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DYNAMIC RESPONSE OF CONTINUOUS BEAM DUE TO MOVING INERTIAL LOAD

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1. Introduction

There are many examples where the structural system of the given object may be represented by the continuous beam with rigid, elastic or combined supports. Certain types of bridges, fluid conveying pipelines, superstructures, etc. are come of the examples.

All of the mentioned structures are acted upon by both dead and moving distributed load. Sometimes the mass of the moving load may not be neglected when compared to the mass of the beam. Consequently, one must take into account both gratational and inertial effects of the moving load with inevitable coupling betwen the moving and fixed masses.

Some results of research in this particular field of structural dynamics may be found in the literarure. See, for example, [2], [3] and [5].

This paper is further contribution to vibration analysis of continuos beam due to moving distributed load where inertial effects of the moving mass are also considered.

2. Basic Assumptions and the Problem

Continuous beam is treated according to Bernoulli-Euler beam theory assuming constant cross section and mass per unit length. Damping is neglected.

Number of supports is arbitrary. Internal supports may be rigid, elastic or combined, but external cross sections are rigidly simply-supported.

Moving load is uniformly distributed and the mass of the load may not be neglected when compared to the mass of the beam. The load is moving at constant speed of the basic motion.

Possible initial imperfection of the beam of whatever cause is also taken into account. Initial imperfection may be convenient way to take into consideration the influence of the dead load simply by treating obtained static deflection as the initial imperfection.

The objective of the paper is to obtain dynamic response of continuous beam under defined initial conditions due to moving distributed inertial load.

3. Notation

δ — variotional operator

T — kinetic energy

A — strain energy

δR — virtual work of nonconservative forces

t — time variable

x — space variable

EJ — flexural stiffness

$u(x, t)$ — dynamic deflection

L — length of the beam

ρ — mass density per unit length of the beam

ρ_0 — mass density per unit length of the moving load

V_0 — speed of basic motion of the moving load

$X(x, t)$ — distributed loading function

$\delta(x)$ — Dirac's function

$R_i(t)$ — dynamic reaction of the rigid support "i"

$\tilde{R}_j(t)$ — dynamic reaction of the elastic reaction "j"

c_j — stiffness of the elastic support "j"

$\eta(x)$ — initial imperfection

k_1 — number of rigid internal supports

k_2 — number of elastic internal supports

N — set of natural numbers

a_i — space coordinate of rigid support "i"

b_j — space coordinate of elastic support "j"

4. Mathematical model

Representative continuous beam with arbitrary configuration of supports is shown in Fig. 1.:

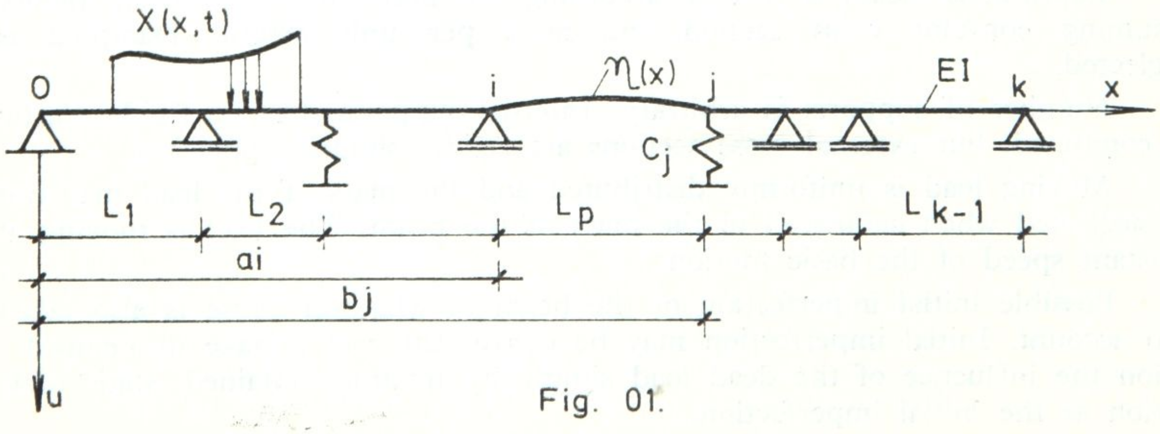

Fig. 01.

Beam element in deformed configuration i presented in Fig. 2.:

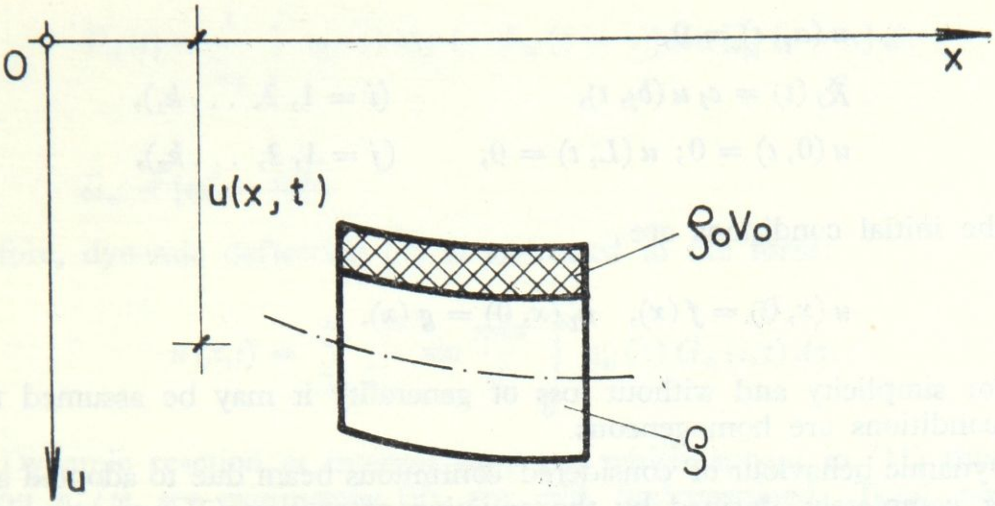

Fig. 02.

Governing equation of undamped transverse vibrations is derived from the Hamilton's principle [1].

$$\int_0^t (\delta T - \delta A)\, dt + \int_0^t \delta R dt = 0, \tag{1}$$

where

$$A = \frac{1}{2}\int_0^L EJ\,[u_{xx} + \eta''_{(x)}]^2\, dx,$$

$$T = \frac{1}{2}\int_0^L [(\rho + \rho_0)\, u_t^2 + 2\,\rho_0\, V_0\, u_t\,(u_x + \eta') + \rho_0\, V_0^2\,(u_x + \eta')^2]\, dx, \tag{2}$$

$$\delta R = \int_0^L \left[X(x,t) - \sum_{i=1}^{k_1} R_{i(t)}\,\delta(x - a_i) - \sum_{j=1}^{k_2} \widetilde{R}_{j(t)}\,\delta(x - b_j)\right]\delta u dx$$

represent strain energy, kinetic energy and virtual work of nonconservative forces. Substituting expressions (2) into the Hamilton's principle (1) partial differential equation of transverse vibrations may be obtained:

$$u_{tt} + \frac{c^2}{1+\beta} - u_{xxxx} + \frac{2\beta V_0}{1+\beta} u_{xt} + \frac{\beta V_0^2}{1+\beta} u_{xx} = q(x,t) - \frac{\beta V_0^2}{1+\beta}\eta''(x) -$$

$$- \frac{c^2}{1+\beta}\eta^{(IV)}_{(x)} - \frac{1}{\rho + \rho_0}\left[\sum_{i=1}^{k_1} R_i(t)\,\delta(x - a_i) + \sum_{j=1}^{k_2} \widetilde{R}_j(t)\,\delta(x - b_j)\right], \tag{3}$$

$$\beta = \frac{\rho_0}{\rho},\quad c^2 = \frac{EJ}{\rho},\quad q\,(x,t) = \frac{X(x,t)}{\rho + \rho_0}. \tag{4}$$

Displacement boundary conditions related to continuous beam Fig. 1. are given by:

$$u(a_i, t) = 0,$$
$$\widetilde{R}_j(t) = c_j u(b_j, t), \qquad (i = 1, 2, \ldots k_1), \tag{5}$$
$$u(0, t) = 0; \; u(L, t) = 0, \qquad (j = 1, 2, \ldots k_2),$$

while the initial conditions are

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \tag{6}$$

For simplicity and without loss of generality it may be assumed that the initial conditions are homogeneous.

Dynamic behaviour of considered continuous beam due to adopted assumptions is completely defined by the equation of motion (3) and boundary and initial conditions (5) and (6).

5. Solution of the Boundary Problem

If the solution of the equation of motion (3) is assumed in the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L} \tag{7}$$

then the displacement boundary conditions at the external supports are satisfied.

Substituting (7) into (3) and elaborating as given in [4], due to orthogonality of natural modes, equation of motion (3) is transformed into the set of ordinary differential equations:

$$\ddot{T}_n(t) + 2\beta_n \dot{T}_n(t) + \omega_n^2 T_n(t) = \eta_n(t), \; n \in N, \tag{8}$$

where

$$\omega_n^2 = \frac{c^2}{1+\beta} \left(\frac{n\pi}{L}\right)^4 \left[1 - \left(\frac{LV_0}{n\pi c}\right)^2\right],$$

$$\beta_n = \frac{\beta V_0}{1+\beta} \frac{n\pi}{L},$$

$$\eta_n(t) = \frac{2}{L} \int_0^L \left[q(x,t) - \frac{\beta V_0^2}{1+\beta} \eta''(x) - \frac{c^2}{1+\beta} \eta_{(X)}^{(IV)}\right] \sin \frac{n\pi x}{L} dx - \tag{9}$$

$$- \frac{2}{L(\rho + \rho_0)} \left[\sum_{i=1}^{k_1} Ri(t) \sin \frac{n\pi a_i}{L} + \sum_{j=1}^{k_2} \widetilde{R}_j(t) \sin \frac{n\pi b_j}{L}\right].$$

Particular solution of the representative equation of the set (8) is given by

$$T_n(t) = \frac{1}{\bar{\omega}_n} \int_0^t \eta_n(\tau)\, exp\,[-\beta_n(t-\tau)] \sin \bar{\omega}_n (t-\tau)\, d\tau$$

$$\bar{\omega}_n = [\omega_n^2 - \beta_n^2]^{\frac{1}{2}} \tag{10}$$

Therefore, dynamic deflection (7) is expressed in the form:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{\bar{\omega}_n} \sin \frac{n\pi x}{L} \int_0^t \eta_n(\tau)\, G_n(t,\tau)\, d\tau. \tag{11}$$

Dynamic reaction at internal supports, which appear in (11) through the function $\eta_n(t)$, see expressions (9), are still undetermined. Total number of unknown dynamic reactions is equal to $k = k_1 + k_2$, i.e. to the number of displacement boundary conditions at internal supports given by (5).

Substituting the solution (11) into the boundary conditions (5) one obtaines the system of integral equations of Volterra type in unknown dynamic reactions $R_i(t)$ and $\tilde{R}_j(t)$:

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_1} A_{nip} \int_0^t R_i(\tau)\, G_n(t,\tau)\, d\tau + \qquad (p = 1,2, \ldots k_1)$$

$$+ \sum_{n=1}^{\infty} \sum_{j=1}^{k_2} \tilde{B}_{njp} \int_0^t \tilde{R}_j(\tau)\, G_n(t,\tau)\, d\tau = A_p(t),$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_1} A_{niq} \int_0^t R_i(\tau)\, G_n(t,\tau)\, d\tau + \qquad (q = 1,2, \ldots k_2) \tag{12}$$

$$+ \sum_{n=1}^{\infty} \sum_{j=1}^{k_2} \tilde{B}_{njq} \int_0^t \tilde{R}_j(\tau)\, G_n(t,\tau)\, d\tau + c_q^{-1} \tilde{R}_p(t) = \tilde{A}_q(t).$$

In equations (12) terms $A_p(t)$ and $\tilde{A}_q(t)$ represent known functions while A_{nip}, B_{njp}, and $\tilde{B}_{njq}$ represent known constant coefficients. The kernel $G_n(t,\tau)$ of integral equations (12) is defined by

$$G_n(t,\tau) = exp\,[-\beta_n(t-\tau)] \sin \bar{\omega}_n(t-\tau). \tag{13}$$

Solving integral equation (12) one obtains dynamic reactions $R_i(t)$ and $\tilde{R}_j(t)$ and consequently, dynamic deflection i.e. motion of the beam, may be computed from the expression (11). Further, starting from obtained deflection $u(x,t)$ all other quantities of interest (velocities of transverse motion, internal forces, stresses etc.) may be readily calculated.

Therefore, the complete dynamic response of the continuous beam is obtained.

As may be noticed, the crucial part of the procedure is solution of integral equations (12) and evaluation of integral given in (11). In computer oriented numerical solution it is convenient to present the system of integral equations (12) in the matrix form:

$$\int_0^t \underset{\sim}{R}(\tau)\,\underset{\sim}{G}(t,\tau)\,d\tau + \underset{\sim}{C}\underset{\sim}{R}^T(t) = \underset{\sim}{A}(t), \tag{14}$$

where

$$\begin{aligned} \underset{\sim}{R}(t) &= \{R_1(t), \dots R_{k1}; \widetilde{R}_1, \dots \widetilde{R}_{k2}\}, \\ \underset{\sim}{A}(t) &= \{A_1(t), \dots A_{k1}; \widetilde{A}_1(t), \dots \widetilde{A}_{k2}\}, \\ \underset{\sim}{C} &= \{0, \dots\dots 0;\quad c_1^{-1}, \dots c_{k2}^{-1}\}, \end{aligned} \tag{15}$$

are the k-dimensional vectors, while the kernel of integral equation (14) is given by the matrix

$$\underset{\sim}{G}(t,\tau) = \left[\begin{array}{c|c} A_{(pi)} & \widetilde{B}_{(qj)} \\ \hline C_{(pi} & \widetilde{D}_{(qj)} \end{array}\right]. \tag{16}$$

As for numerical evaluation of integrals given in (11) one of numerical integration procedures (e. g. Gauss quadrature) may be used.

In the case of continuous beam with internal supports of the same type, i. e. either rigid or elastic, but not combined, the system of integral equations (12) or matrix integral equation (14) is significantly reduced. For example, for rigid internal supports, the obtained Volterra's integral equation is of the first kind. This problem is presented in [4], but for the case where the mass of the moving load is small when compared to the mass of the beam, so that the inertial effects of the moving load are insignificant. Such an assumption may be easily incorporated into the present approach simply by letting $\beta = 0$, sec Eq. (3).

In the case when all of the internal supports are elastic (for example in the analysis of railvay superstructure), integral equations (12) are of the second kind. Dynamic response of u two equal span beam with elastic internal support, due to a moving concentrated force is given in [2].

6. Conclusion

This paper presents a method of analysis of undamped vibrations of continuous beam with rigid and/or elastic supports induced moving distributed load. Inertial effect of a moving load are also taken into account. As opposed to some of the well established methods of dynamic analysis of continuous beams (e. g. the slope-deflection method, the method of initial parameters, etc) the present method is reduced to the solution of integral equations of Volterra fype to obtain dynamic reactions of internal supports. With calculated reaction torces dynamic deflection and other quantities of interest are given by infinite series.

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DYNAMIK DES DURCH BEWEGLICHE TRÄGHEITSBELASTUNG ERREGTEN DURCHLAUFENDEN TRÄGERS

Zusammenfassung

Es wurden in dieser Arbeit die Querschwingungen des durchlaufenden Trägers mit steifen, elastischen und gemischten Unterstützungen untersucht, der durch die Art der beweglichen Trägheitsbelastung erregt wurde. Die anfängliche Unvollkommenheit des Trägers wurde auch berücksichtigt, die aus technologishen Imperfektionen der Konstruktion oder der Auswirkung der statischen Einflüsse auf die Konstruktion im unerregten Zustand resultiert.

Die Idee des Kraftgrössen-Verfahrens folgend, wurde der Einfluss der Zwischenstützen mit equivalenten reaktiver Belastung vertauscht, die mittels der Diracschen Funktion durch die Reihe der dynamischen Reaktionen $R_i(t)$, $(i = 1, 2, \ldots k_1)$ in steifen Zwischenstützen und mit dynamischen Reaktionen $\tilde{R}_j(t), j = 1, 2, \ldots k_2)$ in elastischen Zwischenstützen definiert wurde.

Vom Prinzip klernsten Wirkung von Hamilton ausgehend, wurde die Aufgabe aufs Grenzproblem abgeleitet, das durch partielle Differentialgleichung (3) und Bedingungen in den Zwischenstützen (5) definiert wurde.

Das Problem wurde mittels der Fouriermethode auf Volterra Matrixintegralgleichung abgeleitet:

$$\int_0^t \underset{\sim}{R}(\tau)\, \underset{\sim}{G}(t,\tau)\, d\tau + \underset{\sim}{C}\underset{\sim}{R}^T(t) = \underset{\sim}{A}(t)$$

aus welcher man die dynamischen Reaktionen in den Zwischenstützen direkt, bestimmen kann, die durch k-Dimensionvektor $\underset{\sim}{R}(t)$ definiert wurden, wo $k = k_1 + k_2$ die Anzahl der Zwischenstützen am Träger ist.

Izvod

DINAMIKA KONTINUALNOG NOSAČA POBUĐENOG POKRETNIM INERTNIM OPTEREĆENJEM

U radu su proučavane poprečne vibracije kontinualnog nosača sa krutim, elastičnim ili mešovitim osloncima, koji je pobuđen klasom pokretnog internog opterećenja. U razmatranja je uzeta i početna imperfekcija nosača, koja je rezultat tehnološke nesavršenosti konstrukcije ili delovanja statičkih uticaja na konstrukciju u nepobuđenom stanju.

Sledeći ideju metode sila, uticoj međuoslonaca je zamenjen ekvivalentnim reaktivnim opterećenjem, koje je pomoću Dirac-ove funkcije definisano skupom dinamičkih reakcija $R_i(t)$, $(i = 1, 2, \ldots k_1)$ u krutim međuosloncima i dinamičkim rakcijama $\tilde{R}_j(t)$, $(j = 1, 2, \ldots k_2)$ u elastičnim međuosloncima.

Polazeći od Hamilton-ovog principa, zadatak je sveden na granični problem definisan parcijalnom diferencijalnom jednačinom (3) i uslovima u međuosloncima (5).

Problem je Fourier-ovom metodom sveden na matričnu integralnu jednačinu tipa Volterra:

$$\int_0^t \underset{\sim}{R}(\tau)\, \underset{\sim}{G}(t,\tau)\, d\tau + \underset{\sim}{C}\underset{\sim}{R}^T(t) = \underset{\sim}{A}(t)$$

iz koje je moguće direktno odrediti dinamičke reakcije u međuosloncima, koje su definisane k-dimenzionim vektorom $\underset{\sim}{R}(t)$, gde je $k = k_1 + k_2$ broj međuoslonaca na nosaču.

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