THE TEACHING OF MATHEMATICS
2002, Vol. V, 1, pp. 47–55

A BROADER WAY THROUGH THEMAS OF ELEMENTARY SCHOOL MATHEMATICS, IV

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Abstract. Continuing with the didactical analysis of main topics of school arithmetic, first we cover the technique of using place holders as an indispensable device for realization of aims and objectives present in the contemporary process of learning and teaching arithmetic. To ensure the enforcement of correct responses and clear goals of calculation, we insist on the function of these graphical signs as being holders of blank spaces (and it means that a numerical value should never be assigned to such signs).

Though an ongoing idea of "early algebra" has not yet got its clear contours, it puts forward a more extensive use of numerical expressions in the elaboration of arithmetical topics. In this frame, we consider meaning-based equating of different numerical expressions and establishing of the main rules of arithmetic on that same base. The manner of expressing these rules is of three kinds: procedural (example by example), rhetoric (in words) and symbolic (using letters in the role of variables). The order in which these three manners have just been mentioned is the natural one to be followed in school practice. The procedural manner is also one which is inevitably connected with the use of place holder technique and we treat it here with scrupulous attention to detail.

AMS Subject Classification: 00 A 35

Key words and phrases: Place holders, manners of expressing arithmetical rules.

9. Place holders—an acted-out arithmetic

A variety of graphical signs as number boxes, shorter or longer underlines, etc. are often used to execute some specific tasks in programmed learning. For instance, in arithmetic textbooks these signs are seen applied in the following form

$$3 + 4 = \square, \qquad 5 + \_\_ = 14, \qquad 7 \cdot \_\_ = 63, \qquad ,\text{etc.}$$

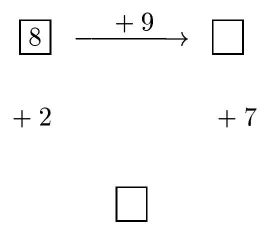
Previous generations of teachers used to express such assignments in words: "Find the sum 3+4", "Which number must be added to 5 to make 14", "How many sevens you must take to make 63" etc. When these assignments are expressed orally, a pupil has to remember the verbal information first, before giving an answer. Set down in the programmed form, such assignments keep the whole information which can be automatically read off, even without converting it in a sentence. Namely, as the school practice shows it, children learn easily what is required to be done, filling in the blank spaces indicated by these graphical signs. Called *place holders* (or space holders), these signs have an evident pedagogical justification, but they can also be used as an invaluable tool for many other purposes in arithmetic learning.

We select here a couple of examples related to the methods of adding up over ten and subtracting down below ten. When these methods have been learnt by visualization (and by the use of a proper piece of didactical apparatus), children are let to practise them accomplishing all intermediate steps. Thus, we have the examples as the following two

$$7+8=7+3+\underline{\quad}=\underline{\quad}+5=\underline{\quad}\,, \qquad 16-9=16-6-\underline{\quad}=\underline{\quad}-3=\underline{\quad}\,.$$

To respond correctly, children fill in writing the numbers which preserve these equalities. In the former example such numbers are: 5, 10, 15 and in the latter: 3, 10, 7.

Another instance of the use of place holders as a means of programmed learning are so called "learning machines". A simple example would be



with the numbers 10 and 17 to be filled in.

As we see it, in different places different numbers are written. Thus we see that place holders are devices used to hold blank spaces and it is wrong to assign them any numerical value. As an example of misunderstanding of their role, the solving of equations for a place holder is sometimes seen:

$$\square+18=35, \qquad \square=35-18, \qquad \square=17.$$

There exist many other examples of misuse of place holders, when the enforcement of correct responses is not ensured or the goal of calculation is not even remotely clear. The section "Atomisation" of Freudenthal's book [4] treats some wrong uses of place holders, including a specimen of such misuses. But our aim here is to demonstrate an inevitable use of place holders in elaboration of an active learning of arithmetic, fostering so both the understanding and the automatic performance of routine tasks.

9.1. Arithmetic ostracized. Grecian *logistica numerosa* was a school subject designed for children to learn to calculate and that art was also considered as being indispensable for everyday life. Nowadays, exactly the same may be said for school arithmetic. What has been changed progressively are the methods of elaboration. At the times of great innovators J. A. Komensky, J. H. Pestalozzi and their followers , learning process was shaped so as to start with sensory experience of a child and to lead to the creation of clear concepts. Accompanied with continuous improvements, that tradition has always been followed in European schools.

In the period of "New Maths", due to the idea to base teaching and learning of mathematics upon fundamental mathematical structures, sometimes the learning process went beyond the limit of logical and didactical acceptability, particularly with an overmathematization at the infra logical level. As the result of the counter-reaction to this trend (and, maybe, of a further compartmentalisation of educational institutions and of the involvement of general educationists and specialists of a rather narrow profile), a contemporary trend is visible, which pushes teaching and learning of elementary school mathematics to the other even more dangerous extreme.

As a result of this new trend, we see bits of arithmetic almost drowned into (often beautiful) illustrations which adorn contemporary school books. When functional, these illustrations are the right way to represent some mathematical structures embedded in reality and to let pupils be aware of them. Iconical representation or word problems carry the ingredients of meaning, and quoting Kant again, without them the conceptualisation would be empty. Of course, there still remains a long process of amounting which leads to the level of reflective intelligence. Hence, without a thought-provoking elaboration, such arithmetic would stay at its embryonic stage, left to care after itself.

For previous generations of teachers, the ability to calculate with numbers of big size was considered as a culmination of arithmetic teaching. Thanks to computers, this skill in doing long calculations has lost its practical value and it is no longer regarded as an essential task of learning. For the public and unfortunately, for specialists in education lacking the ability to analyse conceptually the content of arithmetic, these wonderful machines are taken as a substitute for people when any kind of calculation is concerned. Overlooking so developmental importance of learning arithmetic, such specialists not only banished calculation drills from school curriculum but the normal process of learning arithmetic itself.

9.2. Following a reasonable innovation. If nowadays children who learn arithmetic have got rid of the need to be too much trained to carry lengthy calculations with mechanical speed and accuracy, then it is normal to ask which new tasks have to be undertaken.

As a result of a long study of the problem of linking arithmetic to algebra and so of overcoming a previously existing semantic jump in transition from arithmetic to algebra, an idea of "early algebra" has arisen as a set of specific topics intentionally involved in the content of arithmetic for the first grades. In a series of absorbing papers, C. Kieran considers the ways of facilitation of the transition from arithmetical to algebraic thinking, concluding firmly that "algebra builds upon the understandings and skills that have been developing in arithmetic". (See the papers: Kieran, C., *Concepts associated with the equality symbol*, Educ. Studies in Math. **12**, pp. 317–326, 1981; Kieran, C., *The learning and teaching of school algebra*, In: D. A. Grows (Eds.), *Handbook of Research on Mathematics Teaching and Learning*, pp. 390–419, New York, Macmillan; Kieran, C., *Looking at the role of technology in facilitating the transition from arithmetic to algebraic thinking through the lens of a model of algebraic activity*, Proceedings of the 12th ICMI Study Conference, pp. 713–720, Melbourne, 2001).

If the idea of gradual building of number blocks has been something almost forgotten that should be renovated, an extensive use of arithmetic expressions and transformational activities of their relating based upon the selected rules of arithmetic is seen as a new and important task of arithmetic teaching and learning. Thus we aim to gather together dispersed patches found in the current textbooks and research papers, forming so a didactical analysis of this matter with a necessary attention to detail.

9.3. Numerical expressions as a new topic in arithmetic teaching. Teaching of arithmetic was (and often still is) too much oriented to calculation. Children learnt (and often still do) to look at the expressions as, for example, these ones are:

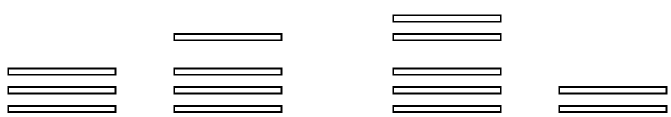
$$7+4, \qquad 6\cdot 8, \quad 63:7, \quad \text{etc.}$$

as being stimuli for calculation. Inspecting a variety of existing "Standards" or other systems of aims and objectives, one easily see that it has not been changed much. And it will stay so until a radically new access to arithmetic has been set up. The following three expressions:

$$3+4, \qquad 5+2, \qquad 7$$

denote one and the same number. What makes them different is a different kind of noise present in comprehension of corresponding addition schemes. Namely, in the first case: two groups of 3 and 4 objects have been seen, in the second: two groups of 5 and 2 objects and in the third: one group of 7 objects.

At the very beginning when children start to write and recognize the addition sign, they should be led to compose sums, reacting to the corresponding schemes. After a number of exercises done together in an interplay between the teacher and his/her class, the children should be assigned a number of exercises combined of meaning conveyors and place holders. For instance,



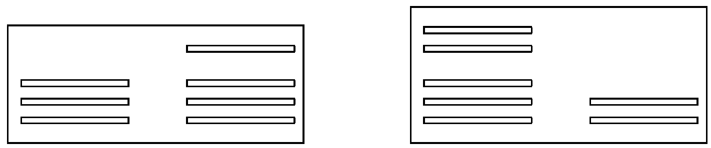
$$\_\_\_ + \_\_\_ \qquad\qquad \_\_\_ + \_\_\_$$

without any suggestion of "how much it gives".

Establishing the practice to equate any two expressions denoting one and the same number and as a subsequent step, children count the sticks and they write the equalities

$$3+4=7, \qquad 5+2=7, \qquad \text{etc.}$$

If the sticks are moved from one place to another and no of them is removed, their total number stays the same. In such situations, we recognize the Piagetian experimental manipulations with the Cantor Principle of Invariance of Number put into action. This activity really deserves a careful attention. Let us consider an example in which, the used frames mean no removing exists. Looking at the following pictures



$$\_\_\_ + \_\_\_ = \_\_\_ + \_\_\_$$

children are supposed to form equalities between numerical expressions.

Such meaning-based transformational activities not only lead a child to the acquirement of invariance principle but they also help him/her to get used to the employing of numerical expressions. Furthermore, rules (laws) of arithmetic are based upon exactly the same kind of activities. Now we turn our consideration to these rules.

9.4. Three manners of expressing arithmetic rules and procedures. The term "procedural knowledge" is usually taken with a negative connotation. And this connotation stays whenever a procedure is carried out without necessary understanding. But if a piece of knowledge is based upon a meaningful ground and expressed example by example, then this manner of expressing we will call *procedural*, without packing any negative connotation. As a classic example, let us remind that prior to Vieta, procedures of solving equations were exposed by using specific examples of equations having numerical coefficients. Once such an example was done, it was expected all other similar examples should be. Procedural manner of expressing is also when children transform numerical expressions, one into another, without knowing to explain rhetorically or symbolically the applied arithmetical rules. Instead of it, they have a feeling that the equality sign preserves in each specific case.

We all know that school geometry is expressed in words rather than in symbols. The same was the case with algebra during a long historical period (lasting until 17th century). Such algebra is called rhetoric and that is why we call a manner of expressing in words *rhetoric*. For instance, we express commutative law rhetorically when we say: interchanging summands the sum preserves its value.

The manner of expressing by means of mathematical symbols is called symbolic. Thus, for instance, when we write the equality: $m + n = n + m$, we express symbolically the commutative law.

The procedural manner is the first and the most important way of expressing knowledge at early stages of arithmetic learning. Skill in operating on numerical expressions formed at early stage of arithmetic learning transcendence the facts of arithmetic and becomes an indispensable faculty for learning algebra.

The following addendum is written as a short review of the historical linkage between arithmetic and algebra.

Addendum 8.

Against the fact that Vieta's *logistica speciosa* had been applied more and more, it lacked a solid logical foundation until the first half of 19th century when G. Peacock tackled the problem of justifying the operations with literal expressions. He made a distinction between arithmetical algebra and symbolic algebra. The former embraced the relations between numerical expressions in which natural numbers figured and the latter dealt with such expressions when natural numbers are replaced by letters. Peacock's principle of the permanence of form meant that a true relation in arithmetical algebra converts to a true relation in symbolic algebra by replacement of the involved natural numbers by letters which could stand for each real (or complex) number.

Not each true relation in arithmetical algebra transfers that way into a true relation in symbolic algebra. As an example, let us take: $3 \cdot 8 = 24$, which transfers to: $a \cdot b = c$, obtaining so a relation which could not be considered as generally true. To protect his principle, Peacock added the phrase "when the symbols are general in form". He certainly was the first who anticipated the idea that algebra like geometry is a deductive science based on its own postulates. Following this idea, in the course of the 19th century, generally accepted laws of symbolic algebra were reduced to those which we call now the axioms of ordered field.

When we write $5+2$ instead of $3+4$, connecting these two expressions by the equality sign, we get a relation which is true being, for instance, meaning based, as it was done in 9.3. On the other hand, $3+4$ can be transformed into $5+2$ applying the rule: $a+b = (a+c)+(b-c)$. Rhetoric form of this rule is: a number may be added to one summand and subtracted from the other without altering the sum. Such a statement we call *a rule of arithmetic*, avoiding to call it a law of arithmetic. In a deductive exposition of arithmetic, the latter term is reserved for the fundamental laws (axioms). But in the school arithmetic, several rules should be selected without caring of their logical interdependence. The only criterion should be how much they are operative. In many textbooks on arithmetic, we can see two commutative laws (for addition and multiplication), two associative laws and the distributive law being singled out. Since they alone are in no way sufficient for transformational activities, they stay there as a mere decor.

9.5. Invariant manners of expressing arithmetic rules procedurally. In each of the following cases, we transform an expression evidently applying some rules of arithmetic but so obtained equalities have been written differently:

a) $3+4 = 5+2,\ \ 3+4 = (3+2)+(4-2)$,
b) $(3+4)+7 = 10+4,\ \ (3+4)+7 = (3+7)+4$,
c) $6 \cdot (5+2) = 30+12,\ \ 6 \cdot (5+2) = 6 \cdot 5 + 6 \cdot 2$,
d) $7 \cdot (10-1) = 7 \cdot 10 - 7,\ \ 7 \cdot (10-1) = 7 \cdot 10 - 7 \cdot 1$.

Replacing the involved natural numbers by letters, in none of these cases the first equality yields a rule of arithmetic and in each of them the second one does. For example, in the case under d), we apply the rule of multiplication of a difference by a number and by the mentioned replacing , we get

$$a \cdot (b - c) = a \cdot b - a, \quad a \cdot (b - c) = a \cdot b - a \cdot c.$$

The first of these two relations is not a rule of arithmetic and the second is. The latter means that the second equality holds true no matter which particular (natural) numbers we substitute for a, b and c. When a numerical relation is expressed in such a form that it yields a rule of arithmetic by replacement of the involved numbers by letters, we call that manner of expressing *invariant*.

Now the right question is how we recognize invariant expressing, staying within the "arithmetical algebra", that is having a relation which involves specific natural numbers. The answer is very simple: such a relation holds true when the involved numbers are replaced by any others. For instance, these two relations

$$7 \cdot (10 - 1) = 7 \cdot 10 - 7 \cdot 1, \quad 7 \cdot (10 - 1) = 7 \cdot 10 - 7,$$

are true, but replacing the numbers 7, 10 and 1 by any other three, say, 8, 9 and 2 the first of so obtained relations

$$8 \cdot (9 - 2) = 8 \cdot 9 - 8 \cdot 2, \quad 8 \cdot (9 - 2) = 8 \cdot 9 - 8$$

stays true while the second one is false. The first manner of expressing the rule of multiplication of difference is invariant, while the second one is not.

Expressing rules invariantly, children not only become more aware of these rules, but due to that practice, they easier accept rhetoric, and one day, symbolic expression of such rules.

It would be too much to require of a child to express these rules invariantly, without giving him/her a help. Instead of it , a number of programmed exercises should be given. For instance:

Fill in the missing numbers

1. a) $3 + 8 + 2 = 8 + \underline{\quad} + 3$, b) $17 + 15 + 3 = \underline{\quad} + 3 + 15$, etc.
2. a) $8 \cdot (5 + 2) = 8 \cdot \underline{\quad} + \underline{\quad} \cdot 2$, b) $7 \cdot (10 + 3) = 7 \cdot \underline{\quad} + 7 \cdot \underline{\quad}$, etc.

Only a lack of appreciation of this didactical task leads to a wrong consideration of such exercises as being formal. On the other hand, when these rules are taken as a ground for quicker calculation, children perform operations, leaving out the intermediate steps. For example, they do like this:

$$17 + 15 + 3 = 20 + 15 = 35, \quad 7 \cdot (10 + 3) = 70 + 21 = 91, \quad \text{etc.}$$

Without the use of place holders, we can only imagine how the replacing rigmarole of words would be fatal for clear learning.

9.6. Meaning-based acceptance of arithmetical rules. Common faults seen in textbooks from several countries are the attempts to establish some arithmetical rules on the basis of simple verifications in a number of cases. For instance:

a) $3+4=7,\ \ 4+3=7,\ \ 12+5=17,\ \ 5+12=17,$ etc.

after which the rule of interchange of summands is stated.

b) $3+8+7=11+7=18,\ \ 3+7+8=10+8=18,$
$4+9+6=13+6=19,\ \ 6+4+9=10+9=19,$ etc.

after which the rule of association of summands is stated.

c) $3\cdot 5=5+5+5=15,\ \ 5\cdot 3=3+3+3+3+3=15,$ etc.

with a meaningless attempt to derive this property of multiplication by reduction to addition.

In the school arithmetic such rules should be established on the basis of meaning . As an example, we chose a property of addition which is associated with the number block 1–100, leaving now the other rules to be treated within the context they naturally belong to.

Following this way of elaboration of arithmetic, we must be aware that the new piece of symbolic apparatus is more complex than traditionally and we must think of all difficulties an untrained child could meet. First, at this stage, the brackets are also in the role of place holders. Since majority of exercises should be given in the programmed form, brackets are used to hold expressions and they stand as a command "do first what is held". The best way to tell it to the children is doing of a number of exercises as this one is:

Write the required sums

	First summand	Second summand	Sum
a)	$3+4$	6	$(\_\_\_+\_\_\_)+\_\_\_$
b)	8	$5-3$	$\_\_\_+(\_\_\_-\_\_\_)$
c)	$8-2$	$4-1$	$(\_\_\_-\_\_\_)+(\_\_\_-\_\_\_)$

Rewrite each of these sums and do them.

a) $(\_\_\_+\_\_\_)+\_\_\_=$
b) $\_\_\_+(\_\_\_-\_\_\_)=$
c) $(\_\_\_-\_\_\_)+(\_\_\_-\_\_\_)=$

etc.

Besides the place holders, an appropriate text should also be used to induce children to equate expressions without any calculation. Let us consider an example of that type.

57

| 57 − 19 | 19 |

(57 − 19) + 19

There are 57 marbles in the two boxes. In the second one there are 19 marbles. Then, in the first one there are: $\_\_\_-\_\_\_$. In the first box there are: $57-19$ marbles and in the second 19.

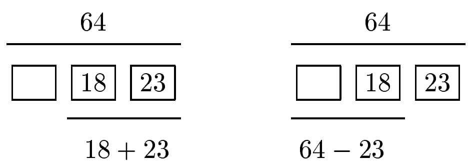
Altogether, it is: $(\underline{\quad} - \underline{\quad}) + \underline{\quad}$ marbles and that number we know to be 57. Equating, we get

$$(\underline{\quad} - 19) + 19 = \underline{\quad}.$$

(As it goes without saying, children are supposed to fill in the missing numbers).

Several similar equalities may be established and their form will be invariant. By replacement of specific numbers with letters, the true algebraic relations are obtained. In the case of example that we have just considered, such a relation is: $(a-b)+b=a$. But this relation (or any similar) is not taken to serve as an arithmetic rule because it has a very limited range of application. On the contrary, in the example that follows we establish and state a rule.

There are 64 marbles in the three boxes, in the second 18 and in the third 23. In the last two boxes there are $\underline{\quad} + \underline{\quad}$ marbles and in the first one: $64 - (\underline{\quad} + \underline{\quad})$. In the first two boxes there are: $64 - \underline{\quad}$ marbles and in the first one: $(64 - \underline{\quad}) - \underline{\quad}$.

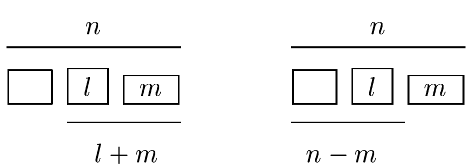


We have written the number of marbles in the first box in two different ways: $64 - (\underline{\quad} + \underline{\quad})$ and $(64 - \underline{\quad}) - \underline{\quad}$. Equating, we get

$$64 - (18+23) = (64-23) - 18.$$

After a number of similar examples have been done, the *rule of subtraction of sums* can be stated: a sum is subtracted from a number subtracting each summand.

Let us remark that in an analogous way the *rule of subtraction of differences* may be established and both of these rules have their right place within the block of numbers up to 100. The labelled boxes are just convenient to "kill" any unnecessary noise. Of course, at this stage, all exercises of this kind should be given in a carefully prepared programmed form. One day (fourth or fifth year of arithmetic learning), when it becomes ripe such rules would be established in full generality using letters but following the same procedure based on meaning. For example, using the same boxes



but then labelled by letters, the symbolic form

$$n - (l+m) = (n-m) - l$$

of that rule will be established.

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