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On the mean rotation in finite deformation

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Abstract

In this paper we define the mean rotation angle for deformable body. Then making use of this definition we show that such a measure of a rotation may be used to define mean rotation in sense of Cauchy and Truesdell and Toupin uniquely for each plane at any point of the body without any restriction.

1 Introduction

From the fundamental theorem it is known that the deformation at any point $\mathbf{X}$ of a deformable body $\mathcal{B}$ may be regarded as resulting from a translation, a rigid rotation of the principal axes of strain, and stretches along these axes and as such is local one. The translation, rotation and stretches may be applied in any order, but their tensorial measures are independent of this order. It is also known that while the principal axes of strain are not rotated in a pure strain, it does not follows that no linear elements suffer rotation. Because of all of that the measure of rotation of a deformable body is more difficult than that of a rigid body. The theory of finite rotation of a deformable body dates from Cauchy (1841), fourteen years after he completed the theory of strain. He took as a measure of rotation the mean values of the angles through which all elements in each of three perpendicular planes are turned. To be precise we follow the approach explained by Truesdell and Toupin in [1].

For a given deformation gradient $\mathbf{F}$ each element $d\mathbf{X}$ at $\mathbf{X}$ deformed to an element $d\mathbf{x} = \mathbf{F}d\mathbf{X}$. Their corresponding unit vectors are denoted by $\mathbf{N}$ and $\mathbf{n}$, respectively. Let us denote by $\mathbf{N}_X = \mathbf{j}\cos\Phi + \mathbf{k}\sin\Phi$ unit vector perpendicular to X-axes at $\mathbf{X}$. Then Cauchy's mean rotation angle χ_X about X-axes is given by

$$\chi_X = \frac{1}{2\pi}\int_0^{2\pi} \vartheta_X(\Phi)\, d\Phi, \tag{1}$$

where ϑ_X is the angle between $\mathbf{N}_X$ and $\mathbf{n}_X$, the projection of $\mathbf{n}$ upon YZ-plane. This formula, as was noted in [1], is not sufficient to calculate χ_X. To remedy this difficulty Truesdell and Toupin made a suggestion to replace this formula by $\cos\vartheta_X$. The same conclusion holds for ϑ_Y and ϑ_Z. However, the angles χ_X, χ_Y and χ_Z in any case do not form vector field; also they do not suffice to determine the rotation of the individual element at $\mathbf{X}$.

Although Cauchy's measure of rotation has never been used there where many attempts to calculate it in the original or modified form. Novozhilov has modified Cauchy's definition by putting

$$\tan\tau_X = \frac{1}{2\pi}\int_0^{2\pi} \tan\vartheta_X(\Phi)\, d\Phi, \tag{2}$$

which enable one to perform the integration explicitly. Moreover, if one introduces the additive decomposition of the deformation gradient

$$\mathbf{F} = \mathbf{I} + \mathbf{E} + \mathbf{W}, \tag{3}$$

where $\mathbf{I}$, $\mathbf{E}$ and $\mathbf{W}$ are identity, symmetric and skew-symmetric tensors, respectively, we may put Novozhilov's integral in invariant

$$\tan\tau_{X^1} = -\frac{W_{23}}{\sqrt{1 + I_{{}_1\mathbf{E}} + II_{{}_1\mathbf{E}}}}, \tag{4}$$

where ${}_1\mathbf{E}$ is two dimensional tensor obtained from $\mathbf{E}$ by suppressing all components having index 1. This elegant formula of Novozhilov suggests that $\mathbf{W}$ may be considered as a measure of *mean rotation*. Marzano, [3], had shown that such an interpretation holds only if the class of admissible deformations is severely restricted by the requirement that $\mathbf{F}$ be positive definite. The mechanical counterpart of this mathematical requirement amounts to exclude that even one single linear element may be turned through a right angle, irrespective of its stretch. Following a suggestion made by Truesdell and Toupin he then introduces another measure of mean rotation replacing $\tan \tau_X$ by $\cos \tau_X$, which does not suffer the limitation of Novozhilov. But still there is left restriction on admissible deformation for the angle ϑ_Y to be well defined, namely, the projection of $\mathbf{FN}$ has to be different of zero. He then restricted his investigation to the plane perpendicular to the directions of eigenvectors of $\mathbf{F}^T$, the transpose of $\mathbf{F}$. For deformation gradients being either pure rotation, or pure strain, or else "additively" pure rotations, he performed the explicit calculation of the measure of mean rotation he proposed. Very recently Q.S. Zheng and K.C. Hwang, [4], [5], succeeded in evaluating $\bar{\chi}_X$ with quite a simple formula. Two approaches were given: one is geometrical related to the rotation circle and another one is algebraic. They also showed that Cauchy's mean rotation angle, evaluated with respect to the eigenvector of $\mathbf{Q}$, the finite rotation tensor in polar decomposition of $\mathbf{F}$, is equal to the rotation angle of $\mathbf{Q}$. Furthermore, Cauchy's mean rotation angle can be related to so-called projection polar decomposition. To remedy the difficulty in definition of Cauchy's measure of mean rotation they introduced so-called generalized local mean rotation, a modified Cauchy's mean rotation in a sense that region of ϑ_X should be $(-\pi, \pi]$. They have also shown that the restriction

$$\det \bar{\mathbf{F}} > 0, \tag{5}$$

for the determinant of two-dimensional projection $\bar{\mathbf{F}}$ of $\mathbf{F}$ has to hold if the deformation rotation angle has to be evaluated as a continuous function on their deformation rotation circle.

Inspired by [4] Martins and Podio-Guidugli [6] investigated the problem of measuring mean rotation in a complete intrinsic way, and consistently solved it by formulae of an invariant character. Their approach relied on implementing Cauchy's concept of a rotation-angle function, which, they proved, exists if (5) holds.

Obviously, in any case it comes out that if we want to calculate mean rotation in a way that it makes sense we have to modified Cauchy's definition of its measure. Keeping it in mind we propose the following modified approach:

Let ϑ_X is the angle between $\mathbf{N}_X$ and rotated $\mathbf{n}_R$ of $\mathbf{n}$ upon the YZ-plane.

Definition: *A mean rotation angle, denoted by* χ_X*, is the mean value of* ϑ_Y *for all elements* $\mathbf{N}_X$ *in* YZ*-plane.*

Making use of this definition we are going to show such a measure of a rotation defines mean rotation in sense of Cauchy and Truesdell and Toupin uniquely for each plane at $\mathbf{X}$ of $\mathcal{B}$ without any restriction. From geometrical point of view, in calculating ϑ for an element $\mathbf{N}$ which belongs to a material plane $\mathfrak{M}$ at $\mathbf{X}$ we first define its deformed element $\mathbf{n}$ in plane $\mathfrak{m}$, deformed configuration of $\mathfrak{M}$. Then by rigid rotation of $\mathfrak{m}$ one obtains a plane $\mathfrak{m}_{\mathcal{R}}$ such that $\mathfrak{m}_{\mathcal{R}} \parallel \mathfrak{M}$. This rotation of $\mathfrak{m}$ as well as all its elements $\mathbf{n}$ is defined by an orthogonal tensor $\mathbf{R}$ which depends at given $\mathbf{X}$ of $\mathcal{B}$ on $\mathbf{F}$ and $\mathbf{M}$, unit outward normal vector of $\mathfrak{M}$. Then it turns out that ϑ is defined for each element $\mathbf{N}$ in $\mathfrak{M}$ by the projection of "gradient" of deformation $\mathbf{RF}$ on $\mathfrak{M}$. Since it is always regular ϑ is well defined for all $\mathbf{N}$ in $\mathfrak{M}$. The same is true if we modified our definition of mean rotation in accordance to Truesdell and Toupin's suggestion. In a special case, which is discussed, we may calculate ϑ without rotating $\mathfrak{m}$ into $\mathfrak{m}_{\mathcal{R}}$. Then ϑ as well as χ_X is independent of $\mathbf{R}$ since then $\mathbf{R} = \mathbf{I}$.

The scope of the paper is the following. In Section 2 we give some mathematical preliminaries we need to make the paper self contained. Mostly we are concern with orthogonal projection operator $-\mathbf{P}^2$ and the properties of projected tensors of second order. Particularly we are concerned with projection of gradient of deformation $\mathbf{F}$ and rotation tensor $\mathbf{R}$ as well their product. In Section 3 we state the problem concerning the angle ϑ for any element $\mathbf{N}$ in a plane $\mathfrak{M}$ at given $\mathbf{X}$ of $\mathcal{B}$ defined by its unit vector $\mathbf{M}$ which is also the axis of projection of operator $-\mathbf{P}^2$. In Section 4 we defined modified Truesdell's measure of mean rotation and calculate it. Then we discuss some special cases. Also making use of this procedure we investigate the case when $\mathbf{R} = \mathbf{I}$ without rotating of plane $\mathfrak{m}$. In Section 5 we defined modified Cauchy's measure of mean rotation. Then we calculate the mean rotation angle in two ways, geometrically and algebraically as it was done in [5]. We also have proved that the mean rotation angle can be related to so-called projection polar decomposition. Particularly it was shown that modified Cauchy's mean rotation angle, evaluated with respect to the axes of rotation tensor $\mathbf{Q}$ in polar decomposition of gradient of deformation $\mathbf{F}$, is actually equal to the rotation angle of $\mathbf{Q}$. In the Appendix we derive formulas for some expressions, particularly $\det \bar{\mathbf{F}}$, we needed.

2 Mathematical preliminaries

We include here the review of the concepts and notation employed therein together with results to be used later. Whenever it is possible our notations and terminology will closely follow that of [1] and [5].

The space under consideration will be always Euclidean space $\mathfrak{E}$. Let $\mathbf{a}$ and $\mathbf{b}$ be two arbitrary vectors in $\mathfrak{E}$. Then $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\otimes\mathbf{b}$ denote their inner (scalar) product, vector product and tensor product, respectively. We use the term tensor as a synonym for linear transformation from $\mathfrak{E}$ to $\mathfrak{E}$. We call a tensor $\mathbf{A}$ symmetric if $\mathbf{A}^T = \mathbf{A}$, a skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$. We write $Tr\mathbf{A}$ for the trace of $\mathbf{A}$. A tensor $\mathbf{A}$ is positive definite if

$$\mathbf{v}\cdot\mathbf{A}\mathbf{v} > 0, \tag{6}$$

for all vectors $\mathbf{v} \neq 0$.

We denote by:

$\mathfrak{E}_3(\mathfrak{E}_2)$ = three (two)-dimensional Euclidean space,

Lin= the set of all tensors,

$\mathbf{Lin}^+$ = the set of all tensors with $\mathbf{S}$ with $\det \mathbf{S} > 0$,

Sym = the set of all symmetric tensors,

Skw = the set of all symmetric tensors,

Psym = the set of all symmetric, positive definite tensors,

Orth = the set of all orthogonal tensors,

$\mathbf{Orth}^+$ = the set of all rotations.

Particularly by **M** we shall always denote a unit vector.

Further, according to spectral theorem we may write

$$\mathbf{C} = \sum \omega_i \mathbf{e}_i \otimes \mathbf{e}_i, \quad (i = 1, 2, 3), \tag{7}$$

for any $\mathbf{C} \in \mathbf{Sym}$, where $\mathbf{e}_i$ $(i = 1, 2, 3)$, are eigenvectors of **C** corresponding to eigenvalues ω_i of **C**; they are orthonormal.

Also, it known that for any $\mathbf{P} \in \mathbf{Skw}$ there is a unique vector **p** such that

$$\mathbf{Pp} = 0, \quad \mathbf{Pv} = \mathbf{p} \times \mathbf{v}, \tag{8}$$

for every vector **v**. Vector **p** is called axial vector of $\mathbf{P} \in \mathbf{Skw}$. For unit vector **p**

$$\mathbf{P}^2 = \mathbf{p} \otimes \mathbf{p} - \mathbf{I}, \ \left(\mathbf{P}^2\right)^{\mathrm{T}} = \mathbf{P}^2, \quad \text{and} \quad \mathbf{P}^3 = -\mathbf{P}, \ \mathbf{P}^4 = -\mathbf{P}^2, \ \mathbf{P}^5 = \mathbf{P}. \tag{9}$$

From geometric point of view $-\mathbf{P}^2$ is an orthogonal projector on the plane $\{\mathbf{p}\}^\perp$ defined by **p** as its outward unit normal vector. Generally, for any $\mathbf{G}, \mathbf{W} \in \mathbf{Skw}$, and their axial vectors **g** and **w**, not necessarily unit vectors, we have

$$\mathbf{GW} = \mathbf{w} \otimes \mathbf{g} - (\mathbf{w} \bullet \mathbf{g})\,\mathbf{I} \Rightarrow \mathbf{w} \bullet \mathbf{g} = -\frac{1}{2} Tr \mathbf{GW}. \tag{10}$$

We are going to use these expressions very often.

Definition 1:

$$\mathbf{v}^* = -\mathbf{p}^2 \mathbf{v}, \quad \mathbf{A}^* = \mathbf{P}^2 \mathbf{A} \mathbf{P}^2 \tag{11}$$

are the projections of a vector $\mathbf{v}$ and a tensor $\mathbf{A}$.

From this definition and (9) trivially we have

a) The projection of a transpose tensor $\mathbf{A}$ is transpose of its projection i.e.

$$\left(\mathbf{A}^{\mathrm{T}}\right)^{*} = (\mathbf{A}^{*})^{\mathrm{T}},$$

b) The projection of symmetric (skew-symmetric) tensor $\mathbf{A}$ is symmetric (skew-symmetric) tensor, i.e. symmetric properties of a tensor is invariant under the projection.

Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{p})$ be a right-handed orthonormal basis with $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{p}$. Then

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{12}$$

where the usual summation convention over repeated indices has been adopted; here and further Latin indices will run from 1 to 3, and Greek from 1 to 2. From (11), (12) and (8) we have

$$\mathbf{A}^{*} = \begin{Bmatrix} \bar{\mathbf{A}} & 0 \\ 0 & 0 \end{Bmatrix}, \tag{13}$$

where by definition

$$\bar{\mathbf{A}} = A_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta. \tag{14}$$

For definiteness we shall call $\bar{\mathbf{A}}$ a two dimensional projection tensor of $\mathbf{A}$ in $\{\mathbf{p}\}^{\perp}$.

By definition $\mathbf{A}^{*}$ is always singular although $\mathbf{A}$ or $\bar{\mathbf{A}}$ a may be regular.

Particularly

$$\mathbf{I}^{*} = -\mathbf{P}^2 = \begin{Bmatrix} \bar{\mathbf{I}} & 0 \\ 0 & 0 \end{Bmatrix}, \tag{15}$$

where $\bar{\mathbf{I}}$ is unit tensor in $\{\mathbf{p}\}^{\perp}$.

Definition 2: We call **A** and **v** projection invariant under the projector $-\mathbf{P}^2$ if

$$\mathbf{A}^* = \mathbf{A}, \quad \text{and} \quad \mathbf{v}^* = \mathbf{v}. \tag{16}$$

Trivially $\mathbf{P}^* = \mathbf{P}$, i.e. we say **P** is projection self-invariant.

Generally

$$(\mathbf{AB})^* \neq \mathbf{A}^*\mathbf{B}^*,$$

i.e. the projection of the product of two tensors is different from the product of their projections. This immediately suggests a question: When does $(\mathbf{AB})^* \neq \mathbf{A}^*\mathbf{B}^*$? To answer this question we proceed with

Proposition 1.

$$(\mathbf{AB})^* = \mathbf{A}^*\mathbf{B}^*, \tag{17}$$

if the axial vector **p** of **P** is eigenvector of **A** or $\mathbf{B}^T$.

Proof. Necessity. Then from (17), (9), (10) and (11) we have

$$\begin{aligned}
\mathbf{P}^2\mathbf{ABP}^2 &= \mathbf{P}^2\mathbf{AP}^2\mathbf{P}^2\mathbf{BP}^2, \\
&= -\mathbf{P}^2\mathbf{AP}^2\mathbf{BP}^2 = -\mathbf{P}^2\mathbf{A}\,(\mathbf{p}\otimes\mathbf{p}-\mathbf{I})\,\mathbf{BP}^2, \\
&= -\mathbf{P}^2\mathbf{Ap}\otimes\mathbf{pBP}^2 + \mathbf{P}^2\mathbf{ABP}^2 \Rightarrow \mathbf{P}^2\mathbf{Ap}\otimes\mathbf{pBP}^2 = 0, \\
&\Rightarrow \mathbf{P}^2\mathbf{Ap} = \mathbf{0}, \quad \text{or} \quad \mathbf{P}^2\mathbf{B}^T\mathbf{p} = \mathbf{0}, \\
&\Rightarrow \mathbf{Ap} = \sigma\mathbf{p}, \quad \text{or} \quad \mathbf{B}^{\mathbf{T}}\mathbf{p} = \tau\mathbf{p}.
\end{aligned}$$

Sufficiency. Let $\mathbf{Ap} = \sigma\mathbf{p}$. Then

$$\begin{aligned}
\mathbf{Ap}\otimes\mathbf{p} &= \sigma\mathbf{p}\otimes\mathbf{p} \Rightarrow \mathbf{A}\left(\mathbf{I}+\mathbf{P}^2\right) = \sigma\mathbf{p}\otimes\mathbf{p}, \\
&\Rightarrow \mathbf{P}^2\mathbf{A}\left(\mathbf{I}+\mathbf{P}^2\right) = 0 \Rightarrow \mathbf{P}^2\mathbf{A} = -\mathbf{P}^2\mathbf{AP}^2.
\end{aligned}$$

because of (8) and (9) and

$$(\mathbf{AB})^* = \mathbf{P}^2\mathbf{ABP}^2 = -\mathbf{P}^2\mathbf{AP}^2\mathbf{BP}^2 = \mathbf{P}^2\mathbf{AP}^2\mathbf{P}^2\mathbf{BP}^2 = \mathbf{A}^*\mathbf{B}^*.$$

The same conclusion follows from $\mathbf{B}^T\mathbf{p} = \tau\mathbf{p}$.

Corollary 1.

$$\left(\mathbf{A}^2\right) = (\mathbf{A}^*)^2, \tag{18}$$

if $\mathbf{p}$ is eigenvector of $\mathbf{A}$.

Of importance is also

Proposition 2: $\mathbf{AP}^2 = \mathbf{P}^2\mathbf{A}$ if the axial vector $\mathbf{p}$ of $\mathbf{P}$ is an eigenvector of $\mathbf{A}$ and $\mathbf{A}^T$. Then

$$\mathbf{A}^* = -\mathbf{AP}^2 = -\mathbf{P}^2\mathbf{A} = -\mathbf{A}^*\mathbf{P}^2 = -\mathbf{P}^2\mathbf{A}^*. \tag{19}$$

Proof. The first part of proposition. Necessity. From $\mathbf{AP}^2 = \mathbf{P}^2\mathbf{A}$ and (9) we have

$$\begin{aligned}\mathbf{A}(\mathbf{p}\otimes\mathbf{p}-\mathbf{I}) &= (\mathbf{p}\otimes\mathbf{p}-\mathbf{I})\mathbf{A} \Rightarrow \mathbf{Ap}\otimes\mathbf{p} = \mathbf{p}\otimes\mathbf{A}^T\mathbf{p},\\ &\Rightarrow \mathbf{Ap} = \lambda\mathbf{p} = \mathbf{A}^T\mathbf{p}, \quad (\lambda \equiv \mathbf{p}\cdot\mathbf{Ap}).\end{aligned}$$

Sufficiency. Then $\mathbf{Ap} = \lambda\mathbf{p} = \mathbf{A}^T\mathbf{p}$ so that

$$\begin{aligned}\mathbf{AP}^2 &= \mathbf{A}(\mathbf{p}\otimes\mathbf{p}-\mathbf{I}) = \mathbf{Ap}\otimes\mathbf{p} - \mathbf{A} = \lambda\mathbf{p}\otimes\mathbf{p} - \mathbf{A},\\ &= \mathbf{p}\otimes\lambda\mathbf{p} - \mathbf{A} = \mathbf{p}\otimes\mathbf{A}^T\mathbf{p} - \mathbf{A} = (\mathbf{p}\otimes\mathbf{p}-\mathbf{I})\mathbf{A} = \mathbf{P}^2\mathbf{A}.\end{aligned}$$

The second part of proposition follows immediately from $\mathbf{AP}^2 = \mathbf{P}^2\mathbf{A}$ when one multiplies it by $\mathbf{P}^2$ from the left or side and make use of (9) and (11).

Corollary 2: If $\mathbf{p}$ is an eigenvector of **Sym** or **Orth** then

$$\mathbf{SP}^2 = \mathbf{P}^2\mathbf{S}, \quad \text{or} \quad \mathbf{RP}^2 = \mathbf{P}^2\mathbf{R}.$$

We are interesting mostly in the projection of $\mathbf{F} \in \mathbf{Lin}^+$. Then, according to the polar decomposition theorem there exist positive definite, symmetric tensors **U, V** and a rotation **Q** such that

$$\mathbf{F} = \mathbf{QU} = \mathbf{VQ}. \tag{20}$$

Moreover, each of these decompositions is unique.

Corollary 3:

a) If $\mathbf{p}$ is an eigenvector of $\mathbf{Q}$ then

$$\mathbf{F}^* = \mathbf{Q}^*\mathbf{U}^* = \mathbf{V}^*\mathbf{Q}^*, \tag{21}$$

b) If $\mathbf{p}$ is an eigenvector of $\mathbf{U}$ (or $\mathbf{V}$), then $\mathbf{F}^* = \mathbf{Q}^*\mathbf{U}^*$, (or $\mathbf{F}^* = \mathbf{V}^*\mathbf{Q}^*$).

The projection of an orthogonal tensor is, generally, tensor which is not orthogonal. Indeed, since $\mathbf{RR}^T = \mathbf{I}$ we have

$$\mathbf{R}^* (\mathbf{R}^*)^T = \mathbf{I}^* - \mathbf{P}^2\mathbf{Rp} \otimes \mathbf{P}^2\mathbf{Rp}, \tag{22}$$

where we made use of (9-11) and (15). From these and (22) we have

Proposition 3: Two-dimensional projection of $\mathbf{R}$ is an orthogonal tensor in $\{\mathbf{p}\}^\perp$, i.e. $\bar{\mathbf{R}}\bar{\mathbf{R}}^T = \bar{\mathbf{I}}$ on $\{\mathbf{p}\}^\perp$ if $\mathbf{Rp} = \mathbf{p}$.

Proposition 4: Projection of a positive definite tensor $\mathbf{A}$ is a positive definite two-dimensional tensor $\bar{\mathbf{A}}$ in $\{\mathbf{p}\}^\perp$.

Proof. Since (6) holds for all $\mathbf{v} \neq 0$ it also has to hold for any such $\mathbf{v} \in \{\mathbf{p}\}^\perp$, i.e. for any $\mathbf{v} = -\mathbf{P}^2\mathbf{v}$. Then

$$0 < \mathbf{v} \cdot \mathbf{Av} = \mathbf{P}^2\mathbf{vAP}^2\mathbf{v} = \mathbf{vP}^2\mathbf{AP}^2\mathbf{v} = \mathbf{v} \cdot \mathbf{A}^*\mathbf{v} = \mathbf{v} \cdot \bar{\mathbf{A}}\mathbf{v}.$$

for any $\mathbf{v} \in \{\mathbf{p}\}^\perp, \mathbf{v} \neq 0$.

3 Statement of the problem

Let $\mathbf{X}$ and $\mathbf{x}$ be the position of the typical particle $\mathbf{X}$ of a body $\mathcal{B}$ with respect to the reference configuration and current configuration of $\mathcal{B}$. The basic mathematical idea of a body motion is that it can be described by a continuous point transformation

$$\mathbf{x} = \Xi(\mathbf{X}, t). \tag{23}$$

The requirement that the body not penetrate itself is expressed by the assumption that Ξ be one-to-one. Then the fundamental kinematic tensor $\mathbf{F}$, underlying the local analysis of deformation, must satisfy the condition

$$\det \mathbf{F} > 0, \tag{24}$$

for all time t, i.e. necessarily $\mathbf{F} \in \mathbf{Lin}^+$ and the polar decomposition given by (20) holds. We also need the right and the left Cauchy-Green tensors $\mathbf{C} \in \mathbf{Psym}$ and $\mathbf{B} \in \mathbf{Psym}$, respectively, defined by

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}. \tag{25}$$

At a given $\mathbf{X}$ we consider $\mathbf{F}$ as a homogeneous deformation. Then the linear element $d\mathbf{X}$ at $\mathbf{X}$ is deformed into $d\mathbf{x}$ at $\mathbf{x}$ through the relation

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}, \tag{26}$$

and

$$\mathbf{n} = \frac{1}{\lambda_{(N)}} \mathbf{F}\mathbf{N}, \tag{27}$$

where $\mathbf{n}$ and $\mathbf{N}$ are unit vectors of $d\mathbf{x}$ and $d\mathbf{X}$, respectively

$$\lambda_{(N)} = \sqrt{\mathbf{N} \cdot \mathbf{C}\mathbf{N}} > 0, \tag{28}$$

is the stretch in the direction of $\mathbf{N}$. Let us denote by $\mathbf{M}$ and $\mathbf{m}$ the unit vectors of a plane $\mathfrak{M}$ and $\mathfrak{m}$ through $\mathbf{X}$ and $\mathbf{x}$, respectively.

Proposition 5. Material plane $\mathfrak{M}$ is deformed into material plane $\mathfrak{m}$ by $\mathbf{F}$ if

$$\mathbf{M} = \mu \mathbf{F}^{\mathrm{T}} \mathbf{m}, \tag{29}$$

i.e. if $\mathbf{M}$ and $\mathbf{F}^{\mathrm{T}}\mathbf{m}$ are collinear.

Proof. By proposition $\mathfrak{M}$: $\mathbf{M} \cdot d\mathbf{X} = 0$ and $\mathfrak{m}$: $\mathbf{m} \cdot d\mathbf{x} = 0$ are material planes. Then

$$0 = \mathbf{m} \cdot d\mathbf{x} = \mathbf{m} \cdot \mathbf{F} d\mathbf{X} = \mathbf{F}^{\mathrm{T}} \mathbf{m} \cdot d\mathbf{X},$$

and the proof is straightforward.

Corollary 4: Material planes $\mathfrak{M}$ and $\mathfrak{m}$ are parallel if $\mathbf{F}^{\mathrm{T}}\mathbf{m} = \mathbf{v}\mathbf{m}$, [3].

Since $\mathbf{F} \in \mathbf{Lin}^+$ there is at least one real eigenvalue ν which defines unit normal vector $\mathbf{m}$ of parallel planes $\mathfrak{M}$ and $\mathfrak{m}$. Moreover, the same eigenvalues of $\mathbf{F}$ define invariant directions of deformation as well as parallel planes, [1].

Let us denote by ψ the angle between planes $\mathfrak{M}$ and $\mathfrak{m}$ as the angle between their normal unit vectors $\mathbf{M}$ and $\mathbf{m}$, respectively. If we rotate $\mathbf{m}$ around the axis defined by the unit vector $\mathbf{t}$ for the angle ψ (so that rotated $\mathbf{m}$ is equal to $\mathbf{M}$) we will obtain a plane $\mathfrak{m}_R$ parallel to $\mathfrak{M}$, Fig. 1. Then the performed rotation is uniquely defined by the proper orthogonal tensor $\mathbf{R}$ given in its canonical form

$$\mathbf{R} = \mathbf{I} + \mathbf{T} \sin\psi + \mathbf{T}^2 (1 - \cos\psi). \tag{30}$$

where $\mathbf{T} \in \mathbf{Skw}$, $\mathbf{T}\mathbf{v} = \mathbf{t} \times \mathbf{v}$, and

$$\cos\psi = \mathbf{m} \cdot \mathbf{M}, \quad \mathbf{t} = \frac{\mathbf{m} \times \mathbf{F}^{\mathrm{T}}\mathbf{m}}{|\mathbf{m} \times \mathbf{F}^{\mathrm{T}}\mathbf{m}|}.$$

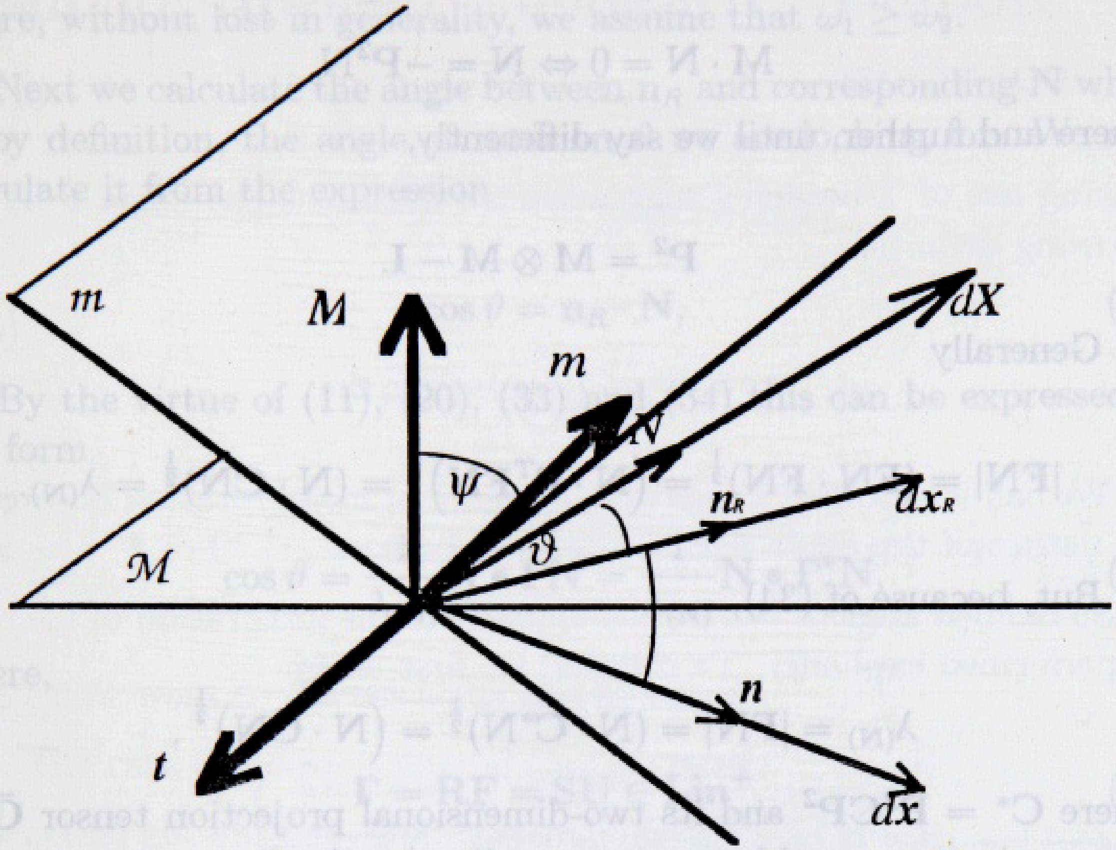

Fig. 1. Planes $\mathfrak{m}$ and $\mathfrak{M}$ and angles ψ and ϑ.

Generally $\mathbf{R}$ depends on $\mathbf{F}$ and $\mathbf{M}$ or $\mathbf{m}$ since, by construction

$$\mathbf{M} = \mathbf{R}\mathbf{m}, \quad \text{and} \quad \mathbf{R}\mathbf{m} = \mu\mathbf{F}^{\mathrm{T}}\mathbf{m}. \tag{31}$$

where we make use of (29).

But

$$\mathfrak{M} \parallel \mathfrak{m} \Leftrightarrow \mathbf{M} = \mathbf{m} \Leftrightarrow \psi = 0 \Leftrightarrow \mathbf{R} = \mathbf{I}. \tag{32}$$

Obviously any linear element $d\mathbf{x} \in \mathfrak{m}$ rotates by $\mathbf{R}$ into $d\mathbf{x}_R \in \mathfrak{m}_R$. Particularly, for $\mathbf{n}_R$, rotated $\mathbf{n}$, we have

$$\mathbf{n_R} = \mathbf{R}\mathbf{n} = \frac{\mathbf{RFN}}{|\mathbf{FN}|}, \tag{33}$$

Here and throughout the paper we keep in mind that if $\mathbf{N} \in \mathfrak{M}$ then

$$\mathbf{M}\cdot\mathbf{N}=0\Leftrightarrow\mathbf{N}=-\mathbf{P}^2\mathbf{N}, \tag{34}$$

where and further, until we say differently,

$$\mathbf{P}^2=\mathbf{M}\otimes\mathbf{M}-\mathbf{I}. \tag{35}$$

Generally

$$|\mathbf{FN}|=(\mathbf{FN}\cdot\mathbf{FN})^{\frac{1}{2}}=\left(\mathbf{N}\cdot\mathbf{F}^T\mathbf{FN}\right)^{\frac{1}{2}}=(\mathbf{N}\cdot\mathbf{CN})^{\frac{1}{2}}=\lambda_{(\mathbf{N})}.$$

But, because of (34),

$$\lambda_{(\mathbf{N})}=|\mathbf{FN}|=(\mathbf{N}\cdot\mathbf{C}^*\mathbf{N})^{\frac{1}{2}}=\left(\mathbf{N}\cdot\bar{\mathbf{C}}\mathbf{N}\right)^{\frac{1}{2}}, \tag{36}$$

where $\mathbf{C}^*=\mathbf{P}^2\mathbf{C}\mathbf{P}^2$ and its two-dimensional projection tensor $\bar{\mathbf{C}}$ are symmetric tensors. Moreover, according to the Proposition 5, $\bar{\mathbf{C}}$ is positive definite tensor in $\mathfrak{M}$. Then, by the spectral theorem,

$$\bar{\mathbf{C}}=\sum\omega_\alpha\mathbf{e}_\alpha\otimes\mathbf{e}_\beta,\quad \omega_\alpha>0, \tag{37}$$

where $\mathbf{e}_\alpha$, $(\alpha=1,2)$, are orthonormal eigenvectors of $\bar{\mathbf{C}}$. The set $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3=\mathbf{e}_1\mathbf{e}_2=\mathbf{M})$ represents the basis of orthonormal vectors. In that basis we write

$$\mathbf{N}=\cos\phi\mathbf{e}_1+\sin\phi\mathbf{e}_2. \tag{38}$$

Then from (35), (36) and (37) we have

$$\begin{aligned}\lambda_{(\mathbf{N})} &= \sqrt{\omega_1\cos^2\phi+\omega_2\sin^2\phi}=\sqrt{\omega_1}\sqrt{1-\left(1-\frac{\omega_2}{\omega_1}\right)\sin^2\phi}, \quad (39)\\ &= \sqrt{\omega_1}\sqrt{1-k^2\sin^2\phi},\\ k^2 &= 1-\frac{\omega_2}{\omega_1},\end{aligned}$$

where, without lost in generality, we assume that $\omega_1 \geq \omega_2$.

Next we calculate the angle between $\mathbf{n}_R$ and corresponding $\mathbf{N}$ which is, by definition, the angle of rotation ϑ we are looking for. We may calculate it from the expression

$$\cos\vartheta = \mathbf{n}_R \cdot \mathbf{N}, \tag{40}$$

By the virtue of (11), (20), (33) and (34) this can be expressed in the form

$$\cos\vartheta = \frac{1}{\lambda_{(\mathbf{N})}} \mathbf{N} \bullet \mathbf{\Gamma N} = \frac{1}{\lambda_{(\mathbf{N})}} \mathbf{N} \bullet \mathbf{\Gamma}^* \mathbf{N}, \tag{41}$$

where,

$$\mathbf{\Gamma} = \mathbf{RF} = \mathbf{SU} \in \mathbf{Lin}^+, \tag{42}$$

and

$$\mathbf{S} = \mathbf{RQ} \in \mathbf{Orth}^+. \tag{43}$$

More convenient form of (41) is

$$\cos\vartheta = \frac{1}{\lambda_{(\mathbf{N})}} Tr \mathbf{\Gamma}^* \mathbf{N} \otimes \mathbf{N}. \tag{44}$$

We are now in position to define a measure of mean rotation making use of the angle ϑ.

Remark. In the case when

$$\cos\vartheta = 0, \quad \text{or} \quad Tr\mathbf{\Gamma N} \otimes \mathbf{N} = 0, \tag{45}$$

for some $\mathbf{N}$. Modified Novozhilov measure of mean rotation is not well defined as can be seen from (2). Since $\mathbf{\Gamma N} \neq 0$ for any $\mathbf{N}$ it comes out that in this case $\mathbf{\Gamma N}$ must be orthogonal to $\mathbf{N}$, or equivalently

$$\mathbf{\Gamma N} = a\mathbf{M} + b\mathbf{PN}. \tag{46}$$

4 Modified Truesdell's measure of mean rotation

Making use of Truesdell's suggestion we introduce the angle $\bar{\vartheta}$ by the following definition

$$\cos\bar{\vartheta} = \frac{1}{2\pi}\int_0^{2\pi}\cos\vartheta\left(\phi\right)d\phi, \tag{47}$$

We call it modified Truesdell's measure of mean rotation because of the nature of the angle $\vartheta\left(\phi\right)$ introduced here.

So defined angle $\bar{\vartheta}$ can be calculate since the integration in (47) can be performed explicitly. To this end we first write

$$\mathbf{N}\otimes\mathbf{N} = \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}, \tag{48}$$

in the basis $(\mathbf{e}_1, \mathbf{e}_2)$; this follows from (38). Then from (47), (48) and (44) we have

$$\begin{aligned} \cos\bar{\vartheta} &= \frac{1}{2\pi}Tr\mathbf{\Gamma}\int_0^{2\pi}\frac{1}{\lambda_{(\mathbf{N})}}\mathbf{N}\otimes\mathbf{N}d\phi, \\ &= \frac{1}{2\pi\sqrt{\omega_1}}Tr\mathbf{\Gamma}\int_0^{2\pi}\frac{\mathbf{e}_1\otimes\mathbf{e}_1\cos^2\phi + \mathbf{e}_2\otimes\mathbf{e}_2\sin^2\phi}{\sqrt{1-k^2\sin^2\phi}}d\phi. \end{aligned}$$

With the aid of the Legendre's complete elliptic integrals of the first and second kind, respectively, [7],

$$\mathbf{K}\left(k\right) = \int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \quad \mathbf{E}\left(k\right) = \int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\phi}d\phi, \tag{49}$$

the above integral can be written as

a)

$$\omega_1 = \omega_2 = \omega,$$

$$\cos\bar{\vartheta} = \frac{1}{2\sqrt{\omega}} Tr\mathbf{\Gamma}^* = \frac{1}{2\sqrt{\omega}} Tr\left(\mathbf{RF}\right)^*,$$

b)

$$\omega_1 > \omega_2$$

$$\cos \frac{1}{2\pi\sqrt{\omega_1}} Tr\mathbf{\Gamma}^* \left\{ \mathbf{e}_1 \otimes \mathbf{e}_1 \mathbf{K}\left(k\right) + \frac{\mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_1}{k^2} \left[\mathbf{K}\left(k\right) - \mathbf{E}\left(k\right)\right] \right\}. \tag{50}$$

We now want to discuss some special cases.

I. The case a) above is special case by itself.

It is well known that an ellipsoid may be associated with any second order tensor. Generally an ellipsoid has two central circular sections - of cause in the case of a spheroid (ellipsoid of revolution) there is only one central circular section whilst in the case of a sphere there is infinity of such sections, [6]. It is of value to consider these central circular sections because generally they have special identifiable properties. For example, it may be shown, [8], that material elements in these planes suffer no shear-the angle α (say) between a pair of material elements before deformation is also the angle between the stretched elements after the deformation. This follows straightforward from (37) and a) since in this case

$$\bar{\mathbf{C}} = \omega\bar{\mathbf{I}}, \tag{51}$$

where ω may be related to the minimax stretch of $\mathbf{C}$. Then it follows that the angle between their rotated elements by rotation $\mathbf{R}$ is also α. As a consequence of it he angle ϑ, given by (40) is independent of $\mathbf{N}$, so that

$$\bar{\vartheta} = \vartheta, \tag{52}$$

which follows from (47). Its explicit formula generally cannot be derived simply from (50), because of conditions imposed in Proposition 2.

But the is can be easily derived in two more specific cases stated below.

Since $\mathbf{U} \in \mathbf{Sym}^+$ from (7) it follows that

$$\mathbf{U} = \sum \lambda_i \mathbf{c}_i \otimes \mathbf{c}_i,$$

where λ_i are the principal stretches and unit vectors $\mathbf{c}_i$ are eigenvectors of $\mathbf{U}$. From (25) it follows that λ_i^2 are eigenvalues of $\mathbf{C}$. Particularly, by definition,

$$\omega \equiv \omega_2 = \lambda_2^2, \tag{53}$$

i) If $\lambda_2 = \lambda_3$, then

$$\mathbf{U} = \lambda_2 \mathbf{I} \left(\lambda_1 - \lambda_2\right) \mathbf{c}_1 \otimes \mathbf{c}_1,$$

and, from (50) and (52) it follows that

$$\cos \vartheta = \frac{1}{2} Tr \left(\mathbf{RQ}\right)^{*}, \tag{54}$$

for $\mathbf{P}^2 = \mathbf{c}_1 \otimes \mathbf{c}_1 - \mathbf{I}$ since $\mathbf{P}^2 \mathbf{c}_1 = 0$.

ii) If $\lambda_1 = \lambda_2 = \lambda_3$, then

$$\mathbf{U} = \lambda_2 \mathbf{I},$$

and from (50) and (31) it follows that

$$\cos \bar{\vartheta} = \frac{1}{2} Tr \left(\mathbf{RQ}\right)^{*}, \tag{55}$$

for any $\mathbf{P}^2 = \mathbf{p} \otimes \mathbf{p} - \mathbf{I}$. More specifically, for $\mathbf{R} = \mathbf{I}$

$$\cos \bar{\vartheta} = Tr \left(\mathbf{Q}\right)^{*} = \frac{1}{2} \left[Tr \mathbf{Q} - 1\right], \tag{56}$$

which is in agreement with [3].

Trivially, $\mathbf{Q} = \mathbf{I} \Rightarrow \mathbf{R} = \mathbf{I} \Rightarrow \cos\bar{\vartheta} = 1$.

II. The case **b)** was investigated in [3] when $\mathbf{S} = \mathbf{I}$. Here we have other possibilities but with the same form. These are the cases when $\mathbf{R} = \mathbf{I}$ or $\mathbf{Q} = \mathbf{I}$. In any of these cases nothing much can be simplified.

III. This procedure allow us the possibility to investigate the case when $\mathbf{R} = \mathbf{I}$, but generally $\mathbf{M} \neq \mathfrak{m}$, i.e. when the plane $\mathfrak{m}$ is not rotated at all, (see Fig. 2).

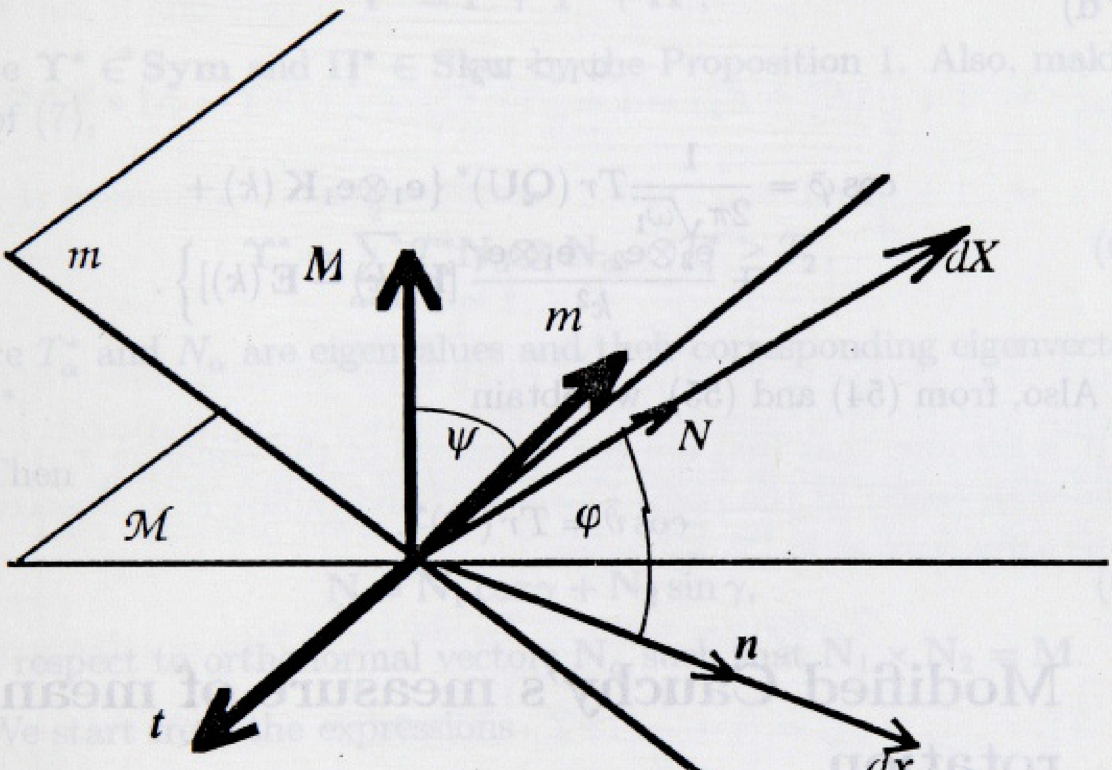

Fig. 2. Planes $\mathfrak{m}$ and $\mathfrak{M}$ and angles ψ and φ.

Then the angle, say φ, between $d\mathbf{x}$ and $d\mathbf{X}$ may be calculate from the expression

$$\cos\varphi = \mathbf{n} \bullet \mathbf{N}, \tag{57}$$

so that

$$\cos\bar{\varphi} = \frac{1}{2\pi}\int_0^{2\pi} \cos\varphi\,(\Phi)\,d\Phi, \tag{58}$$

defines the mean rotation angle $\bar{\varphi}$. This two expressions are of the same form as (40) and (47). Proceeding in the same way and applying the previous results in this case from (50) we obtain

c)

$$\omega_1 = \omega_2 = \omega,$$

$$\cos\bar{\varphi} = \frac{1}{2\sqrt{\omega}} Tr\,(\mathbf{QU})^* = \frac{1}{2\sqrt{\omega}} Tr\,(\mathbf{F})^*, \tag{59}$$

d)

$$\omega_1 > \omega_2$$

$$\cos\bar{\varphi} = \frac{1}{2\pi\sqrt{\omega_1}} Tr\,(\mathbf{QU})^* \left\{ \mathbf{e}_1 \otimes \mathbf{e}_1 \mathbf{K}\,(k) + \right.$$
$$\left. + \frac{\mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_1}{k^2} \left[\mathbf{K}\,(k) - \mathbf{E}\,(k) \right] \right\}.$$

Also, from (54) and (55), we obtain

$$\cos\bar{\vartheta} = Tr\,(\mathbf{Q})^*. \tag{60}$$

5 Modified Cauchy's measure of mean rotation

Following the idea of Cauchy we may now introduce as a measure of rotation about M-axis the mean value of the angles, denoted by ϑ^*, through which all elements in $\mathfrak{M}$ plane are turned relative to their position in $\mathfrak{m}_R$ plane, i.e.

$$\vartheta^* = \frac{1}{2\pi} \int_0^{2\pi} \vartheta\,(\Phi)\, d\Phi = \frac{1}{\pi} \int_0^{\pi} \vartheta\,(\Phi)\, d\Phi. \tag{61}$$

In order to evaluate ϑ^* we proceed by applying the procedure given in [4]. To this end we write

$$\mathbf{\Gamma} = \mathbf{RF}, \tag{62}$$

and, by additive decomposition,

$$\mathbf{\Gamma} = \mathbf{I} + \mathbf{\Upsilon} + \mathbf{\Pi}, \tag{63}$$

where $\mathbf{\Upsilon} \in \mathbf{Sym}$ and $\mathbf{\Pi} \in \mathbf{Skw}$. We denote by $\mathbf{h}$ the axial vector of $\mathbf{\Pi}$. The projection of $\mathbf{\Gamma}$ is given by the expression

$$\mathbf{\Gamma}^* = \mathbf{I}^* + \mathbf{\Upsilon}^* + \mathbf{\Pi}^*, \tag{64}$$

where $\mathbf{\Upsilon}^* \in \mathbf{Sym}$ and $\mathbf{\Pi}^* \in \mathbf{Skw}$ by the Proposition 1. Also, making use of (7),

$$\mathbf{\Upsilon}^* = \sum_{\alpha=1}^{2} T^*_\alpha \mathbf{N}_\alpha \otimes \mathbf{N}_\alpha, \quad T^*_1 \geq T^*_2, \tag{65}$$

where T^*_α and N_α are eigenvalues and their corresponding eigenvectors of $\mathbf{\Upsilon}^*$.

Then

$$\mathbf{N} = \mathbf{N}_1 \cos\gamma + \mathbf{N}_2 \sin\gamma, \tag{66}$$

with respect to orthonormal vectors $\mathbf{N}_\alpha$ such that $\mathbf{N}_1 \times \mathbf{N}_2 = \mathbf{M}$.

We start from the expressions

$$\lambda_{(\mathbf{N})} \cos\vartheta = \mathbf{N} \bullet \mathbf{\Gamma N} = \mathbf{N\Gamma}^*\mathbf{N},$$

$$\lambda_{(\mathbf{N})} \sin\vartheta = \mathbf{M} \bullet (\mathbf{N} \times \mathbf{\Gamma N}) = -\mathbf{N} \bullet \mathbf{P\Gamma}^*\mathbf{N}, \tag{67}$$

which follows from (40), (33), (62), (34), (11) and (8). These expressions are identical in with (16) of [5]. The difference is in the meaning of $\mathbf{\Gamma}$, which is not generally gradient of $\mathbf{\Gamma}$ deformation as it can seen from (62). Having that in mind the application of the result of [5] in our case is strait forward. for instance, substituting $\mathbf{N} \bullet \mathbf{\Gamma N}$ and $M \bullet (\mathbf{N} \bullet \mathbf{\Gamma N})$ (see the Appendix **a**), into (67) we may write, after some calculation,

$$x(\gamma) = \lambda_{(\mathbf{N})}\cos\vartheta = x_M + R_M\cos 2\gamma,$$

and

$$y(\gamma) = \lambda_{(\mathbf{N})}\sin\vartheta = y_M + R_M\sin 2\gamma, \quad (-\pi < \gamma \leq \pi), \tag{68}$$

where

$$R_M = \frac{1}{2}(E_1^* - E_2^*) = \frac{1}{2}\sqrt{2Tr(\mathbf{E}^{*2}) - (Tr\mathbf{E}^*)^2}, \tag{69}$$

$$\begin{aligned} x_M &= 1 + \frac{1}{2}(E_1^* + E_2^*) = 1 + \frac{1}{2}Tr\boldsymbol{\Upsilon}^*1 + \frac{1}{2}(Tr\boldsymbol{\Upsilon} - \mathbf{M}\bullet\boldsymbol{\Upsilon}\mathbf{M})^2, \\ &= 1 + \frac{1}{2}Tr\boldsymbol{\Upsilon} = \frac{1}{2}Tr\boldsymbol{\Gamma}^*, \end{aligned} \tag{70}$$

$$y_M = \mathbf{h}\bullet\mathbf{M} = -\frac{1}{2}Tr\mathbf{P\Pi}. \tag{71}$$

It is obvious that (68) are the equations of the circle on $x - y$ the plane with center at the point $C(x_M, y_M)$ and radius R_M. Proceeding in the same way as in [5], we write

$$T_M\cos\chi_M = x_M, \quad T_M\sin\chi_M = y_M,$$

$$T_M^2 = x_M^2 + y_M^2 = \left[1 + \frac{1}{2}(Tr\boldsymbol{\Upsilon} - \mathbf{M}\bullet\mathbf{M})\right]^2 + (\mathbf{h}\bullet\mathbf{M})^2, \tag{72}$$

which enable one to write (68) in the form

$$\lambda_{(\mathbf{N})}\cos(\vartheta - \chi_M) = R_M\cos(2\gamma - \chi_M),$$

$$\lambda_{(\mathbf{N})}\sin(\vartheta - \chi_M) = -R_M\sin(2\gamma - \chi_M), \quad (-\pi < \gamma \leq \pi) \tag{73}$$

Let $2\delta = 2\gamma - \chi_M$ so that $\delta = 0$ corresponds to some $2\gamma = \chi_M$. Then from (73) it follows that $\vartheta - \chi_M$, $-\pi < \vartheta - \chi_M \leq \pi$ is an odd function of δ. Under this condition Zheng and all, [5], proved their

Theorem 1:

$$\bar{\vartheta}_m = \chi_M, \tag{74}$$

In this form it holds also in our case. Indeed,

$$0 = \frac{1}{2}\int_{-\pi}^{\pi} (\vartheta - \chi_M)\, d\delta = \frac{1}{2}\int_{-\pi}^{\pi} \vartheta d\delta - \chi_M = \vartheta^* - \chi_M.$$

Making use of deformation rotation circle (68) in their case they notice that if and only if

$$T_M > R_M, \tag{75}$$

which is equivalent to the condition

$$\det \bar{\mathbf{F}} > 0,$$

the deformation rotation angle ϑ can be evaluated as a continuous function on the deformation rotation circle. Otherwise, if $T_M \leq R_M$, different choice of fixed angular range of ϑ will result in different mean value of ϑ^2. This constrain never occur in our case. In fact we are going to prove

Proposition 6:

$$\det \bar{\mathbf{\Gamma}} > 0. \tag{76}$$

for all **M** and t.

Proof. Making use of (34), (11) and (13) we may write (33) in the form $\mathbf{n_R}{=}\lambda_{(\mathbf{N})}\bar{\mathbf{\Gamma}}\mathbf{N}$. From this and (28) it follows at once that $\det \bar{\mathbf{\Gamma}} \neq 0$, i.e. $\det \bar{\mathbf{\Gamma}}$ cannot change the sign for any **M** and t. Without loss of generality we may assume that at some time t_0 we have $\mathbf{\Gamma} = \mathbf{I}$ so that $\bar{\mathbf{\Gamma}} = \bar{\mathbf{I}}$ and $\det \bar{\mathbf{\Gamma}} = 1$. But then, $\det \bar{\mathbf{\Gamma}} > 0$ for all **M** and t. From (76) and

$$\det \bar{\mathbf{\Gamma}} = T_M^2 - R_M^2, \tag{77}$$

(see for the proof the Appendix **b)**) the proof of (75) in our case is completed.

6 Projection polar decomposition and generalized Cauchy's mean rotation

In what follows it is helpful to use canonical form for $\mathbf{S} \in \mathbf{Orth}^+$

$$\mathbf{S} = \mathbf{I} + \mathbf{P}\sin\theta + \mathbf{P}^2(1-\cos\theta). \tag{78}$$

Its projection under the projector $-\mathbf{P}^2$ is given by

$$\mathbf{S}^* = \mathbf{P}\sin\theta - \mathbf{P}^2\cos\theta, \tag{79}$$

where we made use of (9) and (11). By means of (13) and (15) this can be write in the form

$$\bar{\mathbf{S}} = \bar{\mathbf{P}}\sin\theta + \bar{\mathbf{I}}\cos\theta. \tag{80}$$

By Proposition 3 it follows that two dimensional projection tensor $\bar{\mathbf{S}}$ of $\mathbf{S}$ is an orthogonal tensor on $\{\mathbf{p}\}^\perp$. Notice that for given $\{\mathbf{p}\}^\perp$ tensor $\bar{\mathbf{S}}$ is fully determined only by the angle of rotation θ. Also since two dimensional projection tensor $\bar{\mathbf{\Gamma}}$ of $\mathbf{\Gamma}$ is regular it has unique right and left projection polar decomposition

$$\bar{\mathbf{\Gamma}} = \bar{\mathbf{S}}\bar{\mathbf{U}} = \bar{\mathbf{V}}\bar{\mathbf{S}}, \tag{81}$$

where $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are positive definite tensors and $\bar{\mathbf{S}}$ an orthogonal tensor on $\mathfrak{M}$.

This turns to be very important in proving

Proposition 7: The angle of rotation θ of $\bar{\mathbf{S}}$, is equal to the modified Cauchy's mean rotation angle ϑ^*.

Proof. Only the right projection polar decomposition (81) will be used. We use (81) and (64) to obtain

$$\bar{\mathbf{\Gamma}} = \bar{\mathbf{S}}\bar{\mathbf{U}} = \bar{\mathbf{I}} + \bar{\mathbf{\Upsilon}} + \bar{\mathbf{\Pi}}. \tag{82}$$

Substituting $\bar{\mathbf{S}}$ into it we have

$$\bar{\mathbf{\Gamma}} = \bar{\mathbf{U}} \cos\theta + \bar{\mathbf{P}}\bar{\mathbf{U}} \sin\theta = \bar{\mathbf{I}} + \bar{\mathbf{\Upsilon}} + \bar{\mathbf{\Pi}}, \tag{83}$$

from which we get

$$Tr\bar{\mathbf{\Gamma}} = Tr\bar{\mathbf{U}} \cos\theta = 2 + Tr\bar{\mathbf{\Upsilon}}. \tag{84}$$

Next, we multiply (83) by $\bar{\mathbf{P}}$ on from the left side so that

$$\bar{\mathbf{P}}\bar{\mathbf{U}} \cos\theta - \bar{\mathbf{U}} \sin\theta = \bar{\mathbf{P}} + \bar{\mathbf{P}}\ \bar{\mathbf{\Upsilon}} + \bar{\mathbf{P}}\bar{\mathbf{\Pi}},$$

and

$$-Tr\bar{\mathbf{U}} \sin\theta = Tr\bar{\mathbf{P}}\bar{\mathbf{\Pi}}. \tag{85}$$

But from (9)-(11) and (13) we have

$$Tr\bar{\mathbf{P}}\bar{\mathbf{\Pi}} = Tr\mathbf{P}^*\mathbf{\Pi}^* = Tr\mathbf{P}\mathbf{\Pi} = -2^{\mathbf{h}} \bullet \mathbf{M}. \tag{86}$$

Also in the same way we obtain

$$Tr\bar{\mathbf{\Upsilon}} = Tr\mathbf{\Upsilon}^* = Tr\mathbf{P}^2\mathbf{\Upsilon}\mathbf{P}^2 = -Tr\mathbf{P}^2\mathbf{\Upsilon} = Tr\mathbf{\Upsilon} - \mathbf{M} \bullet \mathbf{\Upsilon}\mathbf{M}. \tag{87}$$

Then from (83)-(87) it follows that

$$\cos\theta = Tr\bar{\mathbf{\Gamma}}/Tr\bar{\mathbf{U}} = Tr\mathbf{\Gamma}^*/Tr\bar{\mathbf{U}}, \tag{88}$$

and

$$\left(Tr\bar{\mathbf{U}}\right)^2 = 4\left[\left(1 + \frac{1}{2}Tr\bar{\mathbf{\Upsilon}}\right)^2 + (\mathbf{h} \bullet \mathbf{M})^2\right]. \tag{89}$$

From (88), (89), (70) and (72) finally we obtain $\theta = \chi_M = \vartheta^*$.

The Proposition 7 can be put in an other form. Indeed, by (13), we may write (81) as

$$\mathbf{\Gamma}^* = \mathbf{S}^*\mathbf{U}^* = \mathbf{V}^*\mathbf{S}^*.$$

But from (78) and (79) we have

$$\mathbf{S} = \mathbf{S}^* + \mathbf{p} \otimes \mathbf{p},$$

so that

$$\mathbf{\Gamma}^* = \mathbf{S}\mathbf{U}^*,$$

since $\mathbf{p} \otimes \mathbf{p}\mathbf{U}^* = \mathbf{U}^*\mathbf{p} \otimes \mathbf{p} = 0$. Then we may state

Proposition 8. The projection $\mathbf{\Gamma}^*$ of $\mathbf{\Gamma}$ for a plane of projection $\mathfrak{M}$ has unique projection polar decomposition

$$\mathbf{\Gamma}^* = \mathbf{S}\mathbf{U}^* = \mathbf{V}^*\mathbf{S}, \tag{90}$$

where $\mathbf{S}$ is modified Cauchy's mean rotation tensor, its axis of rotation defined by the unit outward normal vector $\mathbf{M}$ of $\mathfrak{M}$, and the rotation angle equal to modified Cauchy's mean rotation angle ϑ^*. Two dimensional tensors of symmetric tensors $\mathbf{U}^*$ and $\mathbf{V}^*$ are positive definite on $\mathfrak{M}$.

From (20), (42), (43), (11) and (90) we have always

$$\mathbf{\Gamma}^* = (\mathbf{S}\mathbf{U})^* = (\mathbf{V}\mathbf{S})^* = \mathbf{S}\mathbf{U}^* = \mathbf{V}^*\mathbf{S}, \tag{91}$$

although, generally,

$$\mathbf{S} \neq \mathbf{S}, \quad \mathbf{U}^* \neq \mathbf{U}^*, \quad \text{and} \quad \mathbf{V} \neq \mathbf{V}^*,$$

as may be seen from the Proposition 2. Only in a case given by Corollary 3a) the sign equality holds. Then we may state

Proposition 9: The finite rotation tensor $\mathbf{S}$ in polar decomposition of tensor $\mathbf{\Gamma}$ can be interpreted as the modified Cauchy's mean rotation tensor with respect to the rotation axis of $\mathbf{S}$.

This is equivalent statement given by the theorem 4 in [5].

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Appendix A

$$\begin{aligned}
\mathbf{N}\bullet\mathbf{\Gamma N} &= 1+\mathbf{N}\bullet\mathbf{\Upsilon N} = 1+\frac{1}{2}Tr\mathbf{\Upsilon}^{*}+\frac{1}{2}\left(E_1^{*}-E_2^{*}\right)\cos 2\gamma, \\
&= 1+\frac{1}{2}Tr\mathbf{\Gamma}^{*}+\frac{1}{2}\left(E_1^{*}-E_2^{*}\right)\cos 2\gamma, \\
&= 1+\frac{1}{2}\left(Tr\mathbf{\Upsilon}-\mathbf{M}\bullet\mathbf{\Upsilon M}\right)+\frac{1}{2}\left(E_1^{*}-E_2^{*}\right)\cos 2\gamma,
\end{aligned}$$

$$
\begin{aligned}
M \bullet (\mathbf{N} \times \boldsymbol{\Gamma}\mathbf{N}) &= \boldsymbol{\Gamma}\mathbf{N} \bullet (M \times \mathbf{N}) = \boldsymbol{\Gamma}\mathbf{N} \bullet \mathbf{P}\mathbf{N} = \mathbf{N} \bullet \mathbf{P}\boldsymbol{\Gamma}\mathbf{N}, \\
&= -\mathbf{N}\mathbf{P}\boldsymbol{\Gamma}^{*}\mathbf{N} = -\mathbf{N}\mathbf{P}\boldsymbol{\Upsilon}^{*}\mathbf{N} - \mathbf{N}\mathbf{P}\boldsymbol{\Pi}\mathbf{N} \\
&= \mathbf{h} \bullet \mathbf{M} - \frac{1}{2}\left(E_1^{*} - E_2^{*}\right)\sin 2\gamma,
\end{aligned}
$$

where we made use of (62)-(66) and (8).

Appendix B

To calculate $\det \bar{\boldsymbol{\Gamma}}$ we start from (64), i.e.

$$
\boldsymbol{\Gamma}^{*} = \mathbf{I}^{*} + \boldsymbol{\Upsilon}^{*} + \boldsymbol{\Pi}^{*}.
$$

From this and (13) we have

$$
\bar{\boldsymbol{\Gamma}} = \bar{\mathbf{I}} + \bar{\boldsymbol{\Upsilon}} + \bar{\boldsymbol{\Pi}}.
$$

But by Cayley-Hamilton theorem

$$
\bar{\boldsymbol{\Gamma}}^{2} - \left(Tr\bar{\boldsymbol{\Gamma}}\right)\bar{\boldsymbol{\Gamma}} + \left(\det \bar{\boldsymbol{\Gamma}}\right)\bar{\mathbf{I}} = 0,
$$

from which we obtain

$$
\det \bar{\boldsymbol{\Gamma}} = \frac{1}{2}\left[\left(Tr\bar{\boldsymbol{\Gamma}}\right)^{2} - Tr\bar{\boldsymbol{\Gamma}}^{2}\right].
$$

Since

$$
\bar{\boldsymbol{\Gamma}}^{2} = \bar{\mathbf{I}} + 2\,\bar{\boldsymbol{\Upsilon}} + 2\bar{\boldsymbol{\Pi}} + \bar{\boldsymbol{\Pi}}\bar{\boldsymbol{\Upsilon}} + \bar{\boldsymbol{\Upsilon}}\bar{\boldsymbol{\Pi}} + \bar{\boldsymbol{\Upsilon}}^{2} + \bar{\boldsymbol{\Pi}}^{2},
$$

then

$$
Tr\bar{\boldsymbol{\Gamma}}^{2} = 2 + 2Tr\bar{\boldsymbol{\Upsilon}} + Tr\bar{\boldsymbol{\Upsilon}}^{2} + Tr\bar{\boldsymbol{\Pi}}^{2},
$$

where we make use of the fact that $Tr\bar{\boldsymbol{\Pi}} = Tr\bar{\boldsymbol{\Upsilon}}\bar{\boldsymbol{\Pi}} = Tr\bar{\boldsymbol{\Pi}}\bar{\boldsymbol{\Upsilon}} = 0$. Then we may write

$$\det \bar{\mathbf{\Gamma}} = 1 + Tr\bar{\mathbf{\Upsilon}} + \frac{1}{2}\left[\left(Tr\bar{\mathbf{\Upsilon}}\right)^2 - Tr\bar{\mathbf{\Upsilon}}^2\right] - \frac{1}{2}Tr\bar{\mathbf{\Pi}}^2.$$

Now making use of (9)-(11) we obtain

$$\begin{aligned} Tr\bar{\mathbf{\Pi}}^2 &= Tr\mathbf{\Pi}^{*2} = Tr\left(\mathbf{P}^2\mathbf{\Pi}\right)^2 = \\ &= Tr\left[\left(M\otimes M - \mathbf{I}\right)\mathbf{\Pi}\right]^2 = -2\left(\mathbf{h}\bullet\mathbf{M}\right)^2, \end{aligned}$$

and finally,

$$\det \bar{\mathbf{\Gamma}} = \left(1 + \frac{1}{2}Tr\ \bar{\mathbf{\Upsilon}}\right)^2 + (\mathbf{\Pi}\bullet\mathbf{M})^2 - \frac{1}{4}\left[2Tr\ \bar{\mathbf{\Upsilon}}^2 - \left(Tr\ \bar{\mathbf{\Upsilon}}\right)^2\right] = T^2_{(M)} - R^2_{(M)}.$$

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O srednjoj rotaciji konačnih deformacija

U radu se definiše ugao srednje rotacije deformabilnog tela. Dalje se pokazuje kako se ova definicija može koristiti za odredjivanje srednje rotacije deformabilnog tela u smislu Cauchy-ja i Truesdell-a i Toupin-a bez ikakvih ograničenja.