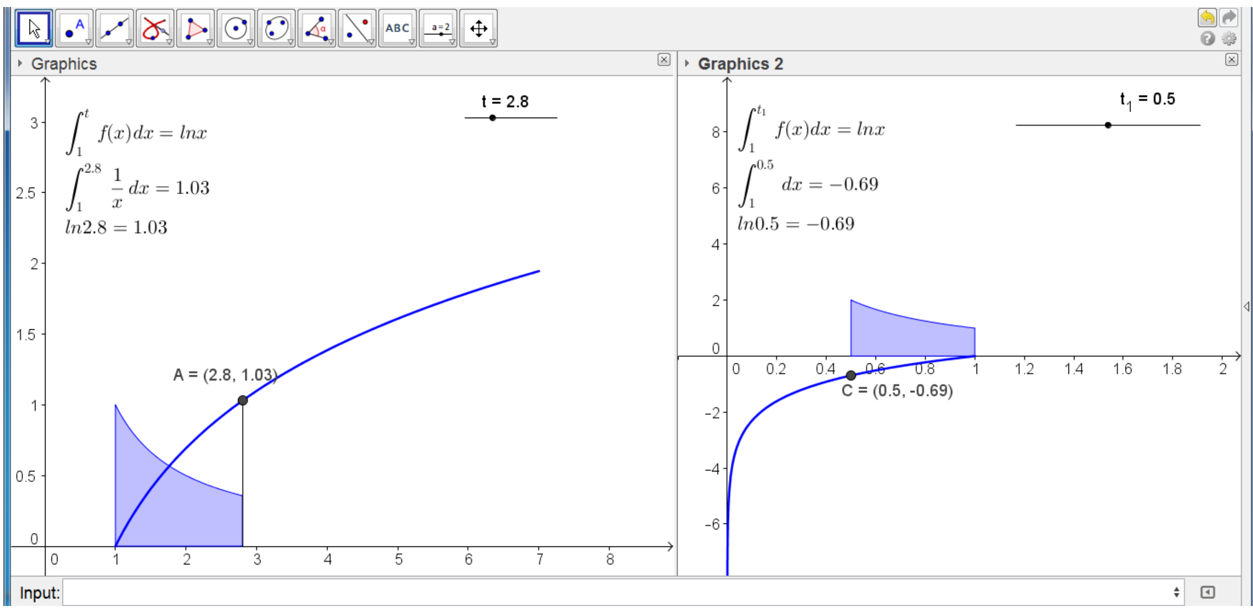

Figure 2.

4. Visualization of the convolution

The composite of integrals $F(F(t))$ of the function $f(x)=x$ is evaluated as:

$$F(F(t))=\int_0^t\left(\int_0^\tau xdx\right)d\tau=\int_0^t\left(\frac{\tau^2}{2}\right)d\tau=\frac{t^3}{3!}.$$

The same result can be obtained by using the formula called convolution of two same functions $f(x)=x$:

$$\int_0^t (t-\tau)\tau d\tau = t\int_0^t \tau d\tau-\int_0^t \tau^2 d\tau=\frac{t^3}{2}-\frac{t^3}{3}=\frac{t^3}{3!}.$$

In general the convolution of two functions f and g is given by:

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau.$$

On Figure 3 the convolution of the functions $f(x)=x$ and $g(x)=e^x$ is visualized. Let us remark that the convolution is the function given by the corresponding integral. Therefore the visualization is presented analogously as in previous cases. The point A belongs to graph of the function representing convolution and its $y-$coordinate is the same as the area under the graph of the function $p(x)=e^{t-x}x.$ By changing of the value t, the values of corresponding area and convolutions are changed. By the click on Figure 3 the interactive applications can be obtained.

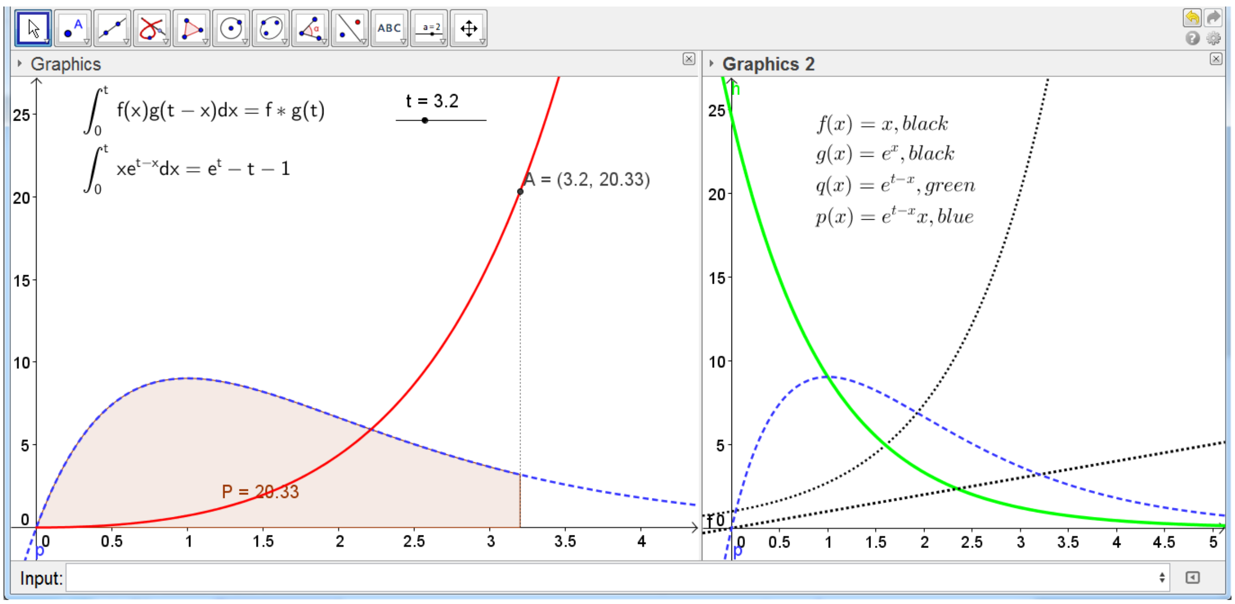

Figure 3.

5. Visualization of the fractional integral

The fractional integral, J^a, of order $a>0$, in the sense Riemann-Liouville, is defined as the following convolution:

$$J^a f(t) = \frac{1}{\Gamma(a)} \int_0^t (t-\tau)^{a-1} f(\tau) d\tau.$$

We put

$$J^0 f(t) = f(t).$$

The main properties of fractional integral operators J^a of order $a > 0$, are:

$$J^a J^b f(t) = J^{a+b} f(t) \quad (a, b > 0), \qquad J^a J^b f(t) = J^b J^a f(t);$$

$$J^a t^c = \frac{\Gamma(c+1)}{\Gamma(a+c+1)} t^{a+c}.$$

On Figure 4, the fractional integral of the function $f(x) = x$, is visualized. By changing a with the slider, from 0 to 2, the graph of fractional integral is changing. For $a = 1$, the fractional integral is primitive function for the function $f(x) = x$, i.e., $h(x) = \frac{x^2}{2}$, and for $a = 2$, it coincide with the function $g(x) = \frac{x^3}{6}$. By the click on Figure 4, the interactive applications can be obtained.

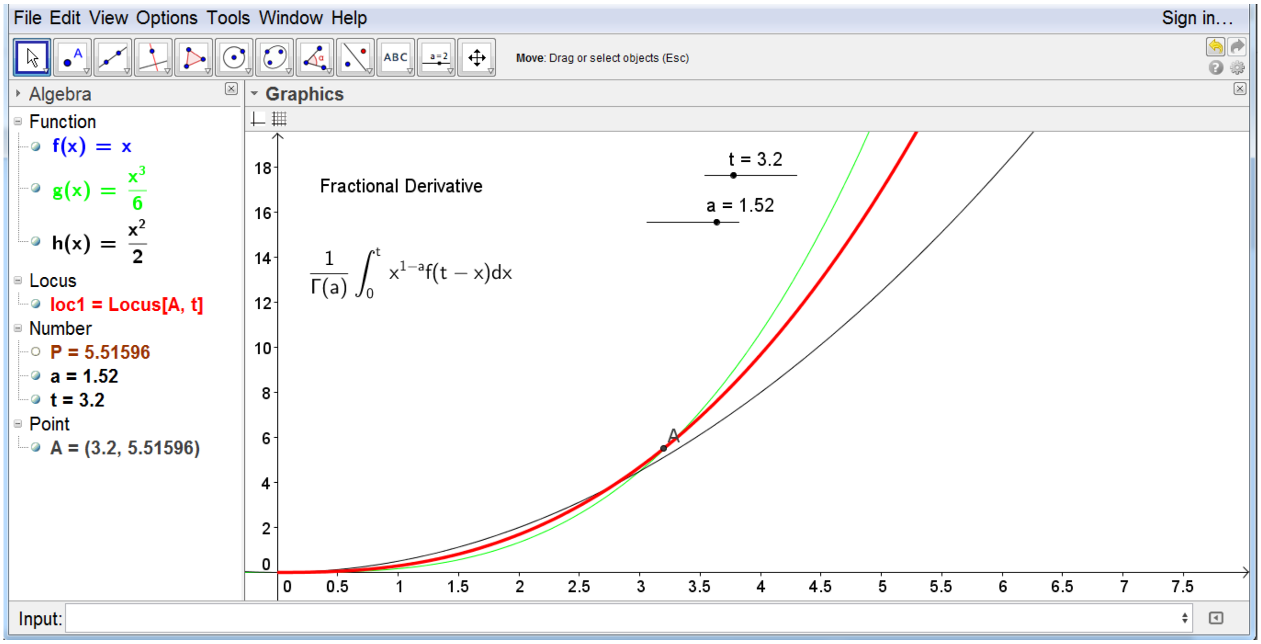

Figure 4.

6. Visualization of fractional derivative

In this paper we shall consider the fractional derivatives in Caputo sense (rather than the Riemann-Liouville one), because it is more suitable for application to the problems with initial and boundary conditions. The Caputo fractional derivative of order $\alpha > 0,$ is defined by:

$$D^a f(t) = \begin{cases} \dfrac{1}{\Gamma(m-a)} \int_0^t (t-\tau)^{m-a-1} f^{(m)}(\tau) d\tau, & m-1 < a < m, \\ \frac{d^m f(t)}{dt^m}, & a = m \end{cases}$$

$(m \in N, \quad t > 0).$

It is important to note that the following relation between J^a and D^a from holds for $m-1 < a < m,$

$$D^a J^a f(t) = f(t);$$
$$J^a D^a f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}$$

On Figure 5, the fractional derivative of the function $f(x) = x,$ is visualized. By changing a, by using slider, from 0 to 1, the graph of fractional derivative is changing from the graph of function $f(x) = x,$ for a=0, to graph of constant for $a = 1$. By the click on Figure 5, the interactive applications can be obtained.

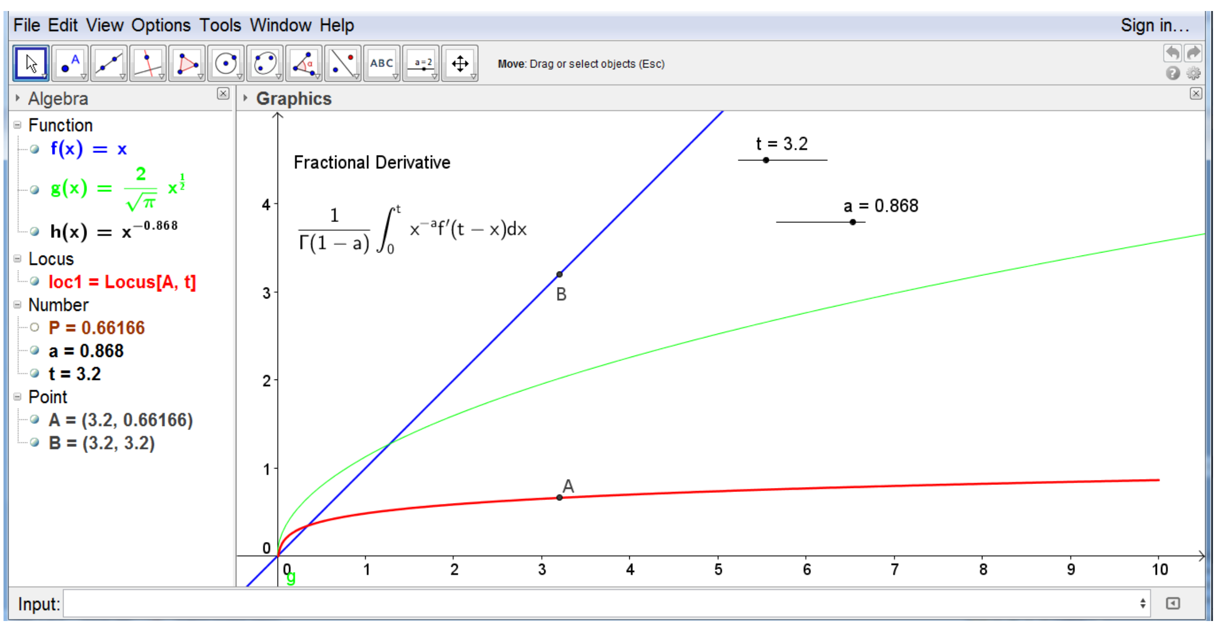

Figure 6.

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