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ON CONSERVATION LAWS IN ELASTODYNAMICS

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1. Introduction

In a paper devoted to the analysis of stress concentrations Rice [1] introduced a "path-independent integral" arising from the field equations of elastostatics and demonstrated its utility in connection with the asymptotic analysis of singular stress field. The physical interpretation of this integral is based on the energetics of quasi-static crack extension

Apart from its inherent theoretical interest, the conservation law made explicit in [1] is of practical importance in connection with the direct asymptotic analysis of geometrically induced singular stress concentrations, such as those occasioned by cracks and notches. The fact that this integral follows from the equilibrium equations, the stress-displacement relations, and the definition of strain-energy density W is easily demonstrated by means of the divergence theorem. For two dimensions, such a proof was given by Rice [1]; it amounts to a verification and, as such, gives no indication of why it holds or whether other path-independent integrals exist.

In a recent paper by Knowles and Sternberg [2] it was shown that the conservation law

$$\int_{\Sigma} (W n_i - \sigma_{kj} U_{k,i} n_j)\, ds = 0 \tag{1.1}$$

and its two-dimensional counterpart follow an application of Noether's theorem on invariant variational principles to the principle of minimum potential energy in elastostatics. This procedure, moreover, yields two additional conservation laws.

Noether's theorem on variational principles invariant under a group of infinitesimal transformations was used by Fletcher [3] to obtain a class of conservation laws associated with linear elastodynamics. These laws represent dynamical generalisations of path-independent integrals in elastostatics. It is shown that the conservation laws obtained are the only ones obtainable by Noether's theorem from invariance under a reasonably general group of infinitesimal transformations. In all the papers above the way of proving and deriving of results is long, complicated and unsystematic.

The aim of this paper is to complete the results of so far derived theories and to systematise the way of derivation. The paper starts with the point that the corresponding coordinate and vector transformations do not depend on material properties of continua. Starting with this requirement, which is quite justified because conservation laws of any kind are valid for allowable elastodynamics processes of all continua, it is very easy and simple to come to corresponding groups of transformations which are in complete accordance with so far obtained results.

2. **Linear Elastodynamics and Noether's Theorem**

For the sake of a better survey and understanding of this paper in this Section we summarized the pertinent field equations and constitutive relations governing the theory to be considered.(1)

Let (x_1, x_2, x_3) be rectangular Cartesian coordinates, and let D be the closed, bounded, regular region in three-dimensional space E_3 occupied by a homogeneous elastic solid in its undeformed state. A particle of the solid in a motion, located at x in the undeformed configuration, is found at time t at the point with position vector $\mathbf{y}(\mathbf{x}, t)$. Then the corresponding displacement vector field u is defined by:

(2.1) $$\mathbf{U}(\mathbf{x}, t) = \mathbf{y}(\mathbf{x}, t) - \mathbf{x} \qquad \mathbf{x} \in D \quad t \geqslant 0.$$

The components of the infinitesimal strain tensor field $\underset{\sim}{\gamma}$ and stress tensor $\underset{\sim}{\sigma}$ are given by

(2.2) $$\gamma_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \qquad \text{on } D \times [0, \infty]$$

(2 3) $$\sigma_{ij} = C_{ijkl}\gamma_{kl}, \qquad \text{on } D \times [0, \infty]$$

where the components C_{ijkl} of the elasticity tensor satisfy the symmetry requirements

(2.4) $$C_{ijkl} = C_{jikl} = C_{klij}.$$

In the absence of body forces, the equations of the motion become[2)]

(2.5) $$\sigma_{ij,j} = \rho \ddot{U}_i, \qquad \text{on } D \times [0, \infty]$$

where ρ is the mass density-assumed constant, σ_{ij} the components of the Piola-Kirchhoff stress tensor and

(2.6) $$\ddot{U}_i(\mathbf{x}, t) = \frac{\partial^2 U_i}{\partial t^2}(\mathbf{x}, t).$$

Using (2.2), (2.3) and (2.4), (2.5) gives the following displacement equations of motion

(2.7) $$C_{ijkl} U_{k,lj} = \rho \ddot{U}_i. \qquad \text{on } D \times [0, \infty]$$

If we set

$$x_0 \equiv t\frac{\partial}{\partial t}(\quad) \equiv (\quad)_0$$

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1) In the paper we strictly used the subscripts and symbols wich wdre used in papers [2] and [3].

2) summation over repeated subscripts is implied
Latin subscripts have the range 1, 2, 3 unless otherwise stated.

The equations of motion can now be written in the more compact fourdimensional notation as follows[3)]:

$$C_{i\alpha k\beta} U_{k,\alpha\beta} = 0, \tag{2.8}$$

where, by definition

$$C_{i\alpha k\beta} = \begin{cases} C_{ijkl} & \text{if} \quad \alpha = j,\ \beta = l \\ -\rho\delta_{ik} & \text{if} \quad \alpha = \beta = 0 \\ 0 & \text{if} \quad \alpha = 0\ \beta \neq 0 \ \text{or}\ \alpha \neq 0\ \beta = 0 \end{cases} \tag{2.9}$$

and

$$U_{i,\alpha} = \begin{cases} U_{i,j} & \text{if} \quad \alpha = j \\ \dot{U}_i & \text{if} \quad \alpha = 0. \end{cases}$$

It is shown in [3] that the formal Euler-Lagrange differential equations associated with the functional

$$\mathscr{L}[\mathbf{U}] = \int_0^T \int_D L(\underset{\sim}{\nabla}\mathbf{U}, \dot{\mathbf{U}})\, d\mathbf{x}\, dt, \tag{2.10}$$

where the Lagrangian density L is given by

$$L(U_{i,\alpha}) = \frac{1}{2} C_{i\alpha k\beta} U_{i,\alpha} U_{k,\beta}, \tag{2.11}$$

are precisely the displacement equations of motion (2.8).

Lagrangian density L can be written in the form

$$L(\underset{\sim}{\nabla}\mathbf{U}, \dot{\mathbf{U}}) = \Gamma(\underset{\sim}{\gamma}) - \frac{1}{2}\rho\, \dot{U}\dot{U} \tag{2.12}$$

where elastic potential $\Gamma(\underset{\sim}{\gamma})$ is defined by

$$\Gamma(\underset{\sim}{\gamma}) = \frac{1}{2} C_{ijkl} U_{i,j} U_{k,l}. \tag{2.13}$$

For an isot opic material

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) \tag{2.14}$$

where the constants λ and μ are the Lame's moduli, so that (2.13) becomes

$$\Gamma(\gamma) = \frac{1}{2}\gamma_{ii}\gamma_{jj} + \mu\gamma_{ij}\gamma_{ij}. \tag{2.15}$$

We now proceed to cite the restricted version of Noether's theorem which has been applied to elastodynamics as well as elastostatics in [2] and [3].

Let $\xi_1, \xi_2, \ldots, \xi_n$ be rectangular Cartesian coordinates in n-dimensional Euclidean space E^n, and let R be a bounded, closed, regular region in E^n. Let

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3) Greek subscripts have the range 0, 1, 2, 3.

$\mathbf{w}=(w_1, w_2, \ldots, w_n)$ be an arbitrary vector field of m components defined on R. Suppose further that $\mathbf{w}$ is twice continuously differentiable, and define a functional on the class of such vector fields $\mathbf{w}$ by the formula

$$\mathcal{F}[\mathbf{w}]=\int_R F(\xi, \mathbf{w}(\xi) \ \underset{\sim}{\nabla} \mathbf{w}(\xi))\, e\, \zeta \tag{2.16}$$

where F is a given real valued scalar function defined and infinitely differentiable for all values of its arguments.

Given the point $\xi \in R$ and a vector field $w \in C^2(R)$, define a family of transformations $(\xi, w(\xi)) \rightarrow (\xi^*, w^*(\xi^*))$ by the formula

$$\xi^*=\mathbf{\Phi}(\xi, \mathbf{w}(\xi), \underset{\sim}{\nabla} \mathbf{w}(\xi)) \tag{2.17a}$$

$$\mathbf{w}^*(\xi^*)=\underset{\sim}{\psi}(\xi, \mathbf{w}(\xi), \underset{\sim}{\nabla} \mathbf{w}(\xi) \tag{2.17b}$$

where $\mathbf{\Phi}$, ψ are respectively, n- and m-dimensional vector-valued functions of their arguments. For $\varepsilon=0$ the transformations are required to reduce to the identity $\zeta^*=\xi$, $\mathbf{w}^*(\zeta^*)=\mathbf{w}(\xi)$.

The functional $\mathcal{F}$ in (2.16) is said to be invariant at $\mathbf{w}$ under the transformation (2.17) if

$$\int_{R^*} \mathbf{F}(\xi^*, \mathbf{w}^*(\xi^*), \underset{\sim}{\nabla}^* \mathbf{w}^*(\xi^*))\, d\xi^*=\int_R \mathbf{F}(\xi, \mathbf{w}(\xi), \underset{\sim}{\nabla} \mathbf{w}(\xi))\, d\xi \tag{2.18}$$

for all sufficiently small values of $|\varepsilon|$. If, for a given w,

$$\left\{\frac{d}{d\varepsilon} \int_{R^*} \mathbf{F}(\xi^*, \mathbf{w}^*(\xi^*), \underset{\sim}{\nabla}^* \mathbf{w}^*(\xi^*))\, d\xi^*\right\}_{\varepsilon=0}=0 \tag{2.19}$$

then $\mathcal{F}$ is said to be infinitesimally invariant at $\mathbf{w}$. Note that if $\mathcal{F}$ is invariant at $\mathbf{w}$, then $\mathcal{F}$ is infinitesimally invariant at $\mathbf{w}$.

Theorem 1. *Let $\hat{R}$ be a domain in E, and suppose the vector field $\mathbf{w}$ satisfies the Euler-Lagrange equations*

$$\mathbf{F}_{|Wi}(\mathbf{X})-\frac{\partial}{\partial \xi}[\mathbf{F}_{|Wi,\alpha}(\mathbf{X})]=0 \tag{2.20}$$

where X stands for

$$\mathbf{X}=(\xi, \mathbf{w}(\xi), \underset{\sim}{\nabla} \mathbf{w}(\xi)), \qquad \xi \in \hat{R}. \tag{2.21}$$

Then $\mathcal{F}$ in (2.16) is infinitesimally invariant at $\mathbf{w}$ under the family of transformations (2.17) for every bounded, regular subregion R of $\hat{R}$ if and only if $\mathbf{w}$ also satisfies

$$\frac{\partial}{\partial \xi_\alpha}\{\mathbf{F}_{|Wi,\alpha}(\mathbf{X}) \cdot \bar{\psi}_i(\mathbf{X})+\mathbf{F}(\mathbf{X}) \cdot \varphi_\alpha(\mathbf{X})\}=0 \tag{2.22}$$

where

$$\varphi_\alpha(\mathbf{X}) = \frac{\partial}{\partial \varepsilon}\Phi_\alpha(\mathbf{X};\ \varepsilon)_{|\varepsilon=0}$$

(2.23)
$$\psi_i(\mathbf{X}) = \frac{\partial}{\partial \varepsilon}\Psi_i(\mathbf{X};\ \varepsilon)_{|\varepsilon=0}$$

$$\bar{\psi}_i(\mathbf{X}) = \psi_i(\mathbf{X}) - w_{i,\alpha}(\xi)\,\varphi_\alpha(\mathbf{X}).$$

If R is a bounded regular subregion of $\hat{R}$, (2.22) and the divergence theorem immediately yields

(2.24)
$$\oint_{\partial R} [\mathbf{F}_{|w_{i,\alpha}}(\mathbf{X}) \cdot \bar{\psi}_i(\mathbf{X}) + \mathbf{F}(\mathbf{X}) \cdot \varphi_\alpha(\mathbf{X})]\, n_\alpha(\xi)\, ds = 0$$

where ∂R is the boundary of R, and n_α is the ξ_α-component of the unit outward normal on ∂R. In general we shall speak of (2.24) — or is equivalent differential form (2.22) — as a conservation law.[4)]

In the application of Noether's theorem to be given in the following section, the vector field $\mathbf{w}$ will always be the displacement field $\mathbf{u}$ so that $m = 4$. In the elastodynamic case $\xi_i = x_i$ for $i = 1, 2, 3$ $\xi_4 = t$, and $n = 4$, and the four-dimensional region R in (2.16) is taken to be $D \times [0, T]$.

3. Transformations under which the Lagrangian functional is infinitesimal invariant

On the basis of here stated Noether's theorem corresponding conservation laws were given in [2] and [3]. The main problem in obtaining these conservation laws was connected with getting corresponding coordinate and vector transformations (2.17) under which function $\mathcal{F}$ is infitesimally invariant.

On the basis of the same theorem one can conclude that the existence of a vector field $\mathbf{w}$ which satisfies (2.20) and infitesimal invariance of $\mathcal{F}$ under (2.17) makes $\mathbf{w}$ satisfy (2.22). These relations are the only data from which the needed transformations can be found. However, from (2.22) and (2.23) one can see that the transformations (2.17) can be defined only to the linear term of ε. By means of a Taylor series expansion we can express (2.17) as follows:

$$\boldsymbol{\Phi}(\zeta,\ \mathbf{w}(\xi),\ \underset{\sim}{\nabla}\,\mathbf{w}(\xi);\ \varepsilon) = \xi + \varepsilon\underset{\sim}{\varphi}(\xi,\ \mathbf{w}(\xi),\ \underset{\sim}{\nabla}\,\mathbf{w}(\xi)) + 0(\varepsilon)$$

as $\varepsilon \to 0$

$$\underset{\sim}{\Psi}(\xi,\ \mathbf{w}(\xi),\ \underset{\sim}{\nabla}\,\mathbf{w}(\xi);\ \varepsilon) = \mathbf{w}(\xi) + \varepsilon\underset{\sim}{\psi}(\xi,\ \mathbf{w}(\xi),\ \underset{\sim}{\nabla}\,\mathbf{w}(\xi)) + 0(\varepsilon)$$

However, the quantities defined with (2.23) are the only ones we need for the corresponding conservation laws.

In this section we shall pay attention to linear elastodynamic problems of isotropic bodies. In this case we can prove the following:

4) The proof of this theorem is given in [4].

Theorem 2. *Suppose F is the function defined by*

$$F(\xi,\ \mathbf{w}(\xi),\ \underset{\sim}{\nabla}\,\mathbf{w}(\xi)) = L(\underset{\sim}{\nabla}\,\mathbf{U},\ \dot{\mathbf{U}}) \tag{3.1}$$

where $L(\nabla u, u)$ *is given by* (2.11) *or* (2.12), *and let* $\mathcal{F}$ *be the admissible functional for D generated by F. Suppose, further,* $\mathcal{F}$ *is infinitesimally invariant at* **u** *with respect to the transformations*

$$\xi^* = \mathbf{\Phi}(\xi,\ \mathbf{w}(\xi);\ \varepsilon)\ \mathbf{w}^*(\xi^*) = \underset{\sim}{\psi}(\xi,\ \mathbf{w}(\xi);\ \varepsilon) \quad \xi = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ t \end{Bmatrix} \tag{3.2}$$

for every elastodynamic state on D corresponding to the elasticity isotropic tensor C_{ijkl}, *given by* (2.14).

Then φ_α and ψ_i must obey

$$\begin{aligned} &\varphi_i(\xi) = \nu\, x_i + \varepsilon_{ijk}\, b_j\, x_k + C_i \\ &\varphi_0(\xi) = \nu\, t + C_0 \\ &\psi_i(\xi) = \nu\, U_i + \varepsilon_{ijk}\, b_j\, U_k + \varepsilon_{ijk}\, a_j x_k + d_i, \end{aligned} \tag{3.3}$$

where ν, C_α, b_i, a_i, and d_i are constant arbitrary real constants.

The proof of this Theorem is given in [3] in the way which is similar to the proof of Theorem 3.2. in [2]. The procedure of proving itself shows the independence of transformations on material constants. However, that demand, in the above mentioned papers, is not explicitly emphasized nor used, which affected the procedure of proving. It is important to note that beside the similarity of proving Theorem 3.2 in [2] and Theorem 2. in [3] there is also an essential difference. In our opinion the proof of theorem 2. in [3] is more correct from the mathematical and physical point of view.

Proof of Theorem 2. If we take that $\mathbf{w} = \mathbf{u}$ and use (3.1) in (2.20) we get

$$\frac{\partial}{\partial \xi_\alpha} L_{U_{i,\alpha}}(U_{j,\beta}) = C_{i\alpha k\beta}\, U_{k,\alpha\beta} = 0. \tag{3.4}$$

It means that Euler equations (2.20) reduce to the Cauchy equations of motion which are satisfied for displacement **u** which defines linear elastodynamic states of bodies. According to the Theorem 2. $\mathcal{F}$, i. e. L, is infitesimally invariant for such displacement **u** with respect to $\mathbf{\Phi}$ and $\underset{\sim}{\psi}$ of (3.2). Then (2.22) is valid on the basis of Theorem 1. This system of equations can be written as

$$\frac{\partial}{\partial \xi_\alpha} [L_{|U_{i,\alpha}}(U_{j,\beta})\, \psi_i(\mathbf{x}) + L(U_{j,\beta})\, \varphi_\alpha(\mathbf{x})] = 0. \tag{3.5}$$

On the basis of (3.4) and (2.23) as well as the expression

$$\bar{\psi}_{i;\alpha} = \psi_{i,\alpha} + \psi_{i,U_n} U_{n,\alpha} - U_{i,\gamma\alpha}\varphi_\gamma - U_{i,\gamma}\varphi_{\gamma,\alpha} - U_{i,\gamma}\varphi_{\gamma,U_n} U_{n,\alpha} \tag{3.6}$$

$$\varphi_{\gamma;\alpha} = \varphi_{\gamma,\alpha} + \varphi_{\gamma,U_n} U_{n,\alpha} \tag{3.7}$$

$$L_{|U_{i,\alpha}} = C_{i\alpha k\beta}\, U_{k,\beta} \tag{3.8}$$

where ";" denote the total derivative on ξ_α, and "," denote the partial derivative on ξ_α, we can write (3.5) in the form

$$C_{i\alpha k\beta} U_{i,\alpha} \psi_{k,\beta} + C_{i\alpha k\beta} U_{i,\alpha} \left[\frac{1}{2} U_{k,\beta} \varphi_{\gamma,\gamma} + \psi_{k,U_n} U_{n,\beta} - U_{k,\rho} \varphi_{\rho,\beta} \right] + \tag{3.9}$$

$$+ C_{i\alpha k\beta} U_{i,\alpha} \left[\varphi_{\gamma,\rho} \left(\frac{1}{2} U_{k,\beta} U_{\rho,\gamma} - U_{k,\gamma} U_{\rho,\beta} \right) \right] = 0.$$

Demanding this expression to be valid without any limitations on $U_{i.\alpha}$ we get three sets of equality

$$C_{i\alpha k\beta} U_{i,\alpha} \psi_{k,\beta} = 0 \tag{3.10}$$

$$C_{i\alpha k\beta} U_{i,\alpha} \left[\frac{1}{2} U_{k,\beta} \varphi_{\gamma,\gamma} + \psi_{k,U_n} U_{n,\beta} - U_{k,\rho} \varphi_{\rho,\beta} \right] = 0 \tag{3.11}$$

$$C_{i\alpha k\beta} U_{i,\alpha} \left[\varphi_{\gamma,U_\rho} \left(\frac{1}{2} U_{k,\beta} U_{\rho,\gamma} - U_{k,\gamma} U_{\rho,\beta} \right) \right] = 0 \tag{3.12}$$

which represent the linear, square and cubic term of the set of equations (3.9) upon $U_{i,\alpha}$. We shall further discuss each of these systems separately.

i. Linear Terms

Differentiating (3.10) with respect to $U_{i,\alpha}$ we get

$$C_{i\alpha k\beta} \psi_{k\beta} = 0.$$

Using (2.9a) this system can be written in the form

$$C_{i\alpha kl} \psi_{k,l} + C_{i\alpha k0} \psi_{k,0} = 0$$

which splits into two systems of equations which are

for $\alpha = j$ $$C_{ijkl} \psi_{k,l} = 0,$$

for $\alpha = 0$ $$C_{i0k0} \psi_{k,0} = 0,$$

With respect to (2.9) these systems of equations can be written as

$$\lambda \delta_{ij} \psi_{k,k} + 2\mu (\psi_{i,j} + \psi_{j,i}) = 0$$

$$- \rho \psi_{i,0} = 0.$$

Since the needed transformations (3.2) do not depend on the properties of elastic bodies, i.e. on material constants λ and μ, from the above equations follows

$$\psi_{i,j} + \psi_{j,i} = 0 \tag{3.13}$$

$$\psi_{k,k} = 0 \tag{3.14}$$

$$\psi_{i,0} = 0. \tag{3.15}$$

It is obvions that (3.14) is contained in (3.13).

ii. Cubic terms

The systems (3.12) can be written in the form

$$C_{i\alpha k\beta}\,\varphi_{\Upsilon, Us}\,(\delta_{\beta\tau}\,\delta_{\Upsilon\sigma}-2\,\delta_{\Upsilon\tau}\,\delta_{\beta\sigma})\,U_{i,\alpha}\,U_{k,\tau}\,U_{s,\sigma}=0$$

which splits into two systems because of (2.9):

(3.16) $$C_{ijkl}\,\varphi_{\Upsilon, Us}\,(\delta_{l\tau}\,\delta_{\Upsilon\sigma}-2\,\delta_{\Upsilon\tau}\,\delta_{l\sigma})\,U_{i,j}\,U_{k,\tau}\,U_{s,\sigma}=0$$

(3.17) $$\varphi_{\Upsilon, Us}\,(\delta_{0\tau}\,\delta_{\Upsilon\sigma}-2\,\delta_{\Upsilon\tau}\,\delta_{0\sigma})\,U_{i,0}\,U_{k,\tau}\,U_{s,\sigma}=0.$$

The equations $(3.17)_2$ can be written in the form

(3.18) $$-\varphi_{0, Us}\,U_{i,0}\,U_{k,0}\,U_{s,0}+\varphi_{j, Us}\,U_{i,0}\,(U_{k,0}\,U_{s,j}-2\,U_{k,j}\,U_{s,0})=0$$

from which follows that

(3.19) $$\varphi_{0. Us}=0.$$

Then (3.18) gives

$$\varphi_{j, Us}\,U_{i,0}\,(U_{k,0}\,U_{s.j}-2\,U_{k,j}\,U_{s,0})=0.$$

If we differentiate this equation with respect to $U_{p,q}$ we get

$$\varphi_{j, Us}\,U_{i,0}\,(U_{k,0}\,\delta_{sp}\,\delta_{jq}-2\,\delta_{kp}\,\delta_{jq}\,U_{s,0})=0.$$

Multiplying this equation with $\delta_{kp}\,\delta_{jq}$ we get

$$-3\,\varphi_{j, Uk}\,U_{i,0}\,U_{k,0}=0$$

from which follows that

(3.20) $$\varphi_{j, Uk}=0.$$

The conditions (3.19) and (3.20) can be written in the compact form

(3.21) $$\varphi_{\Upsilon, Uk}=0.$$

From this expression it follows tha (3.16) is identically satisfied.

iii. Quadratic Terms

According to (2.9) and the assumption that the transformations do not depend on material properties, the equation (3.12) can be split into two equations

(3.22) $$C_{ijkl}\,U_{i,j}\,(\varphi_{\Upsilon,\Upsilon}\,U_{k,l}+2\,\psi_{k, Un}\,U_{n,l}-2\varphi_{\rho,l}\,U_{k,s})=0$$

(3.23) $$U_{k,0}\,(\varphi_{\Upsilon.\Upsilon}\,U_{k,0}+2\,\psi_{k, Un}\,U_{n,0}-2\,\varphi_{i,s}\,U_{k,i}-2\,\varphi_{0,0}\,U_{k,0})=0$$

The equations (3.23) must be satisfied independently with respect to the quadratic and linear terms of $U_{i,0}$, i.e.

(3.24) $$U_{k,0}\,(\varphi_{\Upsilon,\Upsilon}\,U_{k,0}+2\,\psi_{k, Un}\,U_{n,0}-2\,\varphi_{0,0}\,U_{k.0})=0$$

(3.25) $$2\,\varphi_{i,0}\,U_{k,i}\,U_{k,0}=0.$$

From (3.25) it follows that

(3.26) $$\varphi_{i,0}=0.$$

Differentiating (3.24) with respect to $U_{j,0}$ one gets

$$[(\varphi_{\Upsilon,\Upsilon}-2\,\varphi_{0,0})\,\delta_{jk}+(\psi_{j,\,Uk}+\psi_{k,\,Uj})]\,U_{k,0}=0$$

from which follows that

(3.27) $$\psi_{(j,\,Uk)}=-\frac{1}{2}(\varphi_{\Upsilon,\Upsilon}-2\,\varphi_{0,0})\,\delta_{jk}{}^{1)}.$$

It still remains to discuss the equation (3.22). This equation can be written in the following form

$$C_{ijkl}\,U_{i,j}(\varphi_{\Upsilon,\Upsilon}\,U_{k,l}+2\,\psi_{k,\,Un}\,U_{n,l}-2\,\varphi_{m,l}\,U_{k,m})-2\,C_{ijkl}\,U_{i,j}\,\varphi_{0,l}\,U_{k,0}=0.$$

Each part of it must be equal to zero because of $U_{k,0}$ in the second member' i.e.

(3.27a) $$C_{ijkl}\,U_{i,j}(\varphi_{\Upsilon,\Upsilon}\,U_{k,l}+2\,\psi_{k,\,Un}\,U_{n,l}-2\,\psi_{m,l}\,U_{k,m})=0$$

(3.27b) $$C_{ijkl}\varphi_{0,l}\,U_{i,j}\,U_{k,0}=0.$$

Because of the arbitrarity and independence of $U_{i,j}$ and $U_{k,0}$, from (3.27b) follows that

$$C_{ijkl}\,\varphi_{0,l}=0.$$

With the help of (2.14) it immediately follows that

(3.27c) $$\varphi_{0,l}=0.$$

Using (2.14) the equation (3.27a) splits into two parts; one along the coefficient λ, the other along μ, which must be independently satisfied because of the introduced assumption of independence of transformations on these coefficients i.e.

(3.28) $$\varphi_{\Upsilon,\Upsilon}\,U_{i,i}\,U_{i,j}+2\,\psi_{n,\,Uk}\,U_{k,n}\,U_{i,i}-2\,\varphi_{n,k}\,U_{k,n}\,U_{i,i}=0 \qquad \text{along } \lambda$$

(3.29) $$(\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk})\,U_{i,j}\,[\varphi_{\Upsilon,\Upsilon}\,U_{k,l}+2\,\psi_{k,\,Un}\,U_{n,l}-2\,\varphi_{n,l}\,U_{k,n}]=0 \qquad \text{along } \mu.$$

Differentiating (3.28) with respect to $U_{n,k}$ we get

$$(\varphi_{\Upsilon,\Upsilon}\,\delta_{nk}+\psi_{n,\,Uk}-\varphi_{n,k})\,\delta_{ij}\,U_{j,i}+(\psi_{i,\,Uj}-\varphi_{i,j})\,\delta_{nk}\,U_{j,i}=0$$

from which follows that

(3.30) $$(\varphi_{\Upsilon,\Upsilon}\,\delta_{nk}+\psi_{n,\,Uk}-\varphi_{n,k})\,\delta_{ji}+(\psi_{i,\,Uj}-\varphi_{i,j})\,\delta_{n,k}=0.$$

If this system of equations is multiplied with δ_{nk} we get

(3.31) $$3\,(\psi_{i,\,Uj}-\varphi_{i,j})+(\psi_{k,Uk}-\varphi_{k,k}+3\,\varphi_{\Upsilon,\Upsilon})\,\delta_{ij}=0.$$

Contracting the indeces i and j one gets

$$\psi_{k,\,Uk}-\varphi_{k,k}=-\frac{3}{2}\varphi_{\Upsilon,\Upsilon}.$$

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[1] With ( ) we denote the symmetric part of the system on the indices in the brackets. The antisymmetric part of the system on the indices is denoted with.

If we substitute this in (3.31) we obtain

$$\psi_{i,Uj}-\varphi_{i,j}=-\frac{1}{2}\varphi_{\gamma,\gamma}\delta_{ij}. \tag{3.32}$$

From (3.32), with the help of (3.27), we obtain

$$\varphi_{(i,j)}=\frac{1}{3}\varphi_{k,k}\delta_{ij}=\varphi_{0,0}\delta_{ij}. \tag{3.33}$$

It remains to see what one gets from the expression along μ, i.e. from (3.29). Differentiating this equality with respect to $U_{p,q}$ and then rearranging so obtained expression, one gets

$$\varphi_{\gamma,\gamma}(U_{p,q}+U_{q,p})+\psi_{p,Un}U_{n,q}+\psi_{q,Un}U_{n,p}+\psi_{n,Up}(U_{n,q}+U_{q,n})$$
$$+\varphi_{n,q}U_{p,n}-\varphi_{n,p}U_{q,n}-\varphi_{q,n}(U_{n,p}+U_{p,n})=0.$$

If this expression is differentiated with respect to $U_{p,q}$ and the obtained expression is arranged, one obtains

$$(\varphi_{\gamma,\gamma}\delta_{jp}+\psi_{j,Up}-\varphi_{j,p})\delta_{iq}+(\psi_{q,Ui}-\varphi_{q,i})\delta_{jp}+2\psi_{(p,Ui)}\delta_{jq}+$$
$$+[\varphi_{\gamma,\gamma}\delta_{jq}-2\varphi_{(j,q)}]\delta_{ip}=0$$

which is identically satisfied because of (3.30), (3.27) and (3.33). According to this, one can conclude that from (3.29) we do not get any new conditions which the functions $\underset{\sim}{\varphi}$ and $\underset{\sim}{\psi}$ should satisfy. Summarizing all the conditions which must satisfy the functions $\underset{\sim}{\varphi}$ and $\underset{\sim}{\psi}$ we can conclude the following:

$$\varphi_{\gamma}=\varphi_{\gamma}(x,t)$$

or, even more correctly,

$$\varphi_i=\varphi_i(x)$$
$$\varphi_0=\varphi_0(t)$$

which follows from (3.21), (3.26) and (3.27c).

Now, we differentiate (3.32) with respect to x_0 and use (3.15) and (3.26). Then, we get

$$\varphi_{\gamma,\delta 0}=0.$$

This equation has the form

$$\varphi_{0,00}+\varphi_{j,j0}=0$$

from which it follows that

$$\varphi_{0,00}=0 \qquad \text{because of (3.26).}$$

From this one gets

$$\varphi_0=\nu t+a, \tag{3.34}$$

where ν and a are constants.

With the help of (3.34) it is easy to show that (3.33) because

$$\varphi_{(i,j)}=\nu\delta_{ij}. \tag{3.35}$$

Then the symmetrical part of the equation (3.32) can be written in the form

$$\psi_{(j,\,Uk)} = -\nu\delta_{jk}, \tag{3.36}$$

where we have used the relations

$$\varphi_{i,\,i} = 3\,\varphi_{0,\,0} = 3\,\nu$$

and

$$\varphi_{\gamma,\,\gamma} = \varphi_{i,\,i} + \varphi_{0,\,0} = 4\,\nu \tag{3.37}$$

which follows from (3.33).

If (3.35) is differentiated with respect to x_k we get

$$\varphi_{j,\,ki} + \varphi_{k,\,ji} = 0.$$

By cyclic permutation of these indices we get two additional relations

$$\varphi_{k,\,ij} + \varphi_{i,\,kj} = 0$$

$$\varphi_{i,\,jk} + \varphi_{j,\,ik} = 0.$$

By adding these two systems and substracting the so obtained sum from the preceding one, we get

$$\varphi_{i,\,jk} = 0.$$

Integrating this expression, one gets

$$\varphi_{i,\,j} = D_{ij} = D_{(ij)} + D_{[ij]}$$

where D_{ij} is a constant, and $D_{(ij)}$ and $D_{[ij]}$ are its symmetric and skew-symmetric part respectively. With the help of (3.35) this expression can be written in the form

$$\varphi_{i,\,j} = \nu\delta_{ij} + E_{ij}, \tag{3.38}$$

where $E_{ij} = D_{[ij]}$ is skew-symmetric.

By integration of this expression we finally have

$$\varphi_i = \nu\, x_i + E_{ij}\, x_j + E_i \tag{3.39}$$

which determines the function φ_γ. It remains to determine the functions ψ_i.

From (3.15) it follows that ψ_i does not depend on x_0. If (3.13) is differentiated with respect to x_k we have

$$\psi_{i,\,jk} + \psi_{j,\,ik} = 0.$$

By the procedure which is entirely analogous to the previous, one gets

$$\psi_i = C_{ij}(U)\, X_j + D_i(U). \tag{3.40}$$

From (3.40) one can see at once that

$$\psi_{i,\,j} = C_{ij}(U). \tag{3.41}$$

If (3.37) and (3.38) are used in (3.32) we get

$$\psi_{i,\,Uj} = E_{ij} - \nu\delta_{ij}. \tag{3.42}$$

Differentiating (3.42) with respect to x_k we obtain

(3.43) $$\psi_{i,\,Ujk} = \psi_{i,\,kUj} = 0.$$

Differentiating (3.41) with respect to U_k we get

(3.44) $$\psi_{j,\,jUk} = C_{ij,\,Uk} = 0$$

where we have used (3.43). From this it follows that C_{ij} = constant. Taking this into consideration and differentiating (3.40) with respect to U_k and making this expression equal to (3.42) we obtain

$$\psi_{i,\,Uk} = D_{i,\,Uk} = E_{ik} - \nu\delta_{ik},$$

from which it follows that

$$D_i = E_{ij} U_j - \nu U_i + F_i,$$

where F_i is constant.

Replacing this expression in (3.40) we get

(3.45) $$\psi_i = C_{ij} X_j + E_{ij} U_j - \nu U_i + F_i.$$

Because of the skew-symmetry E_{ij} and C_{ij} it is possible to write them in the form

(3.46) $$E_{ij} = \varepsilon_{ijk} A_k$$ $$C_{ij} = \varepsilon_{ijk} B_k.$$

Replacing these expressions in (3.39) and (3.45) together with (3.34) we obtain corresponding transformations in the form which is completely identical to the solutions of Fletcher [3], i.e.

(3.47) $$\varphi_i = \nu x_i + \varepsilon_{ijk} A_k x_j + E_i$$

(3.48) $$\varphi_0 = \nu t + a$$

(3.49) $$\psi_i = -\nu U_i + \varepsilon_{ijk} A_k U_j + \varepsilon_{ijk} B_k x_j + F_i.$$

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ЗАКОНЫ СОХРАНЕНИЯ В ЭЛАСТОДИНАМИКЕ

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Резюме

В статье, пользуясь Теоремой Нетеровой, определены семейства векторных и координатных преобразований которим отвечают законы сохранения в линейной эластодинамике.

Предполагаемся что преобразования не зависят от материяльных качеств тел. Этим способом полученые результаты в полном согласим с результатами Флечера [3].

О ЗАКОНИМА КОНЗЕРВАЦИЈЕ У ЕЛАСТОДИНАМИЦИ

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Резиме

У раду се, користећи Нетерову теорему, одређују фамилије векторских и координатних трансформација којима одговарају закони конзервације у линеарној еластодинамици.

Предпоставља се да трансформације не зависе од материјалних својства тела. На тај начин добијени резултати су у потпуној сагласности са резултатом Флечера [3].