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GLOBAL COMPACT ATTRACTORS AND COMPLETE BOUNDED TRAJECTORIES FOR COMPRESSIBLE MAGNETOHYDRODYNAMIC SYSTEM OF EQUATIONS

Jan Muhammad

ABSTRACT. In this article, we investigate the global behavior of magnetohydrodynamic (MHD) fluid's weak solutions in three-dimensional bounded domain with a compact Lipschitz boundary driven by arbitrary forces. We show that global compact attractors exist under specific limitations on the adiabatic constant γ.

1. Introduction

The study of compressible MHD fluids is crucial because the equations that describe the motion of conducting fluids in the presence of an electromagnetic field have numerous applications ranging from fluid metals to cosmic plasmas, as well as in areas such as geophysics, astrophysics, plasma physics and high-speed aerodynamics. MHD fluids are also related to plenty of engineering challenges, including sustained plasma confined in controlled thermonuclear fusion reaction, liquid-metal cooling processes in nuclear nuclear power plants, magnetohydrodynamic energy production, electro-magnetic manufacturing of metals and the plasma stimulants for ions thrusters as well. In fact, MHD equations are a combination of fluid dynamics' Navier–Stokes equations as well as electromagnetism' Maxwell equations. Here, we consider the compressible MHD fluid to study the problem of the global existence of weak solutions with bounded domain by considering the system of equations, such as

$$
\begin{cases}
\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \\
\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{T} + \nabla p(\rho) = \operatorname{rot} \mathcal{H} \times \mathcal{H} + \rho F, \\
\mathcal{H}_t - \operatorname{rot}(\mathbf{v} \times \mathcal{H}) = \upsilon \Delta \mathcal{H}, \quad \operatorname{div} \mathcal{H} = 0, \\
\mathbb{T}(\nabla \mathbf{v}) = \mu(\nabla \mathbf{v} + \nabla^T \mathbf{v} - \frac{2}{3} \operatorname{div} \mathbf{v} \mathbb{I}) + \daleth \operatorname{div} \mathbf{v} \mathbb{I}
\end{cases} \tag{1.1}
$$

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where $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3)$ symbolizes the velocity vector while the pressure is denoted by $p(\rho) = a\rho^\gamma$, such that γ being the adiabatic constant and $\mathcal{H} = (\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3)$ representing the magnetic field. Here the operators used such as ∇ and div act in term of the variable $x \in \mathbf{R}^3$. Here, we consider the isentropic fluid by taking the letters $\daleth, \mu, \gamma$ as constants that satisfy the relation

$$\mu > 0, \quad \daleth \geqslant 0, \quad \gamma > \frac{3}{2}, \quad a > 0, \quad \text{with} \quad \operatorname{div} \mathbb{T} = \mu \Delta \mathbf{v} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{v},$$

where $\daleth$ is the viscosity constant such that $\daleth + \mu \geqslant 0$. The initial and boundary data are specified by

$$\rho(0, x) = \rho_0 \geqslant 0, \quad \rho\mathbf{v}(0, x) = m_0(x), \quad \mathcal{H}(0, x) = \mathcal{H}_0, \tag{1.2}$$

and

$$\mathcal{H}|_{\partial\Omega} = 0, \quad \mathbf{v}|_{\partial\Omega} = 0. \tag{1.3}$$

MHD equations demonstrate the flow of electric-conducting fluids within the existence of a magnetic field, that must take into account the complicated relationship among the fluid's velocity and the magnetic field. Before describing and proving our desirable results of this work, let us first review some relevant literature results. Incompressible MHD flow is determined by the Navier–Stokes equation along with Maxwell's equation. The mathematical analysis of incompressible MHD equations is investigated in detail in [1–3] and the references therein. With regard to its physical significance, diverse phenomena, complexity and challenges in mathematics, mathematicians and physicists have carried out numerous research on MHD such as; [4–11] and in the citation therein. In [12, 13] the authors presented favorable results regarding the weak solutions' existence with discontinuous large data for compressible MHD equations. It is noteworthy that Feireisl and Ducomet [8] investigated MHD compressible equations along with Poisson's equation with assuming that coefficient of viscosity depending on the magnetic field and the temperature where the pressure term ρ^γ, ($\gamma = \frac{5}{3}$) with large density behave like power law.

In addition, for compressible fluids the global existence behavior was investigated by Feireisl and his collaborators in [14], where they studied global behavior of weak solution in bounded absorbing sets under some additional restriction on the adiabatic constant $\gamma > \frac{5}{3}$. Similarly, they proved in [15] the asymptotic compactness result for global trajectories generated compressible fluid. Moreover, for compressible fluids, Feireisl [16] proved the results of weak solutions compactness result while the density does not require to be square integrable. In [17], Feireisl proved the qualitative behaviors of compressible flows such as long time behaviour, global compact attractors and stabilization. In [18], Wang described global behavior of weak solutions of a specific class known as nematic liquid crystals of compressible flows.

Moreover, Lions [19] and Novotny [20] made contributions and modifications to ease the restrictions to the theory of compressible viscous fluids. By considering an isentropic fluid, they proved a well-known result concerning weak solutions' global existence under an additional condition that $\gamma \geqslant \frac{3}{2}$ and $\gamma > \frac{9}{5}$ in 2D and 3D respectively. Feireisl [21] modified for $\gamma > \frac{3}{2}$ the result of Lions for global existence to three dimensions. Jiang and Zhang [22] investigated the result for weak solutions' global existence regarding the compressible fluids of isentropic type for $\gamma > 1$, under some additional hypothesis related to symmetries on the initial datum. We refer [23–31] for more results related to attractors and global existence of weak solutions of compressible fluids.

We study the behavior of such a fluid that is compressible as well as MHD in a bounded domain motivated by the importance of these types of fluids having numerous applications in engineering, geophysics and applied mathematics. In recent years, mathematicians have focused intensively on studying non-linear fluids, mostly from the perspective of differential equation theory. In view of their practical significance, for decades, MHD fluid problems have been the center of extensive multidisciplinary research. Therefore, in order to gain a comprehensive understanding and improve applications across different manufacturers, it is essential to examine the flow behavior of MHD fluids. Nevertheless, aside from a few straightforward special cases, many problems still remain as open challenges.

In this research work, we investigate the result of global behavior of MHD fluids by considering the weak solution of a compressible MHD fluid with a compact Lipschitz boundary in a bounded three-dimensional domain, by following the same approach as in [15] where the global existence of these weak solutions is proved in [12], without external force.

The remaining article is organized as follows: in Section 2, we define a weak solution of the problem (1.1)–(1.3) and state the main results in the form of Theorems 2.1–2.4. In Section 3, we prove Theorem 2.1 along with the necessary results required to prove the main result. In Section 4, the proof of Theorem 2.2 and some key results related to density are presented. In Section 5, we describe the effective viscous flux of the considered model, by following the same approach as in [15], with some modification as required for MHD fluids. Furthermore, Section 6 contains the results related to density and momenta compactness. Finally, Section 7, outlines the proof of Theorems 2.3–2.4.

2. Main Results

The current section aims to state the main result of this work and define the weak solution of the problem (1.1)–(1.3). Before we state our main result, it is important to note that our weak solution will satisfy the natural energy estimates. In terms of physics, a suitable weak solution must fulfill the conservation law of mass, energy and momentum in terms of distributions. Keeping these fundamental requirements in mind, we characterize our weak solution such as:

Definition 2.1. For $T > 0$, the functions $(\rho, \mathbf{v}, \mathcal{H})$ are said to be the globally finite energy weak solution of (1.1)–1.3 on the interval of time $O \subset \mathbf{R}$, such that

$$\begin{aligned} &\rho \in \big(L^\infty(O; L^\gamma(\Omega)) \cap \big(C(O; L^1(\Omega))\big), \\ &\sqrt{\varrho}\mathbf{v} \in L^\infty\big(O; L^2(\Omega)\big), \quad \mathbf{v} \in L^2\big(O; W_0^{1,2}(\Omega)\big), \\ &\mathcal{H} \in L^2\big(O; W_0^{1,2}(\Omega)\big), \quad \frac{|m_0|^2}{\rho_0} \in L^1(\Omega). \end{aligned} \tag{2.1}$$

Continuity equation $(1.1)_1$ holds in a distributive sense

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \quad \text{in} \quad \mathcal{D}'(O \times \mathbf{R}^3). \tag{2.2}$$

Total mass is invariable, such that

$$\int_\Omega \rho(t) dx = \mathfrak{m} \quad \forall t \in O. \tag{2.3}$$

The renormalized form of $(1.1)_1$ holds in $\mathcal{D}'(O \times \Omega)$. In addition, for any $\xi \in C^1(\mathbf{R})$,

$$(\xi(\rho))_t + \operatorname{div}[\xi(\rho)\mathbf{v}] + (\xi^{'}(\rho)\rho - \xi(\rho)) \operatorname{div} \mathbf{v} = 0, \tag{2.4}$$

where $(\rho, \mathbf{v})$ are assumed to zero outside the domain Ω.

Any weak solution provided in Definition 2.1 will be in the class

$$\rho \in C(O; L_w^\gamma(\Omega)), \quad \rho\mathbf{v} \in C\Big(O; L_w^{\frac{2\gamma}{\gamma+1}}(\Omega)\Big).$$

The functions ρ and $\rho\mathbf{v}$ satisfy

$$\operatorname{ess} \lim_{t \to 0^+} \int_\Omega (\rho, \rho\mathbf{v})\omega(x) dx = \int_\Omega (\rho_0, m_0)\omega \, dx,$$

for any $\omega \in C_0^\infty(\Omega)$.

Moreover, the weak solutions $(\rho, \mathbf{v}, \mathcal{H})$ are constructed to satisfy the integral form of energy inequality, such that

$$\frac{d}{dt}\mathcal{E}(t) + \int_\Omega (\mu|\nabla \mathbf{v}|^2 + (\mu + \lambda)|\operatorname{div} \mathbf{v}|^2 + \upsilon|\nabla \mathcal{H}|^2) dx \leqslant \int_\Omega \rho(t) F(t) \cdot \mathbf{v}(t) dx, \tag{2.5}$$

with

$$\mathcal{E}(t) = \int_\Omega \Big(\frac{1}{2}\rho|\mathbf{v}|^2(t) + \frac{1}{2}|\mathcal{H}|^2 + a\frac{\rho^\gamma(t)}{\gamma - 1}\Big) dx. \tag{2.6}$$

Next, it is assumed that

$$\rho \in L^2_{\text{loc}}(O; L^2(\Omega)). \tag{2.7}$$

We emphasize that every classical solution described in the problem (1.1)–(1.3) can be a weak solution and that any weak solution can solve the problem (1.1)–(1.3) with sufficient regularity in the classical sense. This is done to ensure that our defined weak solution given in Definition 2.1 is accurate. Next, the following theorem outlines the result of absorbing bounded sets.

THEOREM 2.1. *Assuming that $\Omega \subset \mathbf{R}^3$ is a bounded domain of Lipschitz boundary, let $(\rho_0, \mathbf{v}_0, \mathcal{H}_0)$ being the initial datum satisfying* (1.1) *and $\gamma > \frac{5}{3}$. Moreover, let a measurable bounded function*

$$F = (f^j(t, x)), j = 1, 2, 3 \tag{2.8}$$

satisfying the condition

$$\max\Big(\operatorname{ess} \sup_{x \in \Omega, t \in \mathbf{R}} |f^j(t, x)|\Big) \leqslant \mathcal{K}_1, \tag{2.9}$$

and $O = (l, \infty) \subset \mathbf{R}$ *with* $l > -\infty$. *Then, for* $\mathcal{E}_\infty$ *with given* $\mathcal{E}_0$ *the time* $T = T(\mathcal{E}_\infty, l, \mathfrak{m})$ *depending on mass* $\mathfrak{m}$, $\mathcal{K}_1$ *and* γ, *such that*

$$\mathcal{E}(\rho, \mathbf{v}, \mathcal{H})(t) \leqslant \mathcal{E}_\infty \quad \text{for a.e.} \quad t > T. \tag{2.10}$$

Furthermore, the weak solution $(\rho, \mathbf{v}, \mathcal{H})$ *satisfies*

$$\operatorname*{ess\,lim\,sup}_{t \to l} \mathcal{E}(\rho, \mathbf{v}, \mathcal{H})(t) \leqslant \mathcal{E}_0. \tag{2.11}$$

According to the aforementioned Theorem 2.1, the family of trajectories produced by the finite-energy weak solutions of (1.1)–(1.3) defined on interval O, is dissipative in the terms of Levinson, such that it possesses an absorbing bounded set in the energy "norm". In addition, take into consideration the term known as "short trajectory" as defined in [**32**], such as:

$$\begin{aligned}\tilde{U}^S[\mathcal{E}_0, \tilde{\mathcal{F}}, \tilde{M}](t_0, t) = \Big\{&(\rho(\hat{t}), (\rho_m \mathbf{v}_m)(\hat{t})), \hat{t} \in [0,1] \mid \rho(\hat{t}) = \rho(t + \hat{t}), \\ &(\rho\mathbf{v})(\hat{t}) = (\rho\mathbf{v})(t + \hat{t}), \mathcal{H}(\hat{t}) = \mathcal{H}(t + \hat{t}) \text{ with } (\rho, \mathbf{v}, \mathcal{H}) \\ &\text{be the weak solutions of (1.1)–(1.3) on interval } O, (t_0, t_0 + 1) \subset O \\ &\text{with } F \in \tilde{\mathcal{F}} \text{and } \operatorname*{ess\,lim\,sup}_{t \to t_0} \mathcal{E}(\rho, \mathbf{v}, \mathcal{H})(t) \leqslant \mathcal{E}_0, \int_\Omega \rho(t) dx \leqslant \mathfrak{m}\Big\}.\end{aligned} \tag{2.12}$$

Next, the theorem given below describes the finite energy weak solution trajectories' asymptotic behavior.

THEOREM 2.2. *Assume that* $\gamma > \frac{5}{3}$ *and bounded subset* $\tilde{\mathcal{F}} \in L^\infty(\Omega \times \mathbf{R})^3$ *such that*

$$(\rho_m(t_m + t, x), (\rho_m \mathbf{v}_m)(t_m + t, x), \mathcal{H}_m(t_m + t, x)) \in \tilde{U}^S[\mathcal{E}_0, \tilde{\mathcal{F}}, \tilde{M}](l, t), \quad (l \in \mathbf{R}),$$

for some $t_m \to \infty$. *Then, we can find a subsequence (without relabeling) such that*

$$\rho_m(t_m + t) \to \bar{\rho}(t) \quad \text{in} \quad L^\gamma(\Omega \times (0,1)) \quad \text{and in} \quad L^\zeta([0,1]; \Omega) \quad \text{for any} \quad \zeta \in [1, \gamma), \tag{2.13}$$

$$\mathcal{H}_m(t_m + t) \to \bar{\mathcal{H}}(t) \quad \text{in} \quad L^\zeta(\Omega), \tag{2.14}$$

$$\begin{aligned}(\rho_m \mathbf{v}_m)(t_m + t) \to (\bar{\rho}\bar{\mathbf{v}})(t) \quad &\text{in} \quad L^p(\Omega \times (0,1)) \cap C\big([0,1]; (L^{p_1}_{\text{weak}}(\Omega))^3\big), \\ &\text{with} \quad p_1 = \frac{2\gamma}{\gamma+1}, \quad \text{and} \quad p \in \Big[1, \frac{2\gamma}{\gamma+1}\Big).\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}&\mathcal{E}\big[\rho_m(t_m + t, x), \mathbf{v}_m(t_m + t, x), \mathcal{H}_m(t_m + t, x)\big] \\ &\qquad \to \mathcal{E}\big[\bar{\rho}(t, x), \bar{\mathbf{v}}(t, x), \bar{\mathcal{H}}(t, x)\big] \quad \text{in} \quad L^1(0,1),\end{aligned} \tag{2.16}$$

with $(\bar{\rho}, \bar{\mathbf{v}}, \bar{\mathcal{H}})$ *being the weak solution of* (1.1)–(1.3) *defined on* $O = \mathbf{R}$ *where* $\mathcal{E} \in L^\infty(\mathbf{R})$ *and* $F \in \tilde{\mathcal{F}}^+$ *with*

$$\begin{aligned}\tilde{\mathcal{F}}^+ = \Big\{F \mid F = \lim_{\hat{t}_m \to \infty} \hbar_m(t + \hat{t}_m, x) \quad &\text{weakly star in} \quad L^\infty(\Omega \times \mathbf{R}) \\ &\text{for some} \quad \hbar_m \in \tilde{\mathcal{F}} \quad \text{as} \quad \hat{t}_m \to \infty\Big\}.\end{aligned} \tag{2.17}$$

Theorem 2.2 highlights the significance of complete bounded trajectories such that the energy $\mathcal{E}$ is bounded uniformly on $\mathbf{R}$ for weak solutions on $O = \mathbf{R}$. Next, define

$$\tilde{\mathcal{A}}^S[\tilde{\mathcal{F}}] = \big\{(\rho(\hat{t}), (\rho\mathbf{v})(\hat{t}), \mathcal{H}(\hat{t})) \mid (\rho, \mathbf{v}, \mathcal{H}) \text{ is a weak solution of (1.1)–(1.3)} \\ \text{on interval } O = \mathbf{R} \text{ with } F \in \tilde{\mathcal{F}}^+ \text{ and } \mathcal{E}[\rho, \mathbf{v}, \mathcal{H}] \in L^\infty(\mathbf{R})\big\}, \tag{2.18}$$

$$\tilde{U}[\mathcal{E}_0, \tilde{\mathcal{F}}, \tilde{M}](t_0, t) = \Big\{(\rho, \mathbf{v}, \mathcal{H})(t) \Big| (\rho, \mathbf{v}, \mathcal{H}) \text{ is a weak solution of} \\ \text{(1.1)–(1.3) on interval } O, (t, t_0) \subset O \text{ with } F \in \tilde{\mathcal{F}} \text{ and} \\ \operatorname{ess}\lim_{t\to t_0} \sup \mathcal{E}(\rho, \mathbf{v}, \mathcal{H})(t) \leqslant \mathcal{E}_0,\ \int_\Omega \rho(t)dx \leqslant \mathfrak{m}\Big\}. \tag{2.19}$$

Moreover, $\tilde{\mathcal{A}}^S[\tilde{\mathcal{F}}]$, the characteristics as given in the following theorem, the so-called short trajectories of global attractors.

Theorem 2.3. *Assume that $\gamma > \frac{5}{3}$ with bounded subset $\tilde{\mathcal{F}} \in L^\infty(\Omega \times \mathbf{R})$. Then the compact set $\tilde{\mathcal{A}}^S[\tilde{\mathcal{F}}]$ is in $(L^p((0,1) \times \Omega)) \times (L^\gamma((0,1) \times \Omega))$ and*

$$\sup_{(\rho,\mathbf{v})\in\tilde{U}[\mathcal{E}_0,\tilde{\mathcal{F}},\tilde{M}](t_0,t)} \Big[\inf_{(\bar{\rho},\bar{\mathbf{v}})\in\tilde{\mathcal{A}}^S[\tilde{\mathcal{F}}]} (\|\rho - \bar{\rho}\|_{(L^\gamma((0,1)\times\Omega))} + \|(\rho\mathbf{v}) - (\overline{\rho\mathbf{v}})\|_{(L^\gamma(0,1)\times\Omega)})\Big] \to 0, \tag{2.20}$$

as $t \to \infty$ with $p \in \big[1, \frac{2\gamma}{\gamma-1}\big)$.

The set $\tilde{\mathcal{A}}^S[\tilde{\mathcal{F}}]$ is naturally referred as a short trajectories' global attractor based on Theorem 2.3. For a nonempty set $\tilde{\mathcal{F}}$, the set $\tilde{\mathcal{A}}^S[\tilde{\mathcal{F}}]$ is nonempty and compact. If set $\tilde{\mathcal{A}}$ is compact, it is referred to as a global attractor for a general dynamical system that attracts every trajectory and is minimal for a compact set $\tilde{\mathcal{A}}_1$ such that it attracts all the trajectories, then $\tilde{\mathcal{A}}_1 \subset \tilde{\mathcal{A}}$. Regardless of the non-uniqueness of the finite energy weak solution for given initial data, it might make sense to study the global attractor and outline its significant characteristics briefly here. For this, let

$$\tilde{\mathcal{A}}^S[\tilde{\mathcal{F}}] = \big\{(\rho, \mathbf{v}, \mathcal{H}) \mid (\rho = \rho(0), (\rho\mathbf{v}) = (\rho\mathbf{v})(0), \mathcal{H} = \mathcal{H}(0)), \\ \text{where } (\rho, \mathbf{v}, \mathcal{H}) \text{ is a weak solution of (1.1)–1.3} \\ \text{on interval } O = \mathbf{R} \text{ with } F \in \tilde{\mathcal{F}}^+ \text{ and } \mathcal{E}[\rho, \mathbf{v}, \mathcal{H}] \in L^\infty(\mathbf{R})\big\}, \tag{2.21}$$

Next, we describe the result of attractors such that:

Theorem 2.4. *Assume that $\gamma > \frac{5}{3}$ with bounded subset $\tilde{\mathcal{F}} \in L^\infty(\Omega \times \mathbf{R})$. Then the compact set $\tilde{\mathcal{A}}[\tilde{\mathcal{F}}]$ is in $(L^\varsigma(\Omega)) \times (L^{p_1}(\Omega))^3$ and*

$$\sup_{(\rho,\mathbf{v})\in\tilde{U}[\mathcal{E}_0,\tilde{\mathcal{F}},\tilde{M}](t_0,t)} \Bigg[\inf_{(\bar{\rho},\bar{\mathbf{v}})\in\tilde{\mathcal{A}}[\tilde{\mathcal{F}}]} \Bigg(\|\rho - \bar{\rho}\|_{L^\varsigma(\Omega)} + \|\mathcal{H} - \bar{\mathcal{H}}\|_{L^\varsigma(\Omega)} \\ + \bigg|\int_\Omega (\rho\mathbf{v}) - (\overline{\rho\mathbf{v}}) \cdot \varphi\, dx\bigg|\Bigg)\Bigg] \to 0, \tag{2.22}$$

as $t \to \infty$ *with* $\zeta \in [1, \gamma)$, $\varphi \in \big(L^{\frac{2\gamma}{\gamma-1}}(\Omega)\big)^3$ *and the energy* $\mathcal{E}$ *is given such as:*

$$\mathcal{E}(t) = \int_{\rho(x,t)>0} \Big(\frac{1}{2}\rho|\mathbf{v}|^2 + \frac{1}{2}|\mathcal{H}|^2 + a\frac{\rho^\gamma}{\gamma-1}\Big)dx. \tag{2.23}$$

Remark 2.1. In Theorems 2.1–2.3 the energy $\mathcal{E}(t)$ as defined by (2.23) is lower semicontinuous and thus (2.23) implies (2.6) a.e. in O as in [20] and thus the condition $\operatorname{ess\,lim\,sup}_{t\to l} \mathcal{E}(t) \to \mathcal{E}_0$ being equivalent to that of $\limsup_{t\to l} \mathcal{E}(t) \to \mathcal{E}_0$ and hence in the conclusion Theorems 2.1–2.3 hold. In addition, in Theorem 2.1

$$\mathcal{E}(t) \leqslant \mathcal{E}_\infty \quad \text{when} \quad T < t.$$

Lemma 2.1. *Let* (1.1) *be satisfied in* $\mathcal{D}'(0,\infty;\Omega)$ *and allowing that* $\rho, \mathbf{v}$ *be zero on* $\mathbf{R}^3\backslash\Omega$, *such that*

$$\partial_t \rho_m + \operatorname{div}(\rho_m \mathbf{v}_m) = 0 \quad in \quad \mathcal{D}'(O\times\mathbf{R}^3). \tag{2.24}$$

Next, taking $\mathcal{S}_\epsilon = \vartheta_\epsilon * \nu_m$ *with* $\vartheta_\epsilon = \vartheta_\epsilon(x)$ *is a regularizing sequence in* (2.4), *we have*

$$\partial_t \mathcal{S}_\epsilon[\xi(\rho)] + \operatorname{div}(\mathcal{S}_\epsilon[\xi(\rho)\mathbf{v}]) + \mathcal{S}_\epsilon((\xi'(\rho)\rho - \xi(\rho))\operatorname{div}\mathbf{v}) = r_\epsilon^m, \tag{2.25}$$

with

$$r_\epsilon^m \to 0 \ \text{in the space} \ L^2(O; L^{\breve\alpha}(\Omega)) \ \text{as} \ \epsilon \to 0 \ \text{for any prechosen} \ m, \tag{2.26}$$

where

$$\breve\alpha = \frac{2\beta}{2+\beta}, \tag{2.27}$$

and $\xi(\rho) \in L^\infty(0,T;L^\beta(\Omega))$ *provided that* $\beta \geqslant 2$.

Proof. Regarding the proof, we refer [33, Corollary 2.4]. □

Lemma 2.2. *In accordance with all the presumptions of Theorem* 2.1, *on* $\mathbf{R}^+$ *the energy* $\mathcal{E}$ *exhibits a local bounded variation, if required, being modified on a zero-measure set, and*

$$\mathcal{E}(t_+) = \lim_{s\to t_+} \mathcal{E}(s) = \mathcal{E}(t_-), \quad t \in \mathbf{R}^+. \tag{2.28}$$

In addition,

$$(1+\mathcal{E}(t_{1+}))e^{\sqrt{2\mathrm{m}}\mathcal{K}_1(t_2-t_1)} - 1 \geqslant \mathcal{E}(t_{2-}), \quad t_2 > t_1 > 0. \tag{2.29}$$

Proof. To prove the required result, let

$$\frac{d}{dt}\mathcal{E}_1(\rho,\mathbf{v})(t) + \int_\Omega (\mu|\nabla\mathbf{v}|^2 + (\mu+\lambda)|\operatorname{div}\mathbf{v}|^2 + \upsilon|\nabla\mathcal{H}|^2)dx = \int_\Omega \rho(t)F(t)\cdot\mathbf{v}(t)dx, \tag{2.30}$$

then $\mathcal{E}_2 := (\mathcal{E}-\mathcal{E}_1) \in L^1_{\mathrm{loc}}(\Omega)$. In view of (2.5), we have

$$\frac{d}{dt}\mathcal{E}_2(t) \leqslant 0, \quad \text{holds in} \quad \mathcal{D}'(\Omega). \tag{2.31}$$

Next, since $\mathcal{E}$ is regarded as a sum of absolute as well as the sum of nonincreasing functions, this implies the continuity of $\mathcal{E}$ such that (2.28) holds true except for a countable set of points. In addition, by using (2.9), the right side of (2.5), implies that

$$\int_\Omega \rho F\cdot \mathbf{v}\,dx \leqslant \mathcal{K}_1\Big(\int_\Omega \rho\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega \rho|\mathbf{v}|^2 dx\Big)^{\frac{1}{2}} \leqslant \sqrt{2\mathfrak{m}}\mathcal{K}_1\Big(1+\int_\Omega \rho|\mathbf{v}|^2dx\Big) \leqslant \sqrt{2\mathfrak{m}}\mathcal{K}_1(1+\mathcal{E}(t)), \tag{2.32}$$

by applying Gronwall lemma, we get the required result. □

3. Proof of Theorem 2.1

This part focuses on the proof of Theorem 2.1, and the required proof will be completed on the basis of the following results.

PROPOSITION 3.1. *In accordance with all the presumptions of Theorem 2.1, one can find the constant $\mathcal{L}$ depending on the parameters* K_1, γ *and* $\mathfrak{m}$ *satisfying the property, that*

$$\mathcal{E}(T_+) - 1 < \mathcal{E}((T+1)-),\ \textit{for some specific } T \in \mathbf{R}^+. \tag{3.1}$$

Then

$$\sup_{t\in(T,T+1)} \mathcal{E}(t_+) \leqslant \mathcal{L}.$$

The required result will be proved by using some auxiliary results, as described below.

LEMMA 3.1. *In accordance with all the presumptions of Theorem 2.1, and (3.1), one can find a constant $\grave{c}$ depending only on the parameters $\mathcal{K}_1$ and $\mathfrak{m}$, such that*

$$\int_T^{T+1} ||\mathbf{v}||^2_{W_0^{1,2}(\Omega)}dt \leqslant \grave{c}\Big(1+\int_T^{T+1} ||\rho||_{L^{\frac{3}{2}}(\Omega)}dt\Big). \tag{3.2}$$

Moreover,

$$\mathcal{E}(t_+) \leqslant \grave{c}\Big(1+\int_T^{T+1} ||\rho(\hat{z})||^{\gamma}_{L^\gamma}d\hat{z}\Big). \tag{3.3}$$

PROOF. Applying, the energy inequality (2.5), (3.1), Poincare's inequality with the standard embedding theorem, ensure that

$$\int_\Omega |\nabla \mathbf{v}(t)|^2 dx \leqslant \tilde{c}_1\Big(1+\int_T^{T+1}\int_\Omega \rho|\mathbf{v}|dx\,dt\Big). \tag{3.4}$$

Similarly, thanking to Hölder inequality, we have

$$\int_\Omega \rho|\mathbf{v}|dx \leqslant \sqrt{\mathfrak{m}}\Big(\int_\Omega \rho|\mathbf{v}|^2dx\Big)^{\frac{1}{2}} \leqslant \sqrt{\mathfrak{m}}||\rho||^{\frac{1}{2}}_{L^{\frac{3}{2}}(\Omega)}||\mathbf{v}||_{L^6(\Omega)} \tag{3.5}$$

again by using the embedding theorem, implies (3.2). Next, the integration of (2.29) along t_1 and taking $T+1=t_2$, yields that

$$\mathcal{E}(T+1)-) \leqslant \tilde{c}_2\Big(1+\int_T^{T+1}\mathcal{E}(\hat{z})d\hat{z}\Big).$$

Also, we have

$$\mathcal{E}(T_+) < \mathcal{E}((T+1)-) \leqslant \tilde{c}_3\bigg(1+\int_T^{T+1}\mathcal{E}(\hat{z})d\hat{z}\bigg). \tag{3.6}$$

In a similar way, the integration of (2.29) by taking $T = t_1$ and using (3.6), ensures that

$$\mathcal{E}(T^+) \leqslant \tilde{c}_4\bigg(1+\int_T^{T+1}\mathcal{E}(\hat{z})d\hat{z}\bigg). \tag{3.7}$$

Furthermore, applying H$\acute{}$older inequality along with (3.2), infers that

$$\begin{aligned}\int_T^{T+1}\int_\Omega \rho|\mathbf{v}|^2 dx\, dt &\leqslant \sup_{t\in[T,T+1]}||\rho||_{L^{\frac{3}{2}}(\Omega)}\int_T^{T+1}||\mathbf{v}||^2_{W_0^{1,2}(\Omega)}d\hat{s}\\ &\leqslant \tilde{c}_5 \sup_{t\in[T,T+1]}||\rho||_{L^{\frac{3}{2}}}\bigg(1+\int_T^{T+1}||\rho||_{L^{\frac{3}{2}}(\Omega)}dt\bigg).\end{aligned} \tag{3.8}$$

By applying interpolation inequality $\|\rho\|_{L^{\frac{3}{2}}(\Omega)} \leqslant \|\rho\|^{1-\varsigma}_{L^1(\Omega)}\|\rho\|^{\varsigma}_{L^\gamma(\Omega)}$, with $\varsigma = \frac{\gamma}{3\gamma-3}$, we have

$$\int_T^{T+1}\int_\Omega \rho|\mathbf{v}|^2 dxdt \leqslant \tilde{c}_6 \sup_{t\in[T,T+1]}\mathcal{E}(t_+)^{\frac{1}{3\gamma-3}}\bigg(1+\int_T^{T+1}\|\rho(\hat{z})\|^{\varsigma}_{L^\gamma(\Omega)}d\hat{z}\bigg). \tag{3.9}$$

This further implies that

$$\begin{aligned}\sup_{t\in[T,T+1]}\mathcal{E}(t_+) \leqslant \tilde{c}_7\bigg(1+\int_T^{T+1}\|\rho\|^{\gamma}_{L^\gamma(\Omega)}d\hat{s}\\ + \sup_{t\in[T,T+1]}\mathcal{E}(t_+)^{\frac{1}{3\gamma-3}}\bigg(1+\int_T^{T+1}\|\rho(\hat{s})\|^{\varsigma}_{L^\gamma(\Omega)}dt\bigg)\bigg).\end{aligned}$$

By using the condition that $\gamma > \frac{5}{3}$, infers that $\frac{1}{3\gamma-3} < \frac{1}{2}$ and hence the required result follows. □

A Linear Bounded Operator $\mathcal{B}$. Here, some characteristics of operator $\mathcal{B}$, introduced by Bogovskiî [34], may be listed as follows. The operator $\mathcal{B}$ is regarded as the problem's solution

$$\begin{cases}\operatorname{div} g = k\\ k \in L^x(S), \quad \text{with } S \subset \mathbf{R}^3 \text{ is a Lipschitz bounded domain.}\end{cases} \tag{3.10}$$

LEMMA 3.2. [20,34]. *Let $\mathcal{B}$ be a bounded linear operator of the problem (3.7), such that:*

$$\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3] \colon \bigg\{k \in L^x(S) \mid \int_S k\, dx = 0\bigg\} \to W_0^{1,x}(S),$$

and its boundedness is given by $\|\mathcal{B}\{k\}\|_{W_0^{1,x}(S)} \leqslant C_1(x,S)\|k\|_{L^x(S)}$, with $x \in (1,\infty)$.

Also, $g = \mathcal{B}\{k\}$ satisfy $\operatorname{div} g = k$ a.e. in S, and $\|\mathcal{B}(k)\|_{L^y(S)} \leqslant C_2(y,S)\|(k)\|_{L^y(S)}$ with $y \in (1,\infty)$.

Keeping all these results in mind, we can prove Proposition 3.1. To do this, let a test function be such that

$$\phi_j(t,x) = \Psi(t)\mathcal{B}_j\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega \mathcal{S}_\epsilon(\xi(\rho))dx\Big),$$

here $\Psi \in \mathcal{D}(T, T+1)$, $\Psi \in [0,1]$ and the operator $\mathcal{S}_\epsilon$ is defined in Lemma 2.1 where $\xi \in C^1(\mathbf{R})$, for $z > 1, \xi(z) = z^\Theta$ with

$$\Theta = \min\Big\{\frac{1}{4}, \Big(\frac{2}{3} - \frac{1}{\gamma}\Big)\Big\}. \tag{3.11}$$

Moreover, by using $\phi_j(t,x)$ as a test function in $(1.1)_2$ and Lemma 2.1 and 3.2, may be used to get

$$\begin{aligned} a\int_T^{T+1}\int_\Omega \Psi\rho^\gamma\mathcal{S}_\epsilon(\xi(\rho))dx\,dt &= \int_T^{T+1}\Psi\Big(\int_\Omega\rho^\gamma dx\Big)\Big(\frac{1}{|\Omega|}\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))dx\Big)dt \\ &+(\mu+\lambda)\int_T^{T+1}\Psi\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))\operatorname{div}\mathbf{v}\,dx\,dt \\ &-\int_T^{T+1}\Psi_t\int_\Omega\rho\mathbf{v}\mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))dx\Big)dx\,dt \\ &+\mu\int_T^{T+1}\Psi\int_\Omega\nabla\mathbf{v}:\nabla\mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))dx\Big)dx\,dt \\ &-\int_T^{T+1}\Psi\int_\Omega\rho\mathbf{v}\otimes\mathbf{v}:\nabla\mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))dx\Big)dx\,dt \\ +\int_T^{T+1}\Psi\int_\Omega\rho\mathbf{v}\mathcal{B}\Big\{&\mathcal{S}_\epsilon(\xi(\rho)-\xi'(\rho)\rho\operatorname{div}\mathbf{v}) - \frac{1}{|\Omega|}\int_\Omega\mathcal{S}_\epsilon(\xi(\rho)-\xi'(\rho)\rho\operatorname{div}\mathbf{v})dx\Big\}dx\,dt \\ +\int_T^{T+1}\Psi\int_\Omega\rho\mathbf{v}\mathcal{B}\Big(r_\epsilon^m - \frac{1}{|\Omega|}\int_\Omega r_\epsilon^m dx\Big)&dx\,dt - \int_T^{T+1}\Psi\int_\Omega\rho\mathbf{v}\mathcal{B}(\operatorname{div}(\mathcal{S}_\epsilon(\xi(\rho)))\mathbf{v})dx\,dt \\ &-\int_T^{T+1}\Psi\int_\Omega\rho F\mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))dx\Big)dx\,dt \\ +\int_T^{T+1}\Psi\int_\Omega\Big(\mathcal{H}\otimes\mathcal{H}-\frac{1}{2}\mathcal{H}^2\mathbb{I}\Big)&\mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))dx\Big)dx\,dt = \sum_{j=1}^{10}\mathcal{J}_j. \end{aligned} \tag{3.12}$$

Furthermore, we have to evaluate each integral to the respective norms in terms of ρ, $\mathbf{v}$ and $\mathcal{H}$ by using the Sobolev embedding results along with H$\acute{}$older inequality and using the properties of the operator $\mathcal{B}$. For some omitted details one can refer to [14] and [15]. Thus, we get

$$|\mathcal{J}_1| = \Big|\int_T^{T+1}\Psi\Big(\int_\Omega\rho^\gamma dx\Big)\Big(\frac{1}{|\Omega|}\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))dx\Big)\Big|dt \leqslant k_1(\mathfrak{m})\int_T^{T+1}\int_\Omega\rho^\gamma dx\,dt, \tag{3.13}$$

$$|\mathcal{J}_2| = \Big|\int_T^{T+1}\Psi\int_\Omega\mathcal{S}_\epsilon(\xi(\rho))\urcorner(\operatorname{div}\mathbf{v})\operatorname{div}\mathbf{v}\,dx\,dt\Big| \leqslant k_2(\mathfrak{m})\int_T^{T+1}\|\mathbf{v}\|_{W^{1,2}(\Omega)}dt. \tag{3.14}$$

Next, for $\mathcal{J}_3$, we have

$$
\begin{aligned}
(3.15)\qquad |\mathcal{J}_3| &= \left| \int_T^{T+1} \Psi_t \int_\Omega \rho \mathbf{v} \mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega \mathcal{S}_\epsilon(\xi(\rho))dx \Big) dx\, dt \right| \\
&\leqslant k_3 \int_T^{T+1} \|\sqrt{\rho}\mathbf{v}\|_{L^2(\Omega)} |\Psi_t| dt,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.16)\qquad |\mathcal{J}_4| &= \left| \int_T^{T+1} \Psi \int_\Omega \nabla \mathbf{v} : \nabla \mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega \mathcal{S}_\epsilon(\xi(\rho))dx \Big) dx\, dt \right| \\
&\leqslant (\mathfrak{m}) \int_T^{T+1} \|\mathbf{v}\|_{W_0^{1,2}} dt.
\end{aligned}
$$

Similarly, $\mathcal{J}_5$ can be calculated as

$$
\begin{aligned}
(3.17)\qquad |\mathcal{J}_5| &= \left| \int_T^{T+1} \Psi \int_\Omega \rho \mathbf{v}\otimes\mathbf{v} : \nabla \mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|}\int_\Omega \mathcal{S}_\epsilon(\xi(\rho))dx \Big) dx\, dt \right| \\
&\leqslant k_5 \sup \|\rho\|_{L^\gamma(\Omega)} \sup \|\xi(\rho)\|_{L^{q_1}(\Omega)} \int_T^{T+1} \|\mathbf{v}(t)\|^2_{W_0^{1,2}(\Omega)} dt,
\end{aligned}
$$

with $q_1 = \frac{2}{3} - \frac{1}{\gamma}$. Similarly by using (3.11), $\sup_{t\in[T,T+1]} \|\xi(\rho)\| \leqslant k_6(\mathfrak{m})$. Furthermore, we get

$$
\begin{aligned}
(3.18)\qquad |\mathcal{J}_6| &= \bigg| \int_T^{T+1} \Psi \int_\Omega \rho \mathbf{v} \mathcal{B}\Big\{ \mathcal{S}_\epsilon(\xi(\rho) - \xi'(\rho)\rho \operatorname{div} \mathbf{v}) \\
&\qquad\qquad - \frac{1}{|\Omega|}\int_\Omega \mathcal{S}_\epsilon(\xi(\rho) - \xi'(\rho)\rho \operatorname{div} \mathbf{v}) dx \Big\} dx\, dt \bigg| \\
&\leqslant k_7 \sup \|\rho\|_{L^\gamma(\Omega)} \int_T^{T+1} \|\mathbf{v}\|_{W_0^{1,2}(\Omega)} \|\mathcal{B}[\ldots]\|_{L^{q_2}(\Omega)},
\end{aligned}
$$

with

$$
\begin{aligned}
&\left\| \mathcal{B}\Big\{ \mathcal{S}_\epsilon(\xi(\rho) - \xi'(\rho)\rho \operatorname{div} \mathbf{v}) - \frac{1}{|\Omega|}\int_\Omega \mathcal{S}_\epsilon(\xi(\rho) - \xi'(\rho)\rho \operatorname{div} \mathbf{v}) dx \Big\} \right\|_{L^{q_2}(\Omega)} \\
&\qquad\qquad \leqslant \|\xi(\rho) \operatorname{div} \mathbf{v}\|_{L^{q_3}(\Omega)} \leqslant \|\xi(\rho)\|_{L^{\frac{1}{q_1}}(\Omega)} \|\mathbf{v}\|_{W_0^{1,2}(\Omega)}
\end{aligned}
$$

where

$$
\frac{1}{q_2} = \frac{5\gamma - 6}{6\gamma} \quad \text{and} \quad q_3 = \max\Big\{1, \frac{6\gamma}{7\gamma - 6}\Big\}.
$$

Hence, by using (3.11), we have

$$
\begin{aligned}
(3.19)\qquad &\bigg| \int_T^{T+1} \Psi \int_\Omega \rho \mathbf{v} \mathcal{B}\Big\{ \mathcal{S}_\epsilon(\xi(\rho) - \xi'(\rho)\rho \operatorname{div} \mathbf{v}) \\
&\qquad - \frac{1}{|\Omega|}\int_\Omega \mathcal{S}_\epsilon(\xi(\rho) - \xi'(\rho)\rho \operatorname{div} \mathbf{v}) dx \Big\} dx\, dt \bigg| \\
&\qquad\qquad\qquad \leqslant k_8 \|\rho\|_{L^\gamma} \int_T^{T+1} \|\mathbf{v}\|_{W_0^{1,2}(\Omega)} dt.
\end{aligned}
$$

Similarly, for $\mathcal{J}_7$, we get

$$|\mathcal{J}_7| = \left| \int_T^{T+1} \Psi \int_\Omega \rho \mathbf{v} \mathcal{B}\Big(r_\epsilon^m - \frac{1}{|\Omega|} \int_\Omega r_\epsilon^m dx \Big) dx\, dt \right| \leqslant k_9 \int_T^{T+1} \|\rho\|_{L^\gamma(\Omega)} \|\mathbf{v}\|_{W_0^{1,2}(\Omega)} \|r_\epsilon^m\|_{L^{q_3}(\Omega)} dt. \tag{3.20}$$

Next, applying (2.26), implies that

$$\left| \int_T^{T+1} \Psi \int_\Omega \rho \mathbf{v} \mathcal{B}\Big(r_\epsilon^m - \frac{1}{|\Omega|} \int_\Omega r_\epsilon^m dx \Big) dx\, dt \right| \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3.21}$$

Furthermore, for $\mathcal{J}_8$, using the properties of operator $\mathcal{B}$ and applying the same arguments as in (3.17), ensure that

$$|\mathcal{J}_8| = \left| \int_T^{T+1} \Psi \int_\Omega \rho \mathbf{v} \mathcal{B}(\operatorname{div}(\mathcal{S}_\epsilon(\xi(\rho)))\mathbf{v}) dx\, dt \right| \leqslant k_{10}(\mathfrak{m}) \sup_{t\in[T,T+1]} \|\rho\|_{L^\gamma(\Omega)} \int_T^{T+1} \|\mathbf{v}\|^2_{W_0^{1,2}(\Omega)} dt, \tag{3.22}$$

and

$$|\mathcal{J}_9| = \left| \int_T^{T+1} \Psi \int_\Omega \rho F \mathcal{B}\Big(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|} \int_\Omega \mathcal{S}_\epsilon(\xi(\rho)) dx \Big) dx\, dt \right| \leqslant k_{11}(\mathcal{K}_1, \mathfrak{m}), \tag{3.23}$$

$$|\mathcal{J}_{10}| = \left| \int_T^{T+1} \Psi \int_\Omega \Big(\mathcal{H} \otimes \mathcal{H} - \frac{1}{2} \mathcal{H}^2 \mathbb{I} \Big) \mathcal{B}(\mathcal{S}_\epsilon(\xi(\rho)) - \frac{1}{|\Omega|} \int_\Omega \mathcal{S}_\epsilon(\xi(\rho)) dx \Big) dx\, dt \right| \leqslant k_{12}(\mathcal{K}_1, \mathfrak{m}). \tag{3.24}$$

Furthermore, calculating characteristic function on $[T, T+1]$ for a sequence of functions Ψ_ϵ as $\epsilon \to 0$ in the integrals $\mathcal{J}_1 - \mathcal{J}_{10}$, we have

$$\int_T^{T+1} \rho^{\gamma+\Theta} dx\, dt \leqslant k_{12}(\mathcal{K}_1, \mathfrak{m}) \Big\{ (1 + \sup \|\rho\|_{L^\gamma(\Omega)}) \int_T^{T+1} \|\mathbf{v}\|^2_{W_0^{1,2}} dt + \sup_{t\in[T,T+1]} \|\sqrt{\rho}\mathbf{v}\|_{L^2(\Omega)} \Big\}. \tag{3.25}$$

Next, the interpolation of spaces $L^{\gamma+\Theta}, L^1(\Omega)$, implies that

$$\int_T^{T+1} \|\rho\|^\gamma_{L^\gamma} dt \leqslant k_{13} \Big(\int_T^{T+1} \int_\Omega \rho^{\gamma+\Theta} dx\, dt \Big)^{q_4}, \tag{3.26}$$

where $q_4 = \frac{\gamma-1}{\gamma+\Theta-1}$. Taking into account (3.2), we have

$$\left| \|\rho\|_{L^\gamma(\Omega)} \int_T^{T+1} \|\mathbf{v}\|^\kappa_{W_0^{1,\kappa}(\Omega)} \right|^{q_4} \leqslant k_{14}(\mathcal{K}_1, \mathfrak{m})(1 + \|\rho\|_{L^\gamma(\Omega)} \|\rho\|_{L^{\frac{3}{2}}(\Omega)})^{q_4} \leqslant k_{15}(\mathcal{K}_1, \mathfrak{m})(1 + \|\rho\|_{L^\gamma(\Omega)})^{q_5}, \tag{3.27}$$

where $q_5 = \frac{4\gamma-3}{3\gamma+3\Theta-3}$. Finally, we have

$$ess \sup_{t\in[T,T+1]} \|\sqrt{\rho}\mathbf{v}\|_{L^2(\Omega)} \leqslant \sup_{t\in[T,T+1]} \sqrt{2\mathcal{E}(t_+)}. \tag{3.28}$$

Combining the estimation obtained in (3.3) and (3.25)–(3.28), we have

$$\sup \mathcal{E}(t_+) \leqslant k_{16}(\mathcal{K}_1, \mathfrak{m})\big(1 + \sup \sqrt{\mathcal{E}(t_+)} + \sup \|\rho\|_{L^\gamma}^{q_5}\big). \tag{3.29}$$

Referring to predefined Θ, we have $q_5 < \gamma$ and (3.29) indicates the existence $\mathcal{L}$ described in Proposition 3.1.

Based on Propositions 3.1, we prove Theorem 2.1. It is easy to verify that $T(\mathcal{E}_0) = T$ satisfies $\mathcal{E}(t_{0+}) \leqslant \mathcal{L}$ for some $t_0 < T$. In fact, if this mentioned condition is not satisfied, then for a sufficiently large t the energy will be negative. Hence, it contradicts that therefore energy is not negative for $\mathcal{E}(t_{0+}) \leqslant \mathcal{L}$. Furthermore, for any $\mu \geqslant 0$, let

$$\mathcal{E}((t_0 + \mu)+) \leqslant \mathcal{L}. \tag{3.30}$$

Next, we assume by induction that $\mathcal{E}(t_{0+}) \leqslant \mathcal{L}$. Next, by applying Proposition 3.1, either one can get

$$\sup_{t\in[t_0+\mu,t_0+\mu+1]} \mathcal{E}(t_+) \leqslant \mathcal{L},$$

therefore, $\mathcal{E}((t_0 + \mu + 1)-) \leqslant \mathcal{L}$, or

$$\mathcal{E}((t_0 + \mu + 1)+) \leqslant \mathcal{E}((t_0 + \mu + 1)-) \leqslant \mathcal{E}((t_0 + \mu)+) - 1 \leqslant \mathcal{L} - 1.$$

Hence, by using (3.30) and Lemma 2.2, we have

$$\mathcal{E}_\infty = (1 + \mathcal{L})e^{\sqrt{2\mathfrak{m}}\mathcal{K}_1} - 1.$$

This completes the proof of Theorem 2.1.

4. Proof of Theorem

The remaining part of this work is devoted to proving Theorem 2.2. The required result will be proved by applying the analysis of the so-called defect measure

$$\vartheta = \overline{\log(\rho)\rho} - \log(\bar{\rho})\bar{\rho}. \tag{4.1}$$

where $\bar{\rho}$ is used to denote the weak limit of time shifts' sequence

$$\rho_{m,t_m}(t,x) = \begin{cases} \rho_m(t_m + t, x), & \text{when } t_m + t \in O_m, \\ 0 & \text{when } t_m + t \in \mathbf{R} \setminus O_m, \end{cases}$$

where $(l, l+1) \subset O_m$.

Starting with a simple implication of Theorem 2.1, we state the following result:

Proposition 4.1. *In accordance with all the presumptions of Theorem 2.2, the time $T = T(\mathcal{E}_0, \mathfrak{m}, l) > l$ with the constant $\mathcal{L}$ depending on the parameters $\mathfrak{m}$, $\mathcal{K}_1$, $\mathcal{E}_0$ and $F \in \tilde{\mathcal{F}}$ satisfy the estimations:*

$$\sup_{T\leqslant t} \|\rho_m(t)\|_{L^\gamma} \leqslant \mathcal{L}, \tag{4.2}$$

$$\operatorname{ess} \sup_{T\leqslant t_m+t} \big\|\sqrt{\rho_m(t)}\mathbf{v}_m(t)\big\|_{L^2} \leqslant \mathcal{L}, \tag{4.3}$$

$$\sup_{T\leqslant t_m+t} \|\mathcal{H}_m\|_{L^2} \leqslant \mathcal{L}, \tag{4.4}$$

(4.5)
$$\int_K \|\mathbf{v}(t)\|^2_{W_0^{1,2}} \leqslant \mathcal{L},$$

and

(4.6)
$$\int_K \int_\Omega |\rho_m|^{\gamma+\Theta} dx\, dt \leqslant \mathcal{L}, \quad \textit{with } \Theta = \frac{2\gamma}{3} - 1,$$

(4.7)
$$\int_K \int_\Omega (|\nabla \mathbf{v}_m|^2 + |\nabla \times \mathcal{H}_m|^2) dx\, dt \leqslant \mathcal{L},$$

for $|K| \leqslant 1$ *with* $K \subset (T, \infty)$, *independent of the parameter* $m = 1, 2, \dots$.

Proof. The estimations (4.2), (4.3) and (4.7) can easily be obtained by applying the energy inequality and assumptions of Theorem 2.1 (for more details about (4.7), see [35]). While (4.6) can be obtained by a similar way as in (3.25). For this, it is required to take $\gamma > \frac{5}{3}$ and one only needs to know that the boundedness of $\mathcal{E}$ is controlled by $\mathcal{E}_\infty$. □

With the application of the estimation obtained in Proposition 4.1, we are able to pass the sequences without relabeling so that

$$\begin{aligned}
&\rho_{m,t_m} \to \bar{\rho} \ \text{ weakly } \ \star \ \text{ in } \ L^\infty(\mathbf{R}; L^\gamma(\Omega)), \ \ \rho \geqslant 0, \ \ \rho = 0 \ \text{ in } \ \mathbf{R}^3 \setminus \Omega,\\
&\mathbf{v}_{m,t_m} \to \bar{\mathbf{v}} \ \text{ weakly in } \ L^2(K; (W_0^{1,2}(\Omega))^3), \ \ \mathbf{v} = 0 \ \text{ in } \ \mathbf{R}^3 \setminus \Omega,
\end{aligned}$$

(4.8)
$$\mathcal{H}_{m,t_m} \to \bar{\mathcal{H}} \ \text{ weakly in } \ L^2(K; (W_0^{1,2}(\Omega))), \ \ \mathcal{H} = 0 \ \text{ in } \mathbf{R}^3 \setminus \Omega,$$

with $K \subset \mathbf{R}$ and $\bar{\mathbf{v}} \in L^2_{\text{loc}}(\mathbf{R}; (W_0^{1,2}(\Omega)^3))$ such that

(4.9)
$$\int_K \|\bar{\mathbf{v}}\|^2_{W_0^{1,2}(\Omega)} dt \leqslant \mathcal{L} \ \text{ with } \ |K| \leqslant 1, \ \ K \subset \mathbf{R},$$

and

$$\begin{aligned}
&F_{m,t_m} \to \overline{F} \ \text{ weakly } \ \star \ \text{ in } \ L^\infty(\mathbf{R}; \Omega),\\
&\rho_{m,t_m} F_{m,t_m} \to \overline{\rho F} \ \text{ weakly } \ \star \ \text{ in } \ L^\infty(\mathbf{R}; L^\gamma(\Omega)).
\end{aligned}$$

Furthermore, by applying the fact that ρ_{m,t_m} solves the continuity equation $(1.1)_1$ along with estimations obtained in (4.2), (4.5) and using the Arzela–Ascoli Theorem, we have

(4.10)
$$\rho_{m,t_m} \to \bar{\rho},$$

converges in $C(K; L^\gamma_{\text{weak}}(\Omega))$ for the compact interval $K \subset \mathbf{R}$ with

(4.11)
$$\bar{\rho} \in BC(\mathbf{R}; L^\gamma_{\text{weak}}),$$

for more details about (4.11) see [16].

Next, by applying (4.10) results in the strong convergency of the term ρ_{m,t_m} in the space $C\left(K; W^{-1,2}(\Omega)\right)$ and its combination with (4.8) gives that

(4.12)
$$\rho_{m,t_m} \mathbf{v}_{m,t_m} \to \bar{\rho}\bar{\mathbf{v}} \ \text{ weakly in } \ L^2\Big(K; \big(L^{\frac{6\gamma}{6+\gamma}}(\Omega)\big)^3\Big).$$

As a result, we get

(4.13)
$$\partial_t \bar{\rho} + \operatorname{div}(\bar{\rho}\bar{\mathbf{v}}) = 0,$$

in the distributional sense in $\mathcal{D}'\left(O \times \mathbf{R}^3\right)$. By applying the same approach as above, the time derivative of the term $\rho_{m,t_m}\mathbf{v}_{m,t_m}$ with the help of $(1.1)_2$ and taking into account (4.2)–(4.4) implies that

$$\rho_{m,t_m}\mathbf{v}_{m,t_m} \to \bar{\rho}\bar{\mathbf{v}} \ \text{ in } \ C(K; L^{p_1}_{\text{weak}}(\Omega)), \quad (p_1 = \frac{2\gamma}{1+\gamma}) \tag{4.14}$$

where its details can be found in [20, Section 7.10], and

$$\bar{\rho}\bar{\mathbf{v}} \in BC(\mathbf{R}; L^{p_1}_{\text{weak}}(\Omega). \tag{4.15}$$

Here, the estimations obtained in (4.14) result in the strong convergence of the term $\rho_{m,t_m}\mathbf{v}_{m,t_m}$ in $C(K; W^{-1,2}(\Omega))$, exactly the same as in (4.12), we have

$$\rho_{m,t_m}\mathbf{v}_{m,t_m} \otimes \mathbf{v}_{m,t_m} \to \bar{\rho}\bar{\mathbf{v}} \otimes \bar{\mathbf{v}} \ \text{ weakly in } \ L^2(K; L^{p_2}(\Omega)), \ \Big(p_2 = \frac{6\gamma}{4\gamma+3}\Big). \tag{4.16}$$

Moreover, the estimations obtained in (4.6) imply that

$$\rho_{m,t_m} \to \bar{\rho} \ \text{ weakly in } \ L^{\Theta+\gamma}(K \times \mathbf{R}^3) \tag{4.17}$$

with

$$p(\rho_{m,t_m}) \to \overline{p(\rho)} \ \text{ weakly in the space } \ L^{\frac{\Theta+\gamma}{\gamma}}(K \times \Omega). \tag{4.18}$$

Next, we have

$$\mathcal{H}_{m,t_m} \to \bar{\mathcal{H}} \ \text{ weakly } \ \star \ \text{ in } \ L^2(\mathbf{R}; W^{1,2}_0(\Omega)), \quad \operatorname{div} \mathcal{H} = 0, \tag{4.19}$$

in distributional sense $\mathcal{D}'(\mathbf{R} \times \mathbf{R}^3)$, by using (4.8), (4.19) and the compactness result of $W^{1,2}_0 \hookrightarrow L^2$, we have

$$\operatorname{rot}(\mathbf{v}_{m,t_m} \times \mathcal{H}_{m,t_m}) \to \overline{\operatorname{rot}(\mathbf{v} \times \mathcal{H})} \quad \text{in } \mathcal{D}'(\mathbf{R} \times \mathbf{R}^3), \tag{4.20}$$

and

$$\operatorname{rot} \mathcal{H}_{m,t_m} \times \mathcal{H}_{m,t_m} \to \overline{\operatorname{rot} \mathcal{H} \times \mathcal{H}} \quad \text{in } \mathcal{D}'(\mathbf{R} \times \mathbf{R}^3). \tag{4.21}$$

Taking into account the estimations obtained in (4.14)–(4.21), we can conclude that the weak limits of $(\rho, \mathbf{v}, \mathcal{H})$ satisfy the momentum equation $(1.1)_2$ in distributional sense such that

$$\partial_t(\bar{\rho}\bar{\mathbf{v}}) + \operatorname{div}(\bar{\rho}\bar{\mathbf{v}} \otimes \bar{\mathbf{v}}) - \overline{\operatorname{div} \mathbb{T}} + \nabla\overline{p(\rho)} = \overline{\operatorname{rot} \mathcal{H} \times \mathcal{H}} + \overline{\rho F},$$
$$\bar{\mathcal{H}}_t - \overline{\operatorname{rot}(\mathbf{v} \times \mathcal{H})} = \upsilon \Delta \bar{\mathcal{H}}, \quad \operatorname{div} \bar{\mathcal{H}} = 0,$$
$$\bar{\mathbb{T}}(\nabla \bar{\mathbf{v}}) = \mu\Big(\nabla \bar{\mathbf{v}} + \nabla^T \bar{\mathbf{v}} - \frac{2}{3} \operatorname{div} \bar{\mathbf{v}} \mathbb{I}\Big) + \daleth \operatorname{div} \bar{\mathbf{v}} \mathbb{I} \quad \text{in } \ \mathcal{D}'(O \times \Omega). \tag{4.22}$$

4.1. Some key results related to density. By using the assumptions (2.4), ρ_{m,t_m} are said to be the renormalized solution of (2.2) such that

$$(\xi(\rho_{m,t_m}))_t + \operatorname{div}[\xi(\rho_{m,t_m})\mathbf{v}_{m,t_m}] + (\xi'(\rho_{m,t_m})\rho_{m,t_m} - \xi(\rho_{m,t_m})) \operatorname{div} \mathbf{v}_{m,t_m} = 0, \tag{4.23}$$

holds for any global Lipschitz function $\xi \in C^1(\mathbf{R})$ in $\mathcal{D}'(O_{m,t_m} \times \mathbf{R}^3)$, with $O_{m,t_m} = \{x \mid x + t_m \in O_m\}$. In addition, for a compact set $K \subset (-t_m, \infty)$

$$\rho_{m,t_m} \in C(K; L^\zeta), \quad \zeta \in [1, \gamma), \tag{4.24}$$

where its proof is given in [27, Lemma 2.3]. Similarly, this result is true for $\bar{\rho}$, such that

(4.25) $$(\xi(\bar{\rho}))_t + \operatorname{div}[\xi(\bar{\rho})\bar{\mathbf{v}}] + (\xi^{'}(\bar{\rho})\bar{\rho} - \xi(\bar{\rho}))\operatorname{div}\bar{\mathbf{v}} = 0, \quad \bar{\rho} \in C(K; L^{\zeta}), \quad \zeta \in [1,\gamma).$$

Particularly, we have

(4.26) $$\log(\bar{\rho})\bar{\rho} \in BC(O; L^{\zeta}(\Omega)), \quad \zeta \in [1,\gamma).$$

Here, we define some functions

$$\mathfrak{M}_{\kappa}(\grave{z}) = \begin{cases} \grave{z}\log(\grave{z}) & \text{when } \ \grave{z} \in [0,\kappa), \\ \kappa\log(\kappa) + (\log(\kappa)+1)(\grave{z}-\kappa) & \text{when } \ \grave{z} \geqslant \kappa \end{cases}$$

with

$$\mathfrak{T}_{\kappa}(\grave{z}) = \min\{\grave{z},\kappa\}, \quad \grave{z} \geqslant 0, \quad \text{when } \ \kappa > 1.$$

Moreover, $\mathfrak{M}_{\kappa}(\grave{z})$ may be defined as:

$$\mathfrak{M}_{\kappa}(\grave{z}) = (\log\kappa + 1)\grave{z} + L_{\kappa}(\grave{z}), \quad L_{\kappa}(\grave{z}) = \grave{z}(\log\grave{z} - \log\kappa)1_{\{\grave{z}\leqslant\kappa\}} - \grave{z}1_{\{\grave{z}\leqslant\kappa\}} - \kappa 1_{\{\grave{z}\leqslant\kappa\}}.$$

Furthermore, by using Lebesgue convergence theorem the approximation of $\mathfrak{M}_{\kappa}(\grave{z})$ by smooth function's sequences along with the use of (4.23) and (4.24), implies that

(4.27) $$(\mathfrak{M}(\rho_{m,t_m}))_t + \operatorname{div}[\mathfrak{M}(\rho_{m,t_m})\mathbf{v}_{m,t_m}] + \mathfrak{T}_{\kappa}(\rho_{m,t_m})\operatorname{div}\mathbf{v}_{m,t_m} = 0, \quad \text{in } \ \mathcal{D}'((O_{m,t_m},\infty)\times\mathbf{R}^3)$$

and

(4.28) $$(\mathfrak{M}(\bar{\rho}))_t + \operatorname{div}[\mathfrak{M}(\bar{\rho})\bar{\mathbf{v}}] + \mathfrak{T}_{\kappa}(\bar{\rho})\operatorname{div}\bar{\mathbf{v}} = 0, \quad \text{in } \ \mathcal{D}'(\mathbf{R}\times\mathbf{R}^3)$$

Next, by the same way as in (4.10), we get

(4.29) $$\mathfrak{M}_{\kappa}(\rho_{m,t_m}) \to \overline{\mathfrak{M}_{\kappa}(\rho)} \ \text{ in } \ C(K; L^{\gamma}_{\text{weak}}(\Omega)), \ \text{with } \overline{\mathfrak{M}_{\kappa}(\rho)} \in BC(\mathbf{R}; L^{\zeta}_{\text{weak}}(\Omega)),$$

where the boundedness of (4.29) is independent of the parameter κ and only depends on ζ.

LEMMA 4.1. [14] *In accordance with all the presumptions of Theorem 2.2, we state that*

$$\sup_{T-t_m<t}\ \sup_{m} \int_{\Omega} \rho_{m,t_m}(t)\log(\rho_{m,t_m}(t)) - \mathfrak{M}_{\kappa}(\rho_{m,t_m}(t))dx \leqslant \tilde{r}_1(\kappa),$$

where

$$\tilde{r}_1(\kappa) \to 0 \quad \textit{as} \quad \kappa \to \infty.$$

COROLLARY 4.1. *In accordance with all the presumptions of Theorem 2.2, we state that*

$$\rho_{m,t_m}(t)\log(\rho_{m,t_m}) \to \overline{\rho\log(\rho)} \quad \textit{in} \ \ C(K; L^{\zeta}_{\text{weak}}(\Omega))$$

for any fixed $\zeta \in [1,\gamma)$ and each compact interval $K \subset \mathbf{R}$. In addition,

$$\sup_{t\in\mathbf{R}} \int_{\Omega} \overline{\rho\log(\rho)}(t) - \overline{\mathfrak{M}_{\kappa}(\rho)}(t)dx \leqslant \tilde{r}_1(\kappa), \quad \tilde{r}_1(\kappa) \to 0 \quad \textit{as} \quad \kappa \to \infty.$$

Next, by applying (4.23), one can obtain

$$\partial_t \mathfrak{T}_\kappa(\rho_{m,t_m}) + \operatorname{div}(\mathfrak{T}_\kappa(\rho_{m,t_m})\mathbf{v}_{m,t_m}) + \kappa \operatorname{sgn}^+(\rho_{m,t_m} - \kappa) \operatorname{div} \mathbf{v}_{m,t_m} = 0 \quad \text{in } \mathcal{D}'((O_{m,t_m}, \infty) \times \mathbf{R}^3). \tag{4.30}$$

By passing the limit $m \to \infty$, in a similar way as in (4.10)–4.12, we get

$$\mathfrak{T}_\kappa(\rho_{m,t_m}) \to \overline{\mathfrak{T}_\kappa(\rho)} \text{ in } C(K; L^\varpi(\Omega)), \text{ with } \varpi \in [1, \infty),$$

$$\overline{\mathfrak{T}_\kappa(\rho)} \in BC(L^\zeta(\Omega)) \text{ with } \zeta \in [1, \gamma) \text{ independent of } \kappa, \tag{4.31}$$

$$\kappa \operatorname{sgn}^+(\rho_{m,t_m} - \kappa) \operatorname{div} \mathbf{v}_{m,t_m} \to \chi_\kappa \text{ weakly in } L^2(K \times \Omega) \tag{4.32}$$

for each bounded $K \subset \mathbf{R}$, with

$$\partial_t \overline{\mathfrak{T}_\kappa(\rho)} + \operatorname{div} (\overline{\mathfrak{T}_\kappa(\rho)} \bar{\mathbf{v}}) + \chi_\kappa = 0 \quad \text{in } \mathcal{D}'(\mathbf{R} \times \mathbf{R}^3) \tag{4.33}$$

here $\mathfrak{T}_\kappa$ is used to denote the cut-off function as already defined.

5. Effective viscous flux and its properties

In this section, we are going to examine the properties of the quantity

$$p(\rho) - (2\mu + \lambda) \operatorname{div} \mathbf{v}$$

known as effective viscous flux, studied in detail in [19, 21].

LEMMA 5.1. *In accordance with all the presumptions of Theorem 2.2, we state that*

$$\lim_{m\to\infty} \int_K \int_\Omega (p(\rho_{m,t_m}) - (2\mu + \lambda) \operatorname{div} \mathbf{v}_{m,t_m}) \mathfrak{T}_\kappa(\rho_{m,t_m}) dx\, dt = \int_K \int_\Omega (\overline{p(\rho)} - (2\mu + \lambda) \operatorname{div} \bar{\mathbf{v}}) \overline{\mathfrak{T}_\kappa(\rho)} dx\, dt \tag{5.1}$$

for all $\kappa = 1, 2, \dots$ and bounded interval $K \subset \mathbf{R}$.

PROOF. Taking the operators as in [21], we have $\Lambda_i[\nu] = \Delta^{-1}\partial_{x_i}(\nu)$, with $i, j = 1, 2, 3$, in particular,

$$\Lambda_i[\varsigma] = \mathfrak{F}^{-1}\Big\{ \frac{-j\varsigma_i}{|\varsigma|^2} \mathfrak{F}\{\nu\}(\varsigma) \Big\},$$

here $\mathfrak{F}$ is used to denote the Fourier transformation. Furthermore, Mikhlin multiplier theorem implies that $\|\partial_{x_j} \Lambda_i[\nu]\|_{L^q(\Omega)} \leqslant k(q)\|\nu\|_{L^q(\mathbf{R}^3)}$, $q \in (1, \infty)$ and

$$\|\Lambda_i[\nu]\|_{L^s(\Omega)} \leqslant k(a, s)\|\nu\|_{L^a(\mathbf{R}^3)}$$

with $s \in \left[a, \frac{3\gamma}{3-\gamma}\right]$ when $a \in (1, 3)$, for $a = 3$, s is arbitrary finite and $s = \infty$ for $a > 3$.

Next, using the quantities as test functions

$$\Phi_i(x, t) = \varphi(x)\Psi(t)\Lambda_i[\mathfrak{T}(\rho_{m,t_m})], \quad \Phi \in \mathcal{D}(\Omega), \quad \Psi \in \mathcal{D}(K),$$

in the momentum equation $(1.1)_2$ (by prolonging ρ_{m,t_m} outside the domain Ω is zero, as always):

$$
\begin{aligned}
(5.2)\quad & \int_K\int_\Omega \varphi\Psi[p(\rho_{m,t_m})-(2\mu+\lambda)\operatorname{div}\mathbf{v}_{m,t_m}]\mathfrak{T}(\rho_{m,t_m})dx\,dt\\
&=\int_K\int_\Omega \Psi[(\mu+\lambda)\operatorname{div}\mathbf{v}_{m,t_m}-p(\rho_{m,t_m})]\partial_{x_i}\varphi\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]dx\,dt\\
&+\mu\int_K\int_\Omega \Psi\Big\{\nabla\varphi\nabla\mathbf{v}_{m,t_m}\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]-\mathbf{v}_{m,t_m}\partial_{x_j}\varphi\partial_{x_j}\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]dx\,dt\\
&\qquad+\mu\int_K\int_\Omega \Psi\mathbf{v}_{m,t_m}\nabla\varphi\mathfrak{T}(\rho_{m,t_m})\Big\}dx\,dt\\
&-\int_K\int_\Omega \varphi\rho_{m,t_m}\mathbf{v}_{m,t_m}\Big\{\partial_t\Psi\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]+\kappa\Psi\Lambda_i[\operatorname{sgn}^+(\rho_{m,t_m}-\kappa)\operatorname{div}\mathbf{v}_{m,t_m}]\Big\}dx\,dt\\
&-\int_K\int_\Omega \Psi\Big\{\rho_{m,t_m}\mathbf{v}_{m,t_m}\otimes\mathbf{v}_{m,t_m}\partial_{x_j}\varphi\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]+\varphi\rho_{m,t_m}F_{m,t_m}\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]\Big\}dx\,dt\\
&+\int_k\int_\Omega \Psi\mathbf{v}_{m,t_m}\Big\{\mathfrak{T}_\kappa(\rho_{m,t_m})\Re_{i,j}[\varphi\rho_{m,t_m}\mathbf{v}_{m,t_m}]-\varphi\rho_{m,t_m}\mathbf{v}_{m,t}\Re_{i,j}[\mathfrak{T}_\kappa(\rho_{m,t_m})]\Big\}dx\,dt\\
&\qquad-\int_K\int_\Omega \Psi\varphi\operatorname{rot}\mathcal{H}_{m,t_m}\times H_{m,t_m}\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]dx\,dt=\sum_{j=1}^{6}\mathcal{J}_{j,m}
\end{aligned}
$$

here by using (4.30), the operator $\Re_{i,j}$ is defined by

$$\Re_{i,j}[\nu]=\mathfrak{F}^{-1}\Big\{\frac{\varsigma_i\varsigma_j}{|\varsigma|^2}\mathfrak{F}\{\nu\}(\varsigma)\Big\}.$$

Consequently, repeating the same process as above, taking into account (4.22), (4.33) together with the testing function $\Phi_i(x,t)=\Psi\varphi\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]$, we conclude

$$
\begin{aligned}
(5.3)\quad & \int_K\int_\Omega \varphi\Psi[\overline{p(\rho)}-(2\mu+\lambda)\operatorname{div}\bar{\mathbf{v}}]\overline{\mathfrak{T}(\rho)}dx\,dt\\
&=\int_K\int_\Omega \Psi[(\mu+\lambda)\operatorname{div}\bar{\mathbf{v}}-\overline{p(\rho)}]\partial_{x_i}\varphi\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]dx\,dt\\
&+\mu\int_K\int_\Omega \Psi\big\{\nabla\varphi\nabla\bar{\mathbf{v}}\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]-\bar{\mathbf{v}}\partial_{x_j}\varphi\partial_{x_j}\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]+\bar{\mathbf{v}}\cdot\nabla\varphi\overline{\mathfrak{T}(\rho)}\big\}dx\,dt\\
&-\int_K\int_\Omega \varphi\bar{\rho}\bar{\mathbf{v}}\big\{\partial_t\Psi\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]+\Psi\Lambda_i[\chi_\kappa]\big\}dx\,dt\\
&-\int_K\int_\Omega \Psi\big\{\bar{\rho}\bar{\mathbf{v}}\otimes\bar{\mathbf{v}}\partial_{x_j}\varphi\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]+\varphi\overline{\rho F}\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]\big\}dx\,dt\\
&+\int_k\int_\Omega \Psi\bar{\mathbf{v}}\big\{\overline{\mathfrak{T}_\kappa(\rho)}\Re_{i,j}[\varphi\bar{\rho}\bar{\mathbf{v}}]-\varphi\bar{\rho}\bar{\mathbf{v}}\Re_{i,j}[\overline{\mathfrak{T}_\kappa(\rho)}]\big\}dx\,dt\\
&-\int_K\int_\Omega \Psi\varphi\overline{\operatorname{rot}\mathcal{H}\times H}\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]dx\,dt=\sum_{j=1}^{6}\mathcal{J}_{j,m}.
\end{aligned}
$$

Furthermore, by using (4.31), we have $\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]\to\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]$ in $C(K\times\bar{\Omega})$ and thus, by using (4.8) and (4.18), $\mathcal{J}_{2,m}\to\mathcal{J}_2$ as $m\to\infty$. In addition, again by applying (4.8) and (4.31), we get $\partial_j\Lambda_i[\mathfrak{T}_\kappa(\rho_{m,t_m})]\to\partial_j\Lambda_i[\overline{\mathfrak{T}_\kappa(\rho)}]$ in the space

$C(K;W^{-1,2})$ and thus, $\mathcal{J}_{1,m} \to \mathcal{J}_1$ as $m \to \infty$. In a similar way, by using (4.14) for the first term in $\mathcal{J}_{3,m}$, converging to its counterpart in $\mathcal{J}_3$ and similarly as in (4.32), we get $\Lambda_i[\operatorname{sgn}^+(\rho_{m,t_m} - \kappa)\operatorname{div}\mathbf{v}_{m,t_m}] \to \Lambda_i[\chi_\kappa]$ in $L^2(\Omega \times K)$, by using the compactness of $W^{1,2} \hookrightarrow\hookrightarrow L^2$ along with the continuity of $\mathcal{A}_i\colon L^2 \to W^{1,2}$ along with the estimation obtained in (4.8), the second term in $\mathcal{J}_{3,m}$ converging to its counterpart $\mathcal{J}_3$ and we get $\mathcal{J}_{3,m} \to \mathcal{J}_3$. Furthermore, the convergence of $\mathcal{J}_{4,m}$ can be obtained by using (4.14), (4.16), (4.31) and Lemma 3.4 in [21]. Hence, we get

$$\begin{aligned}\mathfrak{T}_\kappa(\rho_{m,t_m})\Re_{i,j} m\varphi\rho_{m,t_m}\mathbf{v}_{m,t_m}] - \varphi\rho_{m,t_m}\mathbf{v}_{m,t}\Re_{i,j}[\mathfrak{T}_\kappa(\rho_{m,t_m})]\\ \longrightarrow \overline{\mathfrak{T}_\kappa(\rho)}\Re_{i,j}[\varphi\rho\mathbf{v}] - \varphi\rho\mathbf{v}\Re_{i,j}[\overline{\mathfrak{T}_\kappa(\rho)}].\end{aligned}$$

Similarly, the convergence of $\mathcal{J}_{5,m} \to \mathcal{J}_5$ can be obtained by using (4.8). Finally, $\mathcal{J}_{6,m} \to \mathcal{J}_6$ is obtained by applying the estimation obtained in (4.19)–(4.21). ☐

6. Density and momenta compactness

The aim of this section is to obtain the propagation of oscillations. The required results can be obtained using the same approach as in [21]. For the reader's convenience, we provide an outline of the proof here. For more details, please refer to [21, Sections 5–7].

Next, take any fixed arbitrary $\tilde{t}_1 < \tilde{t}_2$, then by using (4.24) and (4.27), we have

$$\begin{aligned}\int_\Omega \mathfrak{M}_\kappa(\rho_{m,t_m})(\tilde{t}_2)dx - \int_\Omega \mathfrak{M}_\kappa(\rho_{m,t_m})(\tilde{t}_1)dx\\ + \int_{\tilde{t}_1}^{\tilde{t}_2}\int_\Omega \frac{1}{2\mu+\lambda}p(\rho_{m,t_m})\mathfrak{T}_\kappa(\rho_{m,t_m})dx\,dt\\ = \int_{\tilde{t}_1}^{\tilde{t}_2}\int_\Omega \frac{1}{2\mu+\lambda}p(\rho_{m,t_m})\mathfrak{T}_\kappa(\rho_{m,t_m}) - \operatorname{div}\mathbf{v}_{m,t_m}\mathfrak{T}_\kappa(\rho_{m,t_m})dx\,dt.\end{aligned} \tag{6.1}$$

Furthermore, applying the same approach for (4.28) and using (4.25), yields

$$\begin{aligned}\int_\Omega \mathfrak{M}_\kappa(\bar\rho)(\tilde{t}_2)dx - \int_\Omega \mathfrak{M}_\kappa(\bar\rho)(\tilde{t}_1)dx + \int_{\tilde{t}_1}^{\tilde{t}_2}\int_\Omega \frac{1}{2\mu+\lambda}\overline{p(\rho)\mathfrak{T}_\kappa(\rho)}dx\,dt\\ = \int_{\tilde{t}_1}^{\tilde{t}_2}\int_\Omega \frac{1}{2\mu+\lambda}\overline{p(\rho)\mathfrak{T}_\kappa(\rho)} - \operatorname{div}\bar{\mathbf{v}}\overline{\mathfrak{T}_\kappa(\rho)}dx\,dt.\end{aligned} \tag{6.2}$$

Furthermore, by taking the difference of (6.1), (6.2) and using Lemma 5.1 along with (4.29), we obtain

$$\begin{aligned}\int_\Omega \big(\overline{\mathfrak{M}_\kappa(\rho)} - \mathfrak{M}_\kappa(\bar\rho)(\tilde{t}_2)\big)dx - \int_\Omega \big(\overline{\mathfrak{M}_\kappa(\rho)} - \mathfrak{M}_\kappa(\bar\rho)(\tilde{t}_1)\big)dx\\ + \frac{1}{2\mu+\lambda}\lim_{m\to\infty}\sup\int_{\tilde{t}_1}^{\tilde{t}_2}\int_\Omega p(\rho_{m,t_m})\mathfrak{T}_\kappa(\rho_{m,t_m}) - \overline{p(\rho)\mathfrak{T}_\kappa(\rho)}dx\,dt\\ \leqslant \int_{\tilde{t}_1}^{\tilde{t}_2}\int_\Omega \overline{\mathfrak{T}_\kappa(\rho)} - \mathfrak{T}_\kappa(\bar\rho)\operatorname{div}\bar{\mathbf{v}}\,dx\,dt.\end{aligned} \tag{6.3}$$

To proceed further, we need the following essential lemmas:

Lemma 6.1. [15] *For any* $\hbar > 0$*, we have*

$$\lim_{m\to\infty} \sup \int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega p(\rho_{m,t_m})\mathfrak{T}_\kappa(\rho_{m,t_m}) - \overline{p(\rho)\mathfrak{T}_\kappa(\rho)} dx\, dt + \tilde{r}_2(\kappa)(\tilde{t}_2 - \tilde{t}_1) \geqslant \hbar \int_{\tilde{t}_1}^{\tilde{t}_2} \Psi\bigg(\int_\Omega (\overline{\mathfrak{M}_\kappa(\rho)} - \mathfrak{M}_\kappa(\bar{\rho}))dx \bigg) dt,$$

where $\tilde{r}_2(\kappa) \to 0$ *as* $\kappa \to \infty$ *and* Ψ *is used to represent the convex function as given in* [15, Lemma 5.3].

Lemma 6.2. [20] *In accordance with all the presumptions of Theorem* 2.2*, we state that*

$$\sup_{\kappa>1} \lim_{m\to\infty} \sup \int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega |\mathfrak{T}_\kappa(\rho_{m,t_m}) - \mathfrak{T}_\kappa(\bar{\rho})|^{1+\gamma} dx\, dt \leqslant \mathcal{L}(\tilde{t}_1, \tilde{t}_2).$$

In order to continue with the remaining computations, we apply Lemma 6.1 and Corollary 4.1 for passing the limit as $\kappa \to \infty$ in (6.3), and obtain

$$\int_\Omega \vartheta(\tilde{t}_2, x)dx - \int_\Omega \vartheta(\tilde{t}_1, x)dx + \frac{\hbar}{2\mu + \lambda} \int_{\tilde{t}_1}^{\tilde{t}_2} \Psi\bigg(\int_\Omega \vartheta(t,x)dx \bigg) dt \leqslant \lim_{\kappa\to\infty} \sup \bigg| \int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega (\overline{\mathfrak{T}_\kappa(\rho)} - \mathfrak{T}_\kappa(\bar{\rho})) \operatorname{div} \bar{\mathbf{v}}\, dx\, dt \bigg|, \tag{6.4}$$

where ϑ is used to denote the "defect measure" as defined in (4.1). Further, from (6.4), we get

$$\begin{aligned} &\bigg| \int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega (\overline{\mathfrak{T}_\kappa(\rho)} - \mathfrak{T}_\kappa(\bar{\rho})) \operatorname{div} \bar{\mathbf{v}}\, dx\, dt \bigg|, \\ &\quad \leqslant \bigg| \int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega (\overline{\mathfrak{T}_\kappa(\rho)} - \mathfrak{T}_\kappa(\bar{\rho}))^2 dx\, dt \bigg|^{\frac{1}{2}} \bigg| \int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega |\operatorname{div} \bar{\mathbf{v}}|^2 dx\, dt \bigg|^{\frac{1}{2}} \\ &\quad \leqslant \bigg(\int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega |\overline{\mathfrak{T}_\kappa(\rho)} - \mathfrak{T}_\kappa(\bar{\rho})| dx\, dt \bigg)^{\frac{\gamma-1}{2\gamma}} \bigg(\int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega |\overline{\mathfrak{T}_\kappa(\rho)} - \mathfrak{T}_\kappa(\bar{\rho})|^{1+\gamma} dx\, dt \bigg)^{\frac{1}{2\gamma}} \\ &\quad\quad \times \bigg(\int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega |\operatorname{div} \bar{\mathbf{v}}|^2 dx\, dt \bigg)^{\frac{1}{2}}. \end{aligned}$$

Now, by applying (4.9), (4.18) and Lemma 6.2 along with the inequality

$$\bigg(\int_{\tilde{t}_1}^{\tilde{t}_2} \int_\Omega \big|\overline{\mathfrak{T}_\kappa(\rho)} - \mathfrak{T}_\kappa(\bar{\rho})\big|^q dx\, dt \bigg)^{\frac{1}{q}} \leqslant \mathcal{L}(\tilde{t}_1, \tilde{t}_2) \kappa^{(\frac{1}{\Theta(\gamma)+\gamma} - \frac{1}{q})(\Theta(\gamma)+\gamma)},$$

with $\kappa > 0, q \in [1, \Theta(\gamma) + \gamma)$ and hence it is proved that as $\kappa \to \infty$, the right-hand-side term of (6.4) tends to zero.

Furthermore, taking into account the estimations stated in (4.26), (4.29) and Corollary 4.1, it is implied that

$$\Xi(t) = \int_\Omega \vartheta(t,x)dx$$

is globally bounded and continuous throughout the entire real line $\mathbf{R}$. Next, by applying (6.4), the following inequality holds such that

$$\Xi(\tilde{t}_2) - \Xi(\tilde{t}_1) + \frac{\hbar}{2\mu+\lambda}\int_{\tilde{t}_1}^{\tilde{t}_2} \Psi(\Xi(t))dt \leqslant 0. \tag{6.5}$$

The continuity of Ξ on O and (6.5) implies that

$$\Xi(\tilde{t}_2) \leqslant \chi(\tilde{t}_2 - \tilde{t}_1), \tag{6.6}$$

where χ is referred to as a unique solution to the problem $\chi'(t) + \frac{\hbar}{2\mu+\lambda}\Psi(\chi(t)) = 0$, $\chi(0) = \Xi(\tilde{t}_1)$. Subsequently, the function χ indicates the uniform rate at which the amplitude of possible oscillations declines over time that is independent of the forcing term's norm and the energy's upper bound.

Hence, it shows that $\Xi \equiv 0$ and thus (2.13) is proved. Moreover, using the strong convergence of ρ_{m,t_m}, it implies that

$$\overline{\rho F} = \bar{\rho}\bar{F}, \quad \text{and} \quad \overline{p(\rho)} = p(\bar{\rho}).$$

Thus taking into account (4.22), $\bar{\rho}, \bar{\mathbf{v}}, \bar{\mathcal{H}}$ is said to be the globally defined weak solutions of the problem (1.1)–(1.3).

Furthermore, with the use of the estimations obtained in (2.13), (4.10) and (4.16), we obtain

$$\begin{aligned}\int_O \mathcal{E}(\bar{\rho}, \bar{\mathbf{v}}, \bar{\mathcal{H}})(t)dt &= \int_O \Big(\frac{1}{2}\|\sqrt{\bar{\rho}}\bar{\mathbf{v}}\|^2_{L^2(\Omega)} + \frac{a}{\gamma-1}\|\bar{\rho}\|^2_{L^\gamma(\Omega)} + \frac{1}{2}\|\mathcal{H}\|^2_{L^2(\Omega)}\Big)dt \\ \leqslant \lim_{m\to\infty}\inf \int_O \Big(\frac{1}{2}\|\sqrt{\rho_{m,t_m}}\mathbf{v}_{m,t_m}\|^2_{L^2(\Omega)} &+ \frac{a}{\gamma-1}\|\rho_{m,t_m}\|^2_{L^\gamma(\Omega)} + \frac{1}{2}\|\mathcal{H}_{m,t_m}\|^2_{L^2(\Omega)}\Big)dt \\ &\leqslant |O|\mathcal{E}_\infty\end{aligned}$$

for any arbitrary O. Hence, $\operatorname{ess\,sup}_{t\in\mathbf{R}} \mathcal{E}(t) \leqslant \mathcal{E}_\infty$.

Next, to prove the required result, it is necessary to obtain the compactness of the momenta as given in the relation (2.15). For this, write that

$$\rho_{m,t_m}\mathbf{v}_{m,t_m} = (\rho_{m,t_m})^{\frac{1}{2}}(\rho_{m,t_m})^{\frac{1}{2}}\mathbf{v}_{m,t_m},$$

further, by using (2.13), we get

$$(\rho_{m,t_m})^{\frac{1}{2}} \to (\bar{\rho})^{\frac{1}{2}} \quad \text{strongly converges in the space} \quad L^2((0,1)\times\Omega) \tag{6.7}$$

whereby applying (4.3), it is shown that

$$(\rho_{m,t_m})^{\frac{1}{2}}\mathbf{v}_{m,t_m} \to (\bar{\rho})^{\frac{1}{2}}(\bar{\mathbf{v}}) \quad \text{weakly in} \quad (L^2((0,1)\times\Omega))^3. \tag{6.8}$$

Next, applying (4.16) implies that

$$\begin{aligned}\|(\rho_{m,t_m})^{\frac{1}{2}}\mathbf{v}_{m,t_m}\|^2_{L^2(K\times\Omega)} &= \int_K\int_\Omega |(\rho_{m,t_m})^{\frac{1}{2}}\mathbf{v}_{m,t_m}|^2 dx\,dt \\ &\longrightarrow \int_K\int_\Omega \bar{\rho}|\bar{\mathbf{v}}|^2 dx\,dt = \|(\bar{\rho})^{\frac{1}{2}}\bar{\mathbf{v}}\|^2_{L^2((0,1)\times\Omega)}.\end{aligned}$$

By applying the strong convergence of (6.8), we get (2.15) and (2.16). This further implies that $\rho_{m,t_m}F_{m,t_m}\cdot\mathbf{v}_{m,t_m} \longrightarrow \bar{\rho}\bar{F}\cdot\bar{\mathbf{v}}$ in $\mathcal{D}(K\times\Omega)$. As a result, the energy inequality (2.5) follows for $\bar{\rho}, \bar{\mathbf{v}}, \bar{\mathcal{H}}$. Thus, it completes the proof of Theorem 2.2.

7. Proof of Theorems 2.3–2.4

It can be seen Theorem 2.3 can be immediately derived from Theorem 2.2 by contradiction. Additionally, we recognize that the proof of Theorem 2.4 follows directly from Theorems 2.1 and 2.2 in accordance with Remark 2.1. Here, we outline the proof of Theorem 2.4.

$$\mathcal{E}(t) \leqslant \mathcal{E}_\infty, \quad \forall T(\mathcal{E}_0, \mathfrak{m}, t_0) \leqslant t,$$

and

$$\begin{aligned} \|\rho_m(t_m) - \bar{\rho}\|_{L^\varsigma(\Omega)} &+ \|\mathcal{H}_m(t_m) - \bar{\mathcal{H}}\|_{L^\varsigma(\Omega)} \\ &+ \left| \int_\Omega [(\rho \mathbf{v})(t_m) - (\overline{\rho \mathbf{v}})] \cdot \varphi \, dx \right| \geqslant k, \quad \forall (\bar{\rho}, \bar{\mathbf{v}}, \bar{\mathcal{H}}) \in \tilde{\mathcal{A}}[\tilde{\mathcal{F}}]. \end{aligned} \tag{7.1}$$

Next, by using Theorem 2.2, we may assume a global trajectory $\big(\bar{\rho}, \bar{\mathbf{v}}, \bar{\mathcal{H}}\big)$ such that

$$\sup_{t \in [0,1]} \|\rho_m(t_m) - \bar{\rho}\|_{L^\varsigma(\Omega)} \to 0, \quad \text{as} \quad t_m \to \infty, \tag{7.2}$$

$$\sup_{t \in [0,1]} \|\mathcal{H}_m(t_m) - \bar{\mathcal{H}}\|_{L^\varsigma(\Omega)} \to 0, \quad \text{as} \quad t_m \to \infty, \tag{7.3}$$

and

$$\sup_{t \in [0,1]} \left| \int_\Omega \big[(\rho \mathbf{v})(t_m) - (\overline{\rho \mathbf{v}})\big] \cdot \varphi q \, dx \right| \to 0, \quad \text{as} \quad t_m \to \infty. \tag{7.4}$$

Thus, in contrast to (7.1), we have (7.2), (7.3) and (7.4). Hence, it completes the proof of Theorem 2.4.

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ГЛОБАЛНИ КОМПАКТНИ АТРАКТОРИ И КОМПЛЕТНЕ ОГРАНИЧЕНЕ ТРАЈЕКТОРИЈЕ ЗА СТИШЉИВИ МАГНЕТОХИДРОДИНАМИЧКИ СИСТЕМ ЈЕДНАЧИНА

Резиме. У овом чланку истражујемо глобално понашање слабих решења магнетохидродинамичког (МХД) флуида у тродимензионалном ограниченом домену са компактном Липшицовом границом на кога делују произвољне силе. Показали смо постојање глобалних компактних атрактора под специфичним ограничењима на адијабатску константу γ.

Department of Mathematics
Shanghai University
Shanghai
P. R. China
janmath@shu.edu.cn
https://orcid.org/0000-0001-8872-2523

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