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A GENERALIZATION OF THE CONTACT TRANSFORMATION AND THEIR APPLICATION IN CONTINUUM MECHANICS

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1. Introduction

It is well known that the research of the infinitesimal transformation which maps the set of solutions of a differential equations into itself and together with considering the invariant of the tangent structure equations, on the one hand, and their applications on geometry and mechanics on the other, leads us to the idea devloped by S. Lie and F. Engel [1]. A direct generalization of these ideas and formulations was given in [2] and [3].

In this paper we try to generalize these results in the sense initiated in [4], namely, we wish to generalize these results to a so-called isoperimetric problem and its applications in continuum mechanics.

2. The contact transformation

Let $x=\{x^i \mid n\}\equiv\{x^1, x^2, \ldots x^n\}\in R^n$, $u^a=\{u^a \mid m\}\equiv\{u^1, u^2, \ldots, u^m\}\in R^m$, $i=\psi(n)$ and $a=\psi(m)$ where $s=\psi(k)$, mean that indices have the range 1, 2, ..., k. A comma (,), will denote partial differentiation with the following conventions $f_{,i}\equiv\dfrac{\partial f(x)}{\partial x^i}$. In this sense we can write $f_{i_1 i_2}\equiv\dfrac{\partial f(x)}{\partial x^{i1} x^{i2}}$ and

$\underset{1}{u}\equiv\{u_{,1}, u_{,2}, \ldots, u_n\}$, $\underset{2}{u}\equiv\{u_{,11}, u_{,12}, u_{,13}, \ldots,\}$, $\underset{S}{u}=\{u_{,i_1 i_2}\ldots\ldots\}$.

Let us consider a r-parameters group G of a point transformation:

$$
(1)\qquad G:\quad
\begin{aligned}
\tilde{x}^i &= \tilde{x}^i(x, u, \underset{1}{u}, \underset{2}{u}, \ldots; a)\\
\tilde{u}^a &= \tilde{u}^a(x, u, \underset{1}{u}, \underset{2}{u}, \ldots; a)\\
\tilde{u}_i^a &= \tilde{u}_i^a(x, u, \underset{1}{u}, \underset{2}{u}, \ldots,; a)\\
&\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots
\end{aligned}
$$

in the infinite dimensional space $(x, u, \underset{1}{u}, \ldots\ldots)$ and together with another group G of the point transformations in the space of independent variafiles $(x, u, \underset{1}{u}, \ldots;$ $dx, du, d\underset{1}{u}, \ldots)$. The group G is obtained by the e x t e n s i o n of the action of the g^r^oup G on the differentials by means of the formulas

(2)
$$\begin{aligned}
d\tilde{x}^i &= \frac{\partial \tilde{x}^i}{\partial x^j}\, dx^j + \frac{\partial \tilde{x}^i}{\partial x^a}\, du^a + \frac{\partial \tilde{x}^i}{\partial u_{,i}^{a}}\, du_{,i}^{a} + \ldots \\
d\tilde{u}^a &= \frac{\partial \tilde{u}^a}{\partial x^i}\, dx^i + \frac{\partial \tilde{u}^a}{\partial u^b}\, du^b + \frac{\partial \tilde{u}^a}{\partial u_i^b}\, du_{,i}^{b} + \ldots \\
d\tilde{u}_{,i}^{a} &= \frac{\partial \tilde{u}_{,i}^{a}}{\partial x^j} dx^j + \frac{\partial \tilde{u}_{,i}^{a}}{\partial u^b} du^b + \frac{\partial \tilde{u}_{,i}^{a}}{\partial u_j^b}\, du_{,j}^{b} + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

It is convenient to express our formulas (2) in terms of the operator

(3)
$$\tilde{D} = dx^i \frac{\partial}{\partial x^i} + du^a\, \frac{\partial}{\partial u^a} + du_{,i}^{a}\, \frac{\partial}{\partial u_{,i}^{a}} + \ldots$$

when equations (2) become

$$d\tilde{x}^i = \tilde{D}\tilde{x}^i, \qquad d\tilde{u}^a = \tilde{D}\tilde{u}^a, \qquad d\tilde{u}_i^a = \tilde{D}\tilde{u}_i^a, \qquad \tilde{D}\tilde{u}_{,i}^{a}$$

or in the compact form

(4)
$$\{d\tilde{x}^i,\, d\tilde{u}^a,\, d\tilde{u}_{,i}^{a}\ \ldots\} = \tilde{D}\, \{\tilde{x}^i,\, \tilde{u}^a,\, \tilde{u}_i^a,\, \ldots\}.$$

At present, the most general definition of contact transfo mation is:

DEFINITION. A group of contact transformation is called group G if the system of the equations

(5)
$$D\, u^a = 0, \qquad D\, u_{,i}^{a} = 0, \qquad \ldots\ldots,$$

is invariant with respect to the operator G which is an infinite extension of G.

It is known from [2] that infinitesimal transformation is characterized by infinitesimal operators of the group G

(6)
$$X_\alpha - \xi_\alpha^i \frac{\partial}{\partial x^i} + \eta_\alpha^a \frac{\partial}{\partial u^a} + \zeta_{\alpha i}^{a} \frac{\partial}{\partial u_{,i}^{a}} + \ldots$$

where

$$\xi_\alpha^i = \frac{\partial \tilde{x}^i}{\partial a^\alpha}\bigg|_{a^\alpha = 0}; \qquad \eta_\alpha^a = \frac{\partial \tilde{u}^a}{\partial Q^\alpha}\bigg|_{a^a = 0}; \qquad \zeta_{\alpha i}^{a} = \frac{\partial \tilde{u}_{,i}^{a}}{\partial a^a}\bigg|_{q^a = 0}, \ \ldots$$

and which satisfy the following relations

(7)
$$[x_\alpha\ x_\beta] = C_{\alpha\beta}^{\gamma}\ x_\gamma$$

where $C^{\gamma}_{\alpha\beta}$ are structural constants. We know that operator of the extended group which denoted by $\tilde{X}$, is given by

$$\tilde{X}_\alpha = X_\alpha + \tilde{\xi}^i_\alpha \frac{\partial}{\partial(dx^i)} + \tilde{\eta}^a_\alpha \frac{\partial}{\partial(du^a)} + \tilde{\eta}^a_{\alpha i} \frac{\partial}{\partial(du,^a_i)} \tag{8}$$

where

$$\tilde{\xi}^a_i = \underset{S}{D}\, \xi^i_a, \quad \tilde{\eta}^a_\alpha = \underset{S}{D}\, \eta^a_\alpha, \quad \tilde{\zeta}^a_{\alpha i} = \underset{S}{D}\, \zeta^a_i, \ldots \tag{9}$$

and where operator $\underset{S}{D}$, has the form

$$\underset{S}{D} = dx^i \frac{\partial}{\partial x^i} + du^a \frac{\partial}{\partial u^a} + \ldots . + du^a_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^a_{i_1 i_2 \ldots i_s}} \tag{10}$$

Now the criterion for the invariance of the equations (5), is given by conditions:

$$\tilde{X} D\, u^a = 0 \qquad \tilde{X} D\, u,^a_i = 0 \qquad \tilde{X}\, D\, u,^a_{i_1 i_2} = 0 \qquad \ldots\ldots \tag{11}$$

If we express du^a, $du,^a{}_i$, ... , in terms of independent quantities $u,^a{}_i$, $u^a{}_{i1\ i2}$, ... , and these expressions and (9) are substituted in (11), we have

$$\begin{aligned} &\tilde{\eta}^a_\alpha - \zeta^a_{\alpha i}\; dx^i - \tilde{\xi}^i_\alpha\; u,^a_i = 0 \\ &\tilde{\zeta}^a_{\alpha i} - \zeta^a_{\alpha ij}\, dx^j - \tilde{\xi}^j_\alpha\; u,^a_{ij} = 0 \\ &\tilde{\zeta}^a_{\alpha\, i_1 i_2 .. i_s} - \zeta^a_{\alpha\, i_1 i_2 .. i_s j}\, dx^j - \tilde{\xi}^j_\alpha\, u,^a_{i_1 \ldots i_s j} = 0 \end{aligned} \tag{12}$$

Finally, after all sufistitutions (9) are carried out, we have

$$\begin{aligned} &\zeta^a_{\alpha i} = D_i\, (\eta^a_\alpha) - u,^a_j\; D_i\; \zeta^j_\alpha \\ &\zeta^a_{\alpha\, i_1 i_2} = D_{i2}\; \zeta^n_{\alpha\, i_1} - u,^a_{i_1 j}\; D_{i2}\; \xi^j_\alpha \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag*{)13)}$$

where $$D_i = \frac{\partial}{\partial x^i} + u,^a_i \frac{\partial}{\partial u^a} + u,^a_{ij} \frac{\partial}{\partial u,^a_j} + u,^a_{ijk} \frac{\partial}{\partial u,^a_{jk}} + \ldots$$

The functions $\zeta^a_{\alpha\, i}$, $\zeta^\alpha_{a\, i_1 i_2}$, represent cordinates of extension operators (8).

3. Generalization and Application

A discussion of the group theoretical analysis of differential equations of mechanics which are Euler-Lagrange equations of some variational formulation, leads to the idea of considering the invariance of the tangent structure (5) in conjuction with a given Lagrangian in the form

$$I_\omega = \int_\Omega l\, (x,\, u,\, \underset{1}{u},\, \ldots)\; d\omega \tag{14}$$

with s derivatives occurring in the integrand. In more complicated cases we can consider a group theoretical analysis for some so-called isoperimetric profilem [4]. In these cases we consider the invariance in conjunction with a given Lagrangian and constraint integral in the form

(15) $$K_\omega = \int_\Omega k\ (x,\ u,\ \underset{1}{u},\ \ldots)\ d\omega$$

In these cases we have a generalized theory of contact transformation applied to the so-called isoperimetrical problem.

If we continue in a similar way as in [4], and after longer calculation we can state that if all functions u^a and for all transformations (2) which for the following relation

(16) $$\int_\Omega l\,(x,\ u,\ \underset{1}{u},\ \ldots)\ d\omega = \int_\Omega l\,(\tilde{x},\ \tilde{u},\ \underset{1}{\tilde{u}},\ \ldots)\ J\,(\tilde{x},\ x)\ d\omega$$

are valid, then the valid are a following relation, to,

(17) $$\left\{\left(\tilde{X} + \zeta_a \frac{\partial}{\partial (d\omega)}\right) \mathcal{L}\right\}_{|\,(5)} = 0$$

where $\tilde{X}$ is operator given by (8), and

(18) $$\mathcal{L} = l + \lambda k$$

and where (4), $\zeta_\alpha = D_i\,(\xi_\alpha{}^i)\,dw$, and coordinates $\xi_\alpha{}^i$, $\eta_\alpha{}^a$, $\zeta_{\alpha i}{}^a$, $\ldots$ $\tilde{\xi}_\alpha{}^i$, $\tilde{\eta}_\alpha{}^a$, $\tilde{\zeta}^a_{\alpha i}$, .. $\tilde{\zeta}^a_{\alpha ij}$ ..., satisfy the equdtions (12) and (13). After some calculations and rearrangements, express ion (16) can be written in the form

(19) $$\mu^a_\alpha \frac{\delta \mathcal{L}}{\delta u^a} + D_i\ (A^i_\alpha) = 0,$$

where we have used the relations (8); λ is Lagrange's multipler. The expression $\frac{\delta L}{\delta u^a}$ is given by

(20) $$\frac{\delta \mathcal{L}}{\delta u^u} \equiv \frac{\partial \mathcal{L}}{\partial u^a} + \sum_{\nu=1}^{\infty} (-1)^\nu\ D_{i1} \ldots\ D_{i\nu} \frac{\partial \mathcal{L}}{\partial u^a_{i1 \ldots i2}},$$

and where

$$A^i_\alpha = \mathcal{L}\xi^i_\alpha + \mu^a_\alpha \frac{\delta \mathcal{L}}{\delta u^a_{,i}} + \sum_{S=1}^{\infty} D_{i1}\ D_{i2} \ldots\ D_{i\nu} \frac{\delta' \mathcal{L}}{\delta u^a_{i,\,i1 \ldots is}},$$

(21) $$\frac{\delta' \mathcal{L}}{\delta u^a_{i,\,i1\,..\,is}} = \frac{\partial \mathcal{L}}{\partial u^a_{,i,i1\,i2..is}} - \sum_{\nu=1}^{\infty} (-1)^\nu\ D_{\partial 1} \ldots\ D_{\partial s} \left(\frac{\partial \mathcal{H}}{\partial u^a_{i,j,j\nu\ i1\ i2\ is}}\right).$$

The vector $A^i{}_\alpha$ is known as conserved-quantities vector, and when the equations of motions are Euler-Lagrange's equations and are valid, then from (19) we have as follows:

$$\frac{\delta L}{\delta u^u} = 0 \tag{22}$$

and

$$D_i\,(A^i_\alpha) = 0 \tag{23}$$

The expression (23), is the generailization of Noether's theorem for contact transformations and for the isoperimetric problem.

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ОБОПЩЕНИЕ КАСАТЕЛЬНИХ ПРЕОБРАЗОВАНИЙ И ИХ ИСПОЛЬЗОВАНИЕ В МЕХАНИКЕ СПЛОШНОЙ СРЕДЫ.

Резюме

В работе рассматривается обобщение теории С. Ли на случай нескольких функций и на изопериметрически проблем в механике сплошной среды.

Ползуясь групой касательних преобразований и ползуясь инвариантностю интеграла действйя как и интеграла „принуждения" формируются закони сохранения.

GENERALIZACIJA KONTAKTNIH TRANSFORMACIJA I NJIHOVA PRIMENA U MEHANICI KONTINUUMA

Izvod

U radu se proučava mogućnost uopštenja Liove teorije kontaktnih transformacija na slučaj funkcija više promenljivih i na slučaj kontakta beskonačnog reda. Proučava se takođe „ograničeni" varijacioni problem, definisan izrazima (14) i (15) pa se tako problem svodi na izoperimetrijski problem mehanike kako je definisan u [4]. Formiraju se konstituutivne jednačine grupe (12) koje uz zahtev (17) daju jednačine (19). Odavde se lako dobiva (23), generalisana Netereva teorema.

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