

THE EFFECT OF A HYPOCYCLOID IRREGULARITY ON TORSIONAL SURFACE WAVES IN A MULTILAYER SYSTEM

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ABSTRACT. The present topic is devoted to investigating the effects of semi four-cusped hypocycloid irregularity on torsional surface waves in an initially stressed anisotropic porous medium situated between two anisotropic non-homogeneous semi-infinite spaces. The irregularity manifests as a semi-four-cusped hypocycloid at the interface where the layer and the lower semi-infinite space are separated. It is assumed that the directional rigidities, density and initial stress vary in the upper and lower semi-infinite spaces in an exponential and hyperbolic manner, respectively. A closed form for the torsional wave dispersion equation is presented. Additionally, the velocity equation is found in the case of no irregularity. The study demonstrates that heterogeneity of lower semi-infinite space, initial stress and directional rigidities of both semi-infinite spaces, as well as of porosity of the layer have a favourable effect on the phase velocity of torsional surface waves. However, the heterogeneity of the upper semi-infinite space, ratio of directional rigidities and initial stress of the layer and irregularity parameter have a negative impact. It has also been observed that, in the case of a uniform media, the velocity equation reduces to the conventional equation for the Love wave.

1. Introduction

Earthquakes are the result of the rapid motion along the faults in the Earth's crust, which is usually caused by the sudden breaking of underground rocks. The seismic waves produced by this sudden change cause the Earth to quiver. Volcanic eruptions, landslides, snowfalls and even rushing rivers can also cause earthquake waves. The study of seismic waves is important in order to understand how heterogeneous and anisotropic the Earth's inner core is. There are two types of seismic waves: body waves and surface waves. The waves which originate at the epicentre of an earthquake and travel through the Earth are body waves. The surface wave is the wave that travels over the Earth's surface. Most conjectures about the Earth's internal structure are made by studying seismic waves. Torsional waves are one

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of several types of surface waves. Their amplitude decreases exponentially as the distance from the free surface increases.

In the crust of the Earth, heterogeneity occurs at all levels. Thus, surface wave propagation in an inhomogeneous layer is very important for seismology and earthquake engineering. The study of surface waves in layered and heterogeneous media has become a major focus for theoretical seismologists in recent years. An iconic and well-known aspect of wave theory is dealing with surface waves propagating over the surface of homogeneous and heterogeneous elastic half-spaces. In a well-known book by Ewing et al. [1], a significant quantity of information regarding seismic wave propagation is documented. Love [2] and Gubbins [3] provided details on the Earth's plate tectonics. According to Gupta et al. [4], while the heterogeneity parameter related to rigidity has the opposite impact, the one related to phase velocity decreases as the density of the crystal layer grows. Abd-Alla et al. [5] analysed shear wave velocity of propagation depending on various factors. In the framework of the three-phase-lag model of thermoelasticity, Kumar et al. [6] investigated wave propagation in an anisotropic viscoelastic medium. According to Kakar et al. [7], the shear wave velocity is directly impacted by parameter variations related to the medium's non-homogeneity. Manna et al. [8] noted how inhomogeneity and reinforcement affected the Love wave velocity. Alam et al. [9] observed the influence hydrostatic stress, heterogeneity and magneto-elastic coupling parameter on Love-type wave.

The solid portion of Earth's crust can be regarded as somewhat anelastic in accordance with the tidal hypothesis (Lowrie [10]). Seabeds and intermediate strata beneath the surface are exposed to initial stress, according to the seismic record. There are numerous causes for the medium's initial stresses to arise: the variations in temperature, pressure because of the overloaded layer, quenching process, sluggish creep process, and gravity. These stresses can have a big impact on elastic waves that are created by impacts, explorations, or earthquakes. Dealing with wave propagation problems under initial stress is therefore essential. Dey et al. [11] concluded that when initial compressive stresses are present, the velocity of torsional surface waves is reduced. Ozturk et al. [12] reported and discussed the effect of initial stress on the phase velocity of a torsional wave. Ahmed et al. [13] determined the dispersion equation using the Fourier transform technique to show the influence on Love waves. The impact of initial stress, corrugation and porosity factor on velocity profile of SH waves was analysed by Gupta et al. [14]. For electrically open and electrically short circuits, Kumar et al. [15] showed the influence of initial stress on Rayleigh wave.

Scientists studying seismology and physics have recently become interested in the phenomenon of wave propagation in fluid-saturated porous media. The empty spaces that exist inside the formation are referred to as pore space or void. When the formation is water-saturated, these pore spaces are occasionally filled with formation water. Otherwise, they are often filled with air. The production of porous materials and the science of geophysics are two potential uses for this idea. Seismologists are interested in how porosity in a medium affects elastic wave propagation. Wang et al. [16] used the Biot's theory to obtain the Love wave dispersion equation.

Son et al. [17] showed that the phase velocity increases with the growth of the pore size. Shekhar et al. [18] investigated the impact of porosity on velocity profile of torsional waves. Alam et al. [19] investigated the effects of initial stresses, heterogeneity, porosity and sandy characteristics on the propagation of torsional surface waves. By using graphical representations and discussions supported by numerical data, the significant impact of numerous influencing elements, such as gravity, initial stress, porosity and viscosity on Love waves has been thoroughly examined by Kumhar et al. [20].

A composite structure in the real world may not always contain a layer with a planar border alone. The surface of the earth is irregular in spite of millions of years of gradation. Since the mass of the Earth is unevenly distributed, certain parts of the world are subject to a stronger gravitational “pull” than others. The “height” of various geoid components varies constantly, rising and falling in reaction to gravity as a result of these fluctuations in gravitational force. So it is more realistic to consider the irregularity effect on seismic waves. The possibility of irregularities has led to an endeavor to investigate the impact of hypocycloid-type irregularities on the torsional surface wave’s propagation behavior in order to gain a practical understanding of the issue. The theoretical formulation of modes and surface waves in multi-layered media with irregular interfaces has been successfully established by Chen [21]. Chattaraj et al. [22] obtained the displacement field in the porous layer using the perturbation method, with the assumption that the irregularity is rectangular. The impact of shifting loads owing to irregularities in ice sheets floating on water was investigated by Chatterjee et al. [23]. Using predetermined boundary conditions, the internal deformations were analytically calculated by Chowdhury et al. [24] to produce the wave frequency equation. In a prestressed irregular composite layer, Singh et al. [25] examined the behavior of Love-type waves.

The purpose of the present manuscript is to investigate torsional surface waves as they move through a porous medium that is positioned between two semi-infinite spaces. A semi-infinite space is a mathematical model used to approximate the Earth when performing some calculations in seismology and bounded in one direction and unbounded in another. In the present model, lower semi-infinite space is taken beneath the layer where z varies from $\varepsilon G(r)$ to ∞ and upper semi-infinite space is above the layer where z varies from $-H$ to $-\infty$. The initial stress, density and directional rigidities vary in the upper and lower semi-infinite spaces in an exponential and hyperbolic manner respectively and the irregularity appears as a semi four-cusped hypocycloid at the interface where the layer and lower semi-infinite space are separated. This study’s consideration of exponential and hyperbolic fluctuations in density, directional rigidities and initial stress in upper and lower media is motivated by the Earth’s crust’s numerous heterogeneous strata, each with its own unique set of geological properties. We have employed the separation of variables method to get the dispersion equation. Thus the article addresses the applicability of initial stress, irregularity, porosity, anisotropy and heterogeneity factor on the velocity profile of torsional surface waves which are graphically illustrated. It has also been noted that the velocity equation simplifies to the Love wave’s classical solution for a uniform media.

Lower semi-infinite space heterogeneity features have a positive impact on torsional wave phase velocities, whereas upper semi-infinite space heterogeneity features have a negative effect. Phase velocity is increasingly influenced by anisotropic parameters of the upper and lower semi-infinite spaces; for porous layers, their impacts are lessening. Porosity is found to have a significant increasing impact. The phase velocity is affected more by initial stresses of the semi-infinite spaces than by those of the porous layer. Distinct irregularity sizes have an impact on the phase velocity of torsional surface waves as well. The present finding can be possibly seen in various practical situations. There is a possibility of finding a geometrical model beneath the Earth's surface. The study of the propagation behaviour of torsional wave through such a medium may help geophysicists and seismologists in the prediction of earthquake and in understanding the damages caused by earthquakes. We have considered a unique model that lies in the Earth's crust.

2. Foundation of the problem

We consider a cylindrical coordinate system in order to explore torsional surface waves. The model is an anisotropic porous layer of thickness H that lies between two anisotropic heterogeneous half-spaces, with one semi four-cusped type irregularity at the lower interface under radially directed compressive initial stresses $P_1 = -\chi_{rr}$. We take into account the semi-ellipse's length h for the semi-minor axis and $2m$ for the major axis.

The interface dividing the layer from the lower semi-infinite region is the origin of the cylindrical coordinate system (r, θ, z) , with z pointing downward. Rigidity, density and initial stress variations as follows are taken into consideration:

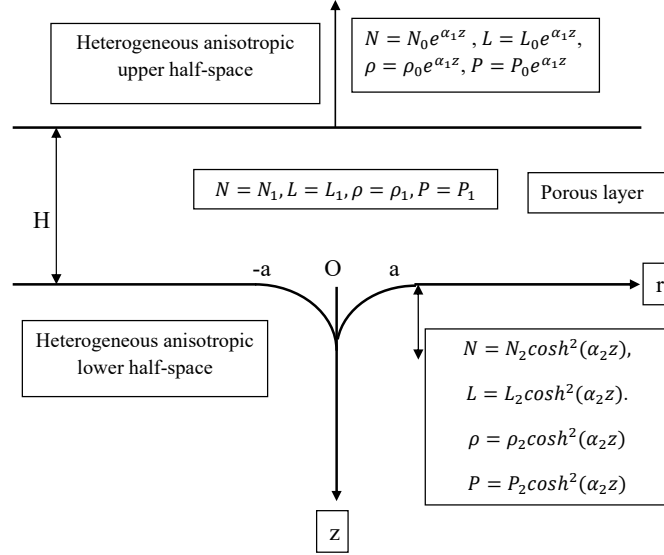


FIGURE 1. Problem's geometric shape

Regarding the upper semi-infinite space,

$$(2.1) \quad \begin{aligned} N &= N_0 e^{\alpha_1 z}, & L &= L_0 e^{\alpha_1 z}, \\ \rho &= \rho_0 e^{\alpha_1 z}, & P &= P_0 e^{\alpha_1 z} \end{aligned}$$

For the porous layer,

$$N = N_1, \quad L = L_1, \quad \rho = \rho_1, \quad P = P_1$$

For lower semi-infinite space,

$$(2.2) \quad \begin{aligned} N &= N_2 \cosh^2(\alpha_2 z), & L &= L_2 \cosh^2(\alpha_2 z), \\ \rho &= \rho_2 \cosh^2(\alpha_2 z), & P &= P_2 \cosh^2(\alpha_2 z) \end{aligned}$$

P represents the initial stress at each location in the medium, the density is ρ , the directional rigidities along radial and axial directions are N , L and constants α_1 , α_2 have inverse dimensions related to length.

The irregularity equation is thought to be

$$z = \varepsilon G(r),$$

where

$$\begin{aligned} G(r) &= b \sqrt[3]{1 - \left(\frac{r}{a}\right)^{\frac{2}{3}}}; & -a \leq r \leq a, \\ &= 0; & \text{otherwise,} \end{aligned}$$

and

$$\varepsilon = \frac{b}{2a}, \quad \varepsilon \ll 1.$$

3. Solution for semi-infinite spaces

The dynamical equation of motion (Biot [26]) of an initially stressed anisotropic semi-infinite space can be written as follows if the wave only travels in a radial direction

$$(3.1) \quad \frac{\partial \chi_{r\theta}}{\partial r} + \frac{\partial \chi_{\theta z}}{\partial z} + \frac{2}{r} \chi_{r\theta} - \frac{\partial}{\partial z} \left(\frac{P}{2} \frac{\partial v}{\partial z} \right) = \rho \frac{\partial^2 v}{\partial t^2}$$

where displacement along azimuthal direction is v , the density is ρ , the radially directed initial stress is P and the stress components are $\chi_{r\theta}$ and $\chi_{z\theta}$. The torsional wave is characterized by the displacements

$$u = 0, \quad w = 0, \quad v = v(r, z, t)$$

The strain-displacement relations are

$$\left. \begin{aligned} e_{rr} &= \frac{1}{2} \frac{\partial u}{\partial r}, & e_{\theta\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right), & e_{zz} &= \frac{1}{2} \frac{\partial w}{\partial z} \\ e_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right), & e_{\theta z} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right), \\ e_{zr} &= \frac{1}{2} \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right) \end{aligned} \right\}$$

The stress-strain relationships are now provided by

$$\left. \begin{aligned} \chi_{rr} &= (A + P)e_{rr} + (A - 2N + P)e_{\theta\theta} + (F + P)e_{zz} + Q_\varepsilon, \\ \chi_{\theta\theta} &= (A - 2N)e_{rr} + Ae_{\theta\theta} + Fe_{zz} + Q_\varepsilon, \\ \chi_{zz} &= Fe_{rr} + Fe_{\theta\theta} + Ce_{zz} + Q_\varepsilon, \\ \chi_{r\theta} &= 2Ne_{r\theta}, \chi_{\theta z} = 2Le_{\theta z}, \chi_{rz} = 2Le_{zr} \end{aligned} \right\}$$

where the medium's elastic constants are A , F , and C . The coupling between the change in the solid and liquid's volumes is measured by Q_ε .

At this point, the displacement component and stress components for anisotropic elastic semi-infinite space are connected by

$$(3.2) \quad \chi_{r\theta} = N \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right), \quad \chi_{z\theta} = L \frac{\partial v}{\partial z}$$

By employing the relationships (3.2), the motion equation (3.1) is transformed into

$$(3.3) \quad N \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) v + \frac{\partial}{\partial z} \left(M \frac{\partial v}{\partial z} \right) = \rho(z) \frac{\partial^2 v}{\partial t^2},$$

where $M = L - \frac{P}{2}$. We consider the following as the form of the solution to equation (3.3):

$$(3.4) \quad v = V(z) J_1(kr) e^{i\omega t}$$

where $J_1(kr)$ represents the first order and first kind Bessel's function, the spatial frequency of wave is k and the angular rate is ω .

Equation (3.3) assumes the following form with the aid of equation (3.4)

$$(3.5) \quad N \left(\frac{k^2 r^2 J_1''(kr) + kr J_1'(kr) - J_1(kr)}{r^2} \right) V e^{i\omega t} + \left(M \frac{d^2 V}{dz^2} + \frac{dM}{dz} \frac{dV}{dz} \right) J_1(kr) e^{i\omega t} = -\rho \omega^2 V J_1(kr) e^{i\omega t}$$

or,

$$N \left(-\frac{k^2 r^2 J_1(kr)}{r^2} \right) V + \left(M \frac{d^2 V}{dz^2} + \frac{dM}{dz} \frac{dV}{dz} \right) J_1(kr) = -\rho \omega^2 V J_1(kr)$$

or,

$$\frac{d^2 V}{dz^2} + \left(\frac{1}{M} \frac{dM}{dz} \right) \frac{dV}{dz} - \frac{k^2 N}{M} \left(1 - \frac{\rho c^2}{N} \right) V = 0$$

The torsional wave velocity is represented by $c = \frac{\omega}{k}$ in the above equation. To remove the expression $\frac{dV}{dz}$ from the previous equation, substitute $V(z) = \frac{\phi(z)}{\sqrt{M}}$. This yields

$$(3.6) \quad \frac{d^2 \phi}{dz^2} + \frac{1}{4M^2} \left(\frac{dM}{dz} \right)^2 \phi - \frac{1}{2M} \frac{d^2 M}{dz^2} \phi - \frac{k^2 N}{M} \left(1 - \frac{\rho c^2}{N} \right) \phi = 0.$$

3.1. Solution for upper semi-infinite space. Using equation (2.1), equation (3.6) reduces to

$$(3.7) \quad \frac{d^2\phi}{dz^2} - m_0^2\phi = 0,$$

where $m_0 = k\sqrt{\frac{1}{4}\left(\frac{\alpha_1}{k}\right)^2 + \frac{N_0/L_0}{1-\xi_0}\left(1 - \frac{c^2}{c_0^2}\right)}$, $c_0 = \sqrt{\frac{N_0}{\rho_0}}$ = shear wave velocity in this half-space and $\xi_0 = \frac{P_0}{2L_0}$ = dimensionless parameter due to initial stress P_0 . When $z \rightarrow -\infty$, the solution to equation (3.7) disappears and can be expressed as

$$\phi(z) = A_1 e^{m_0 z}$$

For the upper semi-infinite space, the displacement component becomes

$$(3.8) \quad v = v_0(\text{say}) = A_1 \frac{e^{m_0 z} J_1(kr) e^{i\omega t}}{e^{\frac{\alpha_1 z}{2}}}$$

3.2. Solution for lower semi-infinite space. Using equation (2.2), equation (3.6) reduces to

$$(3.9) \quad \frac{d^2\phi}{dz^2} - m_2^2\phi = 0,$$

where $m_2 = k\sqrt{\left(\frac{\alpha_2}{k}\right)^2 + \frac{N_2/L_2}{1-\xi_2}\left(1 - \frac{c^2}{c_2^2}\right)}$, $c_2 = \sqrt{\frac{N_2}{\rho_2}}$ = the shear wave velocity in this semi-infinite space and $\xi_2 = \frac{P_2}{2L_2}$ = dimensionless parameter due to initial stress P_2 . The solution of equation (3.9) vanishing at $z \rightarrow \infty$ may be written as

$$\phi(z) = C_1 e^{-m_2 z}$$

The displacement component in this space is so

$$(3.10) \quad v = v_2(\text{say}) = C_1 \frac{e^{-m_2 z} J_1(kr) e^{i\omega t}}{\cosh(\alpha_2 z)}$$

4. Solution for layer

An anisotropic porous medium's dynamical equations of motion (Biot [26]) are as follows when initial stress P is applied without body force:

$$\left. \begin{aligned} \frac{\partial \chi_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \chi_{r\theta}}{\partial \theta} + \frac{\partial \chi_{rz}}{\partial z} + \frac{\chi_{rr} - \chi_{\theta\theta}}{r} - P_1 \frac{\partial w'_\theta}{\partial z} &= \frac{\partial^2}{\partial t^2} (\rho_{rr} u + \rho_{r\theta} U), \\ \frac{\partial \chi_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \chi_{\theta\theta}}{\partial \theta} + \frac{\partial \chi_{\theta z}}{\partial z} + \frac{2\chi_{r\theta}}{r} - P_1 \frac{\partial w'_z}{\partial r} &= \frac{\partial^2}{\partial t^2} (\rho_{rr} v + \rho_{r\theta} V), \\ \frac{\partial \chi_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \chi_{\theta z}}{\partial \theta} + \frac{\partial \chi_{zz}}{\partial z} + \frac{\chi_{rz}}{r} - P_1 \frac{\partial w'_\theta}{\partial r} &= \frac{\partial^2}{\partial t^2} (\rho_{rr} w + \rho_{r\theta} W) \end{aligned} \right\}$$

and

$$(4.1) \quad \left. \begin{aligned} \frac{\partial \chi}{\partial r} &= \frac{\partial^2}{\partial t^2} (\rho_{r\theta} u + \rho_{\theta\theta} U), \\ \frac{\partial \chi}{\partial \theta} &= \frac{\partial^2}{\partial t^2} (\rho_{r\theta} v + \rho_{\theta\theta} V), \\ \frac{\partial \chi}{\partial r} &= \frac{\partial^2}{\partial t^2} (\rho_{r\theta} w + \rho_{\theta\theta} W), \end{aligned} \right\}$$

where the liquid's stress is denoted by χ , the solid's displacement components are (u, v, w) , the liquid's displacement components are (U, V, W) , the respective stress components are χ_{rr} , $\chi_{\theta\theta}$, χ_{zz} , χ_{rz} , $\chi_{r\theta}$, and $\chi_{\theta z}$ and

$$w'_r = \frac{1}{2r} \left(\frac{\partial w}{\partial \theta} - r \frac{\partial v}{\partial z} \right), w'_\theta = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right), w'_z = \frac{1}{2r} \left(\frac{\partial(rv)}{\partial r} - r \frac{\partial v}{\partial \theta} \right)$$

are the components that make up the rotational vector w' . The porous layer's stress-strain relationships are now provided by

$$\left. \begin{aligned} \chi_{rr} &= (A + P_1)e_{rr} + (A - 2N_1 + P_1)e_{\theta\theta} + (F + P_1)e_{zz} + Q_\varepsilon, \\ \chi_{\theta\theta} &= (A - 2N_1)e_{rr} + Ae_{\theta\theta} + Fe_{zz} + Q_\varepsilon, \\ \chi_{zz} &= Fe_{rr} + Fe_{\theta\theta} + Ce_{zz} + Q_\varepsilon, \\ \chi_{r\theta} &= 2N_1e_{r\theta}, \chi_{\theta z} = 2L_1e_{\theta z}, \chi_{rz} = 2L_1e_{zr}, \end{aligned} \right\}$$

where the medium's elastic constants are A , F , and C . The radial and axial shear moduli are N_1 and L_1 , respectively and the coupling between the change in the solid and liquid's volumes is measured by Q_ε . The strain-displacement relations are

$$\left. \begin{aligned} e_{rr} &= \frac{1}{2} \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right), \quad e_{zz} = \frac{1}{2} \frac{\partial w}{\partial z} \\ e_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right), \quad e_{\theta z} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right), \\ e_{zr} &= \frac{1}{2} \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right) \end{aligned} \right\}$$

The fluid pressure P_1 and the stress vector χ have the following relationship:

$$-\chi = fP_1,$$

where f = porosity of medium. In relation to the layer density ρ , solid density ρ_s and water density ρ_w , the mass coefficients ρ_{rr} , $\rho_{r\theta}$, and $\rho_{\theta\theta}$ are

$$\frac{\rho_{rr} + \rho_{r\theta}}{\rho_s} = (1 - f), \quad \frac{\rho_{r\theta} + \rho_{\theta\theta}}{\rho_w} = f.$$

Therefore, the aggregate's mass density is

$$\frac{\rho_1 - \rho_s}{\rho_w - \rho_s} = f.$$

According to Biot, the mass coefficients are positive and

$$\rho_{rr}\rho_{\theta\theta} - \rho_{r\theta}^2 > 0.$$

The torsional surface wave has the property that

$$u = 0, \quad w = 0, \quad v = v(r, z, t), \quad U = 0, \quad W = 0, \quad V = V(r, z, t)$$

These provide two non-zero stress components as

$$(4.2) \quad \chi_{\theta z} = L_1 \frac{\partial v}{\partial z}, \quad \chi_{r\theta} = N_1 \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)$$

By substituting the relations (4.2) in equations (4) and (4.1) one obtains

$$(4.3) \quad \left(N_1 - \frac{P_1}{2} \right) \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) + L_1 \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2}{\partial t^2} (\rho_{rr}v + \rho_{r\theta}V)$$

and

$$\frac{\partial^2}{\partial t^2}(\rho_{r\theta}v + \rho_{\theta\theta}V) = 0.$$

One way to express the final equation is as

$$(4.4) \quad \rho_{r\theta}v + \rho_{\theta\theta}V = k_1(\text{say}).$$

When V is removed from Equations (4.3) and (4.4), one obtains

$$(4.5) \quad \left(N_1 - \frac{P_1}{2}\right) \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + L_1 \frac{\partial^2 v}{\partial z^2} = d_1 \frac{\partial^2 v}{\partial t^2}$$

where

$$d_1 = \rho_{rr} - \frac{\rho_{r\theta}^2}{\rho_{\theta\theta}}.$$

According to Equation (4.5), shear wave's velocity in a radial direction is

$$\sqrt{\frac{N_1 - \frac{P_1}{2}}{d_1}} = \sqrt{\frac{1 - \xi_1}{d}} c_1^2,$$

where

$$c_1 = \sqrt{\frac{N_1}{\rho_1}} = \text{the shear wave velocity in the medium,}$$

$$\xi_1 = \frac{P_1}{2N_1} = \text{initial stress } P_1 \text{ related dimensionless parameter,}$$

$$d = \frac{d_1}{\rho_1} = \gamma_{11} - \frac{\gamma_{12}^2}{\gamma_{22}}.$$

$\gamma_{11} = \frac{\rho_{rr}}{\rho_1}$, $\gamma_{12} = \frac{\rho_{r\theta}}{\rho_1}$ and $\gamma_{22} = \frac{\rho_{\theta\theta}}{\rho_1}$ are the parameters for the porous layer's material which are dimensionless. Once more, the solution to equation (4.5) is considered as

$$(4.6) \quad v = \psi(z) J_1(kr) e^{i\omega t},$$

Equation (4.5) assumes the following form with the aid of (4.6)

$$(4.7) \quad \frac{d^2 \psi}{dz^2} + m_1^2 \psi = 0,$$

where $m_1 = k \sqrt{\frac{N_1}{L_1} d \left(\frac{c_1^2}{c_1^2} - \frac{1 - \xi_1}{d} \right)}$. The solution to (4.7) is

$$\psi(z) = B_1 \cos(m_1 z) + B_2 \sin(m_1 z)$$

In the porous layer, the displacement component is hence

$$(4.8) \quad v(z) = v_1(\text{say}) = [B_1 \cos(m_1 z) + B_2 \sin(m_1 z)] J_1(kr) e^{i\omega t}.$$

5. Boundary conditions

The subsequent prerequisites for boundary conditions need to be fulfilled:

- (1) At the point where the upper semi-infinite space and the layer meet, both the displacement and stress components remain continuous, i.e.,
 - (i) $v_0 = v_1$ at $z = -H$.
 - (ii) $L_0 \frac{\partial v_0}{\partial z} = L_1 \frac{\partial v_1}{\partial z}$ at $z = -H$.
- (2) At the interface irregularity of the layer and lower semi-infinite space, the displacement and the stress components stay continuous, i.e.,
 - (i) $v_1 = v_2$ at $z = \varepsilon G(r)$.
 - (ii) $(\chi_{r\theta})_2 = (\chi_{r\theta})_3$ at $z = \varepsilon G(r)$.

where $(\chi_{r\theta})_2 = L_1 \left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r} \right) (-\epsilon G') + L_1 \frac{\partial v_1}{\partial z}$, $(\chi_{r\theta})_3 = L_2 \left(\frac{\partial v_2}{\partial r} - \frac{v_2}{r} \right) (-\epsilon G') + L_2 \frac{\partial v_2}{\partial z}$. The following equations have been obtained by applying equations (3.8), (3.10), and (4.8) to the aforementioned four boundary conditions.

$$\begin{aligned}
 A_1 e^{-(m_0 - \frac{\alpha_1}{2})H} - B_1 \cos(m_1 H) + B_2 \sin(m_1 H) &= 0, \\
 A_1 \left(m_0 - \frac{\alpha_1}{2} \right) e^{-(m_0 - \frac{\alpha_1}{2})H} - B_1 \left(\frac{L_1}{L_0} m_1 \sin(m_1 H) \right) \\
 &\quad - B_2 \left(\frac{L_1}{L_0} m_1 \cos(m_1 H) \right) = 0, \\
 B_1 \cos(m_1 \varepsilon G(r)) + B_2 \sin(m_1 \varepsilon G(r)) - C_1 \frac{e^{-m_2 \varepsilon G(r)}}{\cosh(\alpha_2 \varepsilon G(r))} &= 0, \\
 B_1 Q_1 + B_2 Q_2 - C_1 \left(\frac{L_2}{L_1} \frac{e^{-m_2 \varepsilon G(r)}}{\cosh(\alpha_2 \varepsilon G(r))} Q_3 \right) &= 0,
 \end{aligned}$$

where

$$\begin{aligned}
 Q_1 &= \varepsilon G' k \cos(m_1 \varepsilon G(r)) Q - m_1 \sin(m_1 \varepsilon G(r)), \\
 Q_2 &= \varepsilon G' k \sin(m_1 \varepsilon G(r)) Q + m_1 \cos(m_1 \varepsilon G(r)), \\
 Q_3 &= \varepsilon G' k Q - (m_2 + \tanh(\alpha_2 \varepsilon G(r))), \\
 Q &= \frac{J_2(kr)}{J_1(kr)}.
 \end{aligned}$$

Removing the arbitrary constants A_1 , B_1 , B_2 , and C_1 from the preceding four equations, we obtain

$$\begin{aligned}
 (5.1) \quad \tan(m_1 H) &= \left[\frac{L_1}{L_0} \frac{m_1}{k} \left\{ \frac{L_2}{L_1} \frac{Q_3}{k} \cos(m_1 \varepsilon G(r)) - \frac{Q_1}{k} \right\} \right. \\
 &\quad \left. + \left(\frac{m_0}{k} - \frac{1}{2} \frac{\alpha_1}{k} \right) \left\{ \frac{L_2}{L_1} \frac{Q_3}{k} \sin(m_1 \varepsilon G(r)) - \frac{Q_2}{k} \right\} \right] \\
 &\quad / \left[\frac{L_1}{L_0} \frac{m_1}{k} \left\{ \frac{L_2}{L_1} \frac{Q_3}{k} \sin(m_1 \varepsilon G(r)) - \frac{Q_2}{k} \right\} \right. \\
 &\quad \left. - \left(\frac{m_0}{k} - \frac{1}{2} \frac{\alpha_1}{k} \right) \left\{ \frac{L_2}{L_1} \frac{Q_3}{k} \cos(m_1 \varepsilon G(r)) - \frac{Q_1}{k} \right\} \right],
 \end{aligned}$$

The torsional surface wave velocity in an initially stressed anisotropic porous layer situated between two anisotropic heterogeneous semi-infinite spaces can be found in this dispersion equation.

6. Particular Cases

CASE I: If $\varepsilon \rightarrow 0$, i.e., if the lower interface is devoid of irregularities, the dispersion equation (5.1) becomes

$$\tan(m_1 H) = \frac{-\frac{L_1}{L_0} \frac{L_2}{L_1} \frac{m_1}{k} \frac{m_2}{k} - \frac{m_1}{k} \left(\frac{m_0}{k} - \frac{1}{2} \frac{\alpha_1}{k} \right)}{-\frac{L_1}{L_0} \left(\frac{m_1}{k} \right)^2 + \frac{L_2}{L_1} \frac{m_2}{k} \left(\frac{m_0}{k} - \frac{1}{2} \frac{\alpha_1}{k} \right)}$$

The torsional surface waves in an initially stressed anisotropic porous layer positioned between two heterogeneous anisotropic semi-infinite spaces are represented by this dispersion equation.

CASE II: In the absence of irregularity if the semi-infinite spaces and layer are homogenous, isotropic, non-porous and free from initial stress, i.e., If $\varepsilon \rightarrow 0$, $\alpha_1 \rightarrow 0$, $\alpha_2 \rightarrow 0$, $N_0 = L_0 = \mu_0$, $N_1 = L_1 = \mu_1$, $N_2 = L_2 = \mu_2$, $d = 1$, $\xi_0 \rightarrow 0$, $\xi_1 \rightarrow 0$ and $\xi_2 \rightarrow 0$, the dispersion equation (6.1) becomes

$$(6.1) \quad \tan(m'_1 H) = \frac{-\frac{\mu_1}{\mu_0} \frac{\mu_2}{\mu_1} \frac{m'_1}{k} \frac{m'_2}{k} - \frac{m'_1}{k} \frac{m'_0}{k}}{-\frac{\mu_1}{\mu_0} \left(\frac{m'_1}{k} \right)^2 + \frac{\mu_2}{\mu_1} \frac{m'_2}{k} \frac{m'_0}{k}}$$

where

$$\frac{m'_0}{k} = \sqrt{1 - \frac{c^2}{c_0^2}}, \quad \frac{m'_1}{k} = \sqrt{\frac{c^2}{c_1^2} - 1}, \quad \frac{m'_2}{k} = \sqrt{1 - \frac{c^2}{c_2^2}}$$

CASE III: The dispersion equation (6.1) now has the following form if we ignore the upper half-space (i.e., $\mu_0 = 0$)

$$(6.2) \quad \tan \left[kH \sqrt{\frac{c^2}{c_1^2} - 1} \right] = \frac{\mu_2}{\mu_1} \frac{\sqrt{1 - \frac{c^2}{c_2^2}}}{\sqrt{\frac{c^2}{c_1^2} - 1}}.$$

Since equation (6.2) is a well-known classical finding of Love waves, the solution to the problem under discussion is validated.

7. Numerical results and discussion

Numerical calculations are executed to illustrate the impact of varying values of $\frac{\alpha_1}{k}$, $\frac{\alpha_2}{k}$, $\frac{N_0}{L_0}$, $\frac{N_1}{L_1}$, $\frac{N_2}{L_2}$, d , ξ_0 , ξ_1 , ξ_2 , ε and $\frac{b}{H}$ on the propagation of torsional waves in the porous layer situated between two heterogeneous anisotropic semi-infinite spaces, utilizing the dispersion equation (5.1). The quantitative information was obtained from (Gubbins [3]). $\frac{c}{c_1}$ varies with kH for various values of those, as seen in the figures. In all of the figures, the values of $\frac{L_1}{L_0} = 0.2$, $\frac{L_2}{L_0} = 0.8064$, $\frac{c_1^2}{c_0^2} = 4.032$,

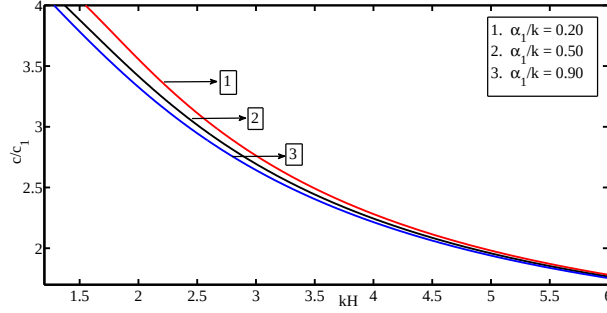


FIGURE 2. Phase velocity variations for the torsional surface wave for various values of $\frac{\alpha_1}{k}$.

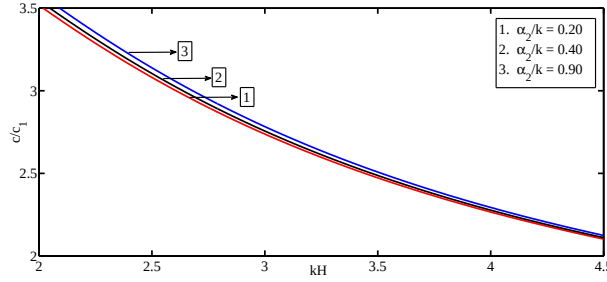


FIGURE 3. Phase velocity variations for the torsional surface wave for various values of $\frac{\alpha_2}{k}$.

$\frac{c_2}{c_1} = 0.004$, $kr = 0.01$, $\frac{r}{a} = 0.01$ and $\frac{b}{a} = 0.02$ have been maintained fixed. All of these figures demonstrate that under postulated condition, dimensionless wave number kH drops with increasing phase velocity $\frac{c}{c_1}$.

For the fluctuations of heterogeneity parameters $\frac{\alpha_1}{k}$ and $\frac{\alpha_2}{k}$ of the upper and lower semi-infinite spaces, respectively, Figures 2 and 3 show the phase velocity variations of torsional surface waves. Figure 2 delineates that when heterogeneity increases, the phase velocity of torsional surface waves drops, but the increasing effect in heterogeneity is found in Figure 3. The curves in this case, which accumulate at a single point, show that phase velocity stays constant for the same wave number even when $\frac{\alpha_1}{k}$ and $\frac{\alpha_2}{k}$ varies.

The effects of ratios of directional rigidities, $\frac{N_0}{L_0}$, $\frac{N_1}{L_1}$ and $\frac{N_2}{L_2}$, on the phase velocities of torsional surface waves are shown in Figures 4, 5 and 6. Figures 4 and 6 show an almost comparable increasing effect of $\frac{N_0}{L_0}$ and $\frac{N_2}{L_2}$ on the phase velocity. However, Figure 5 shows a declining influence of $\frac{N_1}{L_1}$.

The fluctuation of torsional surface wave velocity for the changing of porosity d is shown in Figure 7. The value of d has been determined to be 0.50, 0.55, and 0.60 for curves no. 1, no. 2, and no. 3. It states that phase velocity increases as porosity factor increases.

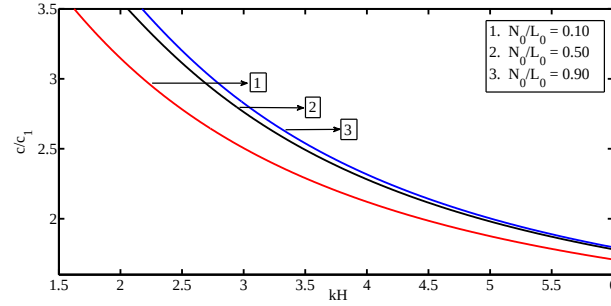


FIGURE 4. Phase velocity variations for the torsional surface wave for various values of $\frac{N_0}{L_0}$.

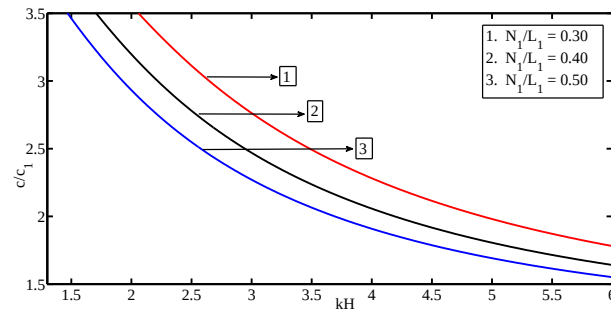


FIGURE 5. Phase velocity variations for the torsional surface wave for various values of $\frac{N_1}{L_1}$.

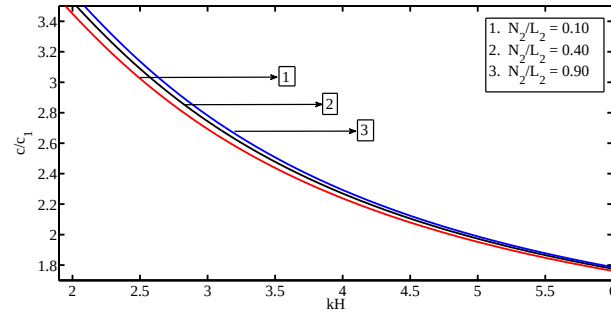


FIGURE 6. Phase velocity variations for the torsional surface wave for various values of $\frac{N_2}{L_2}$.

The effects of initial stresses of upper semi-infinite space, layer and lower semi-infinite space respectively on the phase velocities of torsional surface waves are shown in Figures 8, 9, and 10. In Figures 8 and 10, the growing influence of ξ_0 and ξ_2 on the phase velocity is nearly the same, but Figure 9 shows a decreasing

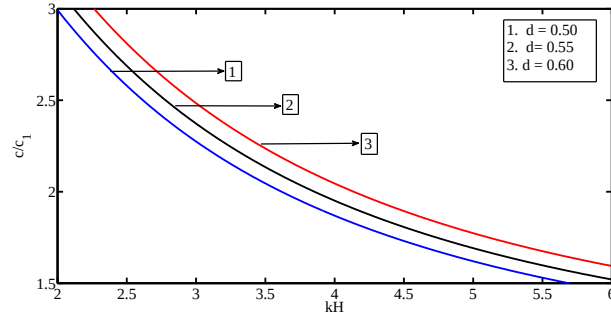


FIGURE 7. Phase velocity variations for the torsional surface wave for various values of d .

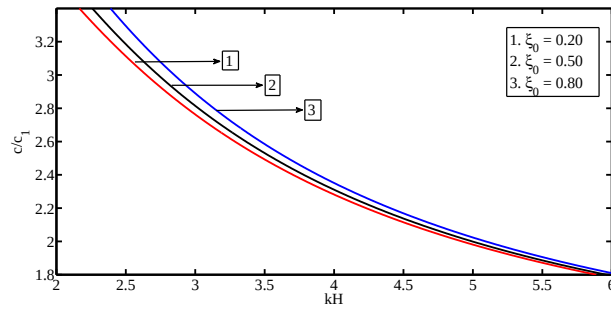


FIGURE 8. Phase velocity variations for the torsional surface wave for various values of ξ_0 .

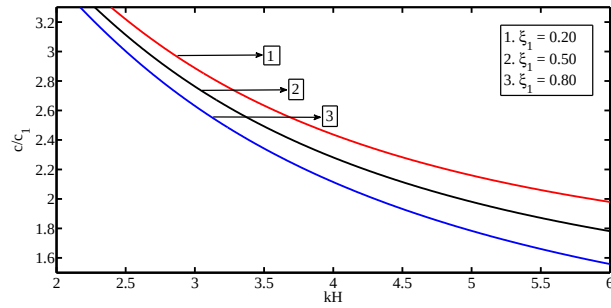


FIGURE 9. Phase velocity variations for the torsional surface wave for various values of ξ_1 .

effect of ξ_1 . Furthermore, the phase velocity is significantly impacted by ξ_0 and ξ_2 at lower frequencies, whereas ξ_1 has a greater impact at higher frequencies.

The effect of various irregularity parameters ε size is discussed in Figure 11. This figure shows that the irregularity parameter significantly affects the torsional

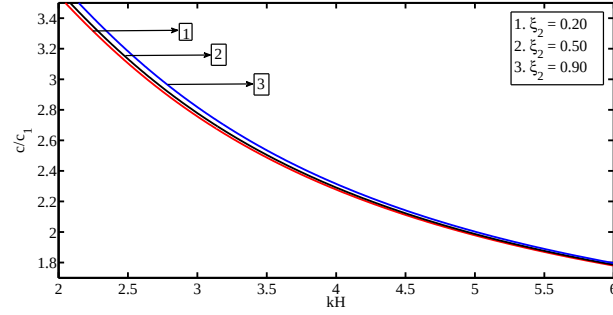


FIGURE 10. Phase velocity variations for the torsional surface wave for various values of ξ_2 .

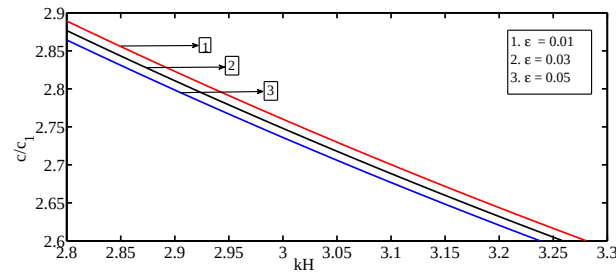


FIGURE 11. Phase velocity variations for the torsional surface wave for various values of ε .

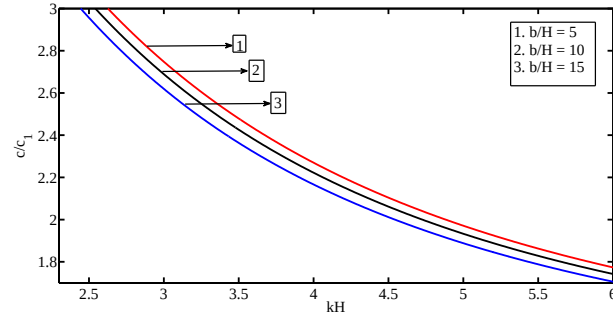


FIGURE 12. Phase velocity variations for the torsional surface wave for various values of $\frac{b}{H}$.

wave's ability to propagate. At a certain frequency, the phase velocity decreases as ε grows.

The impact of the fraction $\frac{b}{H}$ on the torsional wave's propagation in a porous medium is shown in Figure 12. This picture shows that the phase velocity falls at the same frequency as the dimensionless ratio $\frac{b}{H}$ increases.

8. Conclusions

The dispersion equation is determined analytically through the study of torsional surface wave propagation in an initially stressed porous layer sandwiched between two heterogeneous anisotropic semi-infinite spaces. The dispersion equation for a homogeneous layer over a homogeneous half-space has been found to accord with the classical conclusion of the Love wave when the upper semi-infinite space is absent. A graphical representation of the impact of porosity, anisotropic factor, irregularity, non-homogeneity and initial stress on the dimensionless phase velocity has been provided.

Torsional wave phase velocities are positively impacted by the heterogeneity features of the lower semi-infinite space, while they are negatively impacted by those of the upper semi-infinite space. Anisotropic factors of the upper and lower semi-infinite spaces have increasing effects on phase velocity; however, for porous layers, their effects are diminishing. Porosity is observed to have a considerable increasing influence. Phase velocity is affected more by initial stresses of the semi-infinite spaces than those of the porous layer. The phase velocity of torsional surface waves is also affected by different irregularity sizes. As there is no particular method, it has become very difficult to examine the interior of the Earth. Therefore, more research needs to be done about the structure of the earth. Until now, technologies have not provided much vision to explore the interior structure properly. So, it is of primary importance to study about different types of layers and characteristics of the materials to understand the influence of those layers and materials inside the Earth. In porous media, the energy of a seismic wave dissipates rapidly. In this article, torsional surface wave propagation through a porous media situated between two semi-infinite spaces under the effect of irregularity has been explored. While most academics looked at irregularities at the lower interface as triangles or rectangles, the irregularity is taken into account in the current study for the first time as a semi-four-cusped hypocycloid. Additionally, fluctuations in density, initial stress, and directional rigidities are taken into account in both half-spaces and layers, whereas the majority of authors simply took these variations in rigidity and density.

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ЕФЕКАТ ХИПОЦИКЛОИДНЕ НЕПРАВИЛНОСТИ НА ТОРЗИОНЕ ПОВРШИНСКЕ ТАЛАСЕ У ВИШСЛОЈНОМ СИСТЕМУ

РЕЗИМЕ. Ова тема је посвећена истраживању ефеката полу-четворокраке хипоциклоидне неправилности на торзионе површинске таласе у пошетно напрегнutoм анизотропној порозној средини која се налази између два анизотропна нехомогена полубесконачна простора. Неправилност се манифестује као полу-четворокрак хипоциклоид на интерфејсу где су слој и доњи полубесконачни простор раздвојени. Претпоставља се да усмерена крутост, густина и почетни напон варирају у горњем и доњем полу-бесконачном простору експоненцијално и хиперболички, редом. Дисперзиона једначина торзионих таласа се даје у затвореном облику. Додатно, једначина брзине се налази у случају да нема неправилности. Студија показује да хетерогеност доњег полубесконачног простора, почетна напрезања и усмерне крутости оба полубесконачна простора, као и порозност слоја имају повољан утицај на фазну брзину торзионих површинских таласа. Међутим, хетерогеност горњег полубесконачног простора, однос усмерених крутости и почетног напрезања слоја и параметар неправилности имају негативан утицај. Такође је примећено да се, у случају униформне средине, једначина брзине своди на конвенционалну једначину за Ловов талас.

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