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SOME NONSEMISYMMETRIC RICCI-SEMISYMMETRIC WARPED PRODUCT HYPERSURFACES

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ABSTRACT. We investigate curvature properties of some nonsemisymmetric Ricci-semisymmetric hypersurfaces of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 5$, which can be locally realized as a warped product.

1. Introduction

A semi-Riemannian manifold (M, g), $\dim M = n \geqslant 3$, is said to be *semisymmetric* if $R \cdot R = 0$ on M. A semi-Riemannian manifold (M, g), $n \geqslant 3$, is called *Ricci-semisymmetric* if $R \cdot S = 0$ on M. For precise definitions of the symbols used, we refer to Section 2. It is clear that every semisymmetric manifold is Ricci-semisymmetric. The converse statement is not true. Under some additional assumptions both conditions are equivalent to each other. This problem, named the *problem of P. J. Ryan* (cf. [**39**]), was considered among others in: [**1**], [**2**], [**3**], [**4**], [**7**], [**10**], [**12**], [**18**], [**19**], [**23**], [**25**], [**26**], and [**37**] (see also [**17**] and [**28**] and references therein).

A semi-Riemannian manifold (M, g), $n \geqslant 3$, is called a *quasi-Einstein manifold* if at every $x \in M$ its Ricci tensor S has the form

$$S = \alpha g + \beta w \otimes w, \qquad w \in T_x^* M, \qquad \alpha, \beta \in \mathbb{R}. \tag{1}$$

We refer to [**28**] for a review of results on quasi-Einstein manifolds.

Let M be a hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, with signature $(s, n+1-s)$, $n \geqslant 4$. Let U_H be the set of all $x \in M$

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at which the transformation $\mathcal{A}^2$ is not a linear combination of the shape operator $\mathcal{A}$ and the identity transformation Id at x. It is known that if (1) is satisfied at $x \in M - U_H$, then the Weyl tensor C of M vanishes at x or at this point the Ricci tensor S of M is proportional to the metric tensor [**26**, Lemma 4.1(iii)]. With respect to this, we restrict our considerations to the subset $U_H \subset M$. Ricci-semisymmetric quasi-Einstein hypersurfaces in semi-Euclidean spaces $\mathbb{E}^{n+1}_s$, $n \geqslant 4$, were investigated in [**19**] and [**26**]. We have the following

THEOREM 1.1. *Let M be a quasi-Einstein hypersurface of $\mathbb{E}^{n+1}_s$, $n \geqslant 4$, and let* (1) *be satisfied on $U_H \subset M$.*
(i) [**26**, Theorem 5.1] *On U_H any of the following three conditions is equivalent to each other:*

$$(2) \quad (a)\ R \cdot S = 0, \quad (b)\ \mathcal{A}^3 = \mathrm{tr}(\mathcal{A})\mathcal{A}^2 - \frac{\varepsilon\kappa}{n-1}\mathcal{A},\ \ \varepsilon = \pm 1, \quad (c)\ \mathcal{A}(W) = 0,$$

where the vector W is related to w by $g(W, X) = w(X)$, $X \in T_xM$, and w and α are defined by (1).
(ii) [**19**, Theorem 5.1]; [**26**, Corollary 5.2] *If at every $x \in U_H$ one of the conditions:* (2)(*a*), (2)(*b*) *or* (2)(*c*) *is satisfied, then on U_H we have*

$$(a)\ \mathrm{rank}\Big(S - \frac{\kappa}{n-1}g\Big) = 1, \quad (b)\ R \cdot C = Q(S, C), \quad (c)\ C \cdot S = 0.$$

It is clear that every semi-Riemannian semisymmetric as well as conformally flat manifold (M, g), $n \geqslant 4$, realizes trivially at every point of M the following condition

$(*)$ the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent.

Semi-Riemannian manifolds satisfying $(*)$ were investigated among others in: [**22**], [**23**], [**24**] and [**29**]. $(*)$ is equivalent to $R \cdot C = LQ(S, C)$ on $U = \{x \in M \mid Q(S, C) \neq 0 \text{ at } x\}$, where L is some function on U. Examples of nonsemisymmetric and nonconformally flat manifolds satisfying $(*)$ are given in [**21**]. We denote by U_L the set consisting of all points of U at which the function L is nonzero. Combining Theorem 4.1 of [**24**] with the main results of [**25**] we obtain

THEOREM 1.2. [**28**, Theorem 1.3] *If M is a hypersurface of $\mathbb{E}^{n+1}_s$, $n \geqslant 5$, satisfying the condition $R \cdot C = LQ(S, C)$, then at every $x \in U_H \cap U_L \subset M$ we have:*

$$R \cdot S = 0, \quad C \cdot S = 0, \quad R \cdot C = Q(S, C), \quad C \cdot R = \frac{n-3}{n-2}Q(S, R),$$
$$\mathcal{A}^3 = \mathrm{tr}(\mathcal{A})\mathcal{A}^2 - \frac{\varepsilon\kappa}{n-1}\mathcal{A}, \quad \varepsilon = \pm 1, \quad \mathcal{A}(W) = 0,$$
$$S = \frac{\kappa}{n-1}g + \beta w \otimes w, \quad w \in T^*_xM, \quad \beta \in \mathbb{R},$$

where the vector W is related to the covector w by $g(W, X) = w(X)$, $X \in T_xM$.

In Section 2 we fix notations and we give a review of conditions of pseudosymmetry type. In Section 3 we consider Ricci-semisymmetric hypersurfaces M of $N^{n+1}_s(c)$, $n \geqslant 4$. We prove that some curvature conditions of pseudosymmetry type are fulfilled on the subset $U_H \subset M$ of such hypersurfaces (see Theorem 3.2).

In Section 4 we investigate nonsemisymmetric Ricci-pseudosymmetric hypersurfaces M of $\mathbb{E}_s^{n+1}$, $n \geqslant 4$, which are locally warped products. Let g be the metric induced on M from the metric tensor of the ambient space. Further, we assume that for every $x \in U_H \subset M$ at which the tensor $R \cdot R$ is nonzero there exists a coordinate neighbourhood $V \subset U_H$ of x such that $V = \overline{M} \times \widetilde{N}$, $g = \overline{g} \times_F \widetilde{g}$, and $(\overline{M}, \overline{g})$, $\dim \overline{M} = p \geqslant 1$, and $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - p \geqslant 4$, are some semi-Riemannian manifolds and F is a positive smooth function on $\overline{M}$. In addition, we assume that the manifold $(\widetilde{N}, \widetilde{g})$ is not of constant curvature. Now we can prove (see Theorem 4.2) that $(\overline{M}, \overline{g})$, $p \geqslant 2$, is a flat manifold, $(\widetilde{N}, \widetilde{g})$ is a Ricci-pseudosymmetric manifold satisfying the curvature conditions presented in Proposition 3.1 and the function F satisfies some system of differential equations. Finally, by making use of Theorem 3.2, we obtain curvature properties of pseudosymmetry type of the Cartan hypersurfaces of dimension 6, 12 or 24 (Theorem 4.3).

2. Basic notations

Let (M, g), $n \geqslant 3$, be a connected semi-Riemannian manifold of class C^∞ and let ∇ be its Levi-Civita connection. For a symmetric $(0,2)$-tensor A and a $(0,k)$-tensor T, $k \geqslant 2$, we define their Kulkarni–Nomizu product $A \wedge T$ by

$$\begin{aligned}(A \wedge T)&(X_1, X_2, X_3, X_4; Y_3, \dots, Y_k)\\ &= A(X_1, X_4)T(X_2, X_3, Y_3, \dots, Y_k) + A(X_2, X_3)T(X_1, X_4, Y_3, \dots, Y_k)\\ &\quad - A(X_1, X_3)T(X_2, X_4, Y_3, \dots, Y_k) - A(X_2, X_4)T(X_1, X_3, Y_3, \dots, Y_k).\end{aligned}$$

In the special case, when $k = 2$ and the tensor T is a symmetric tensor, the tensor $A \wedge T$ is the standard Kulkarni–Nomizu product of A and T. For a $(0,k)$-tensor field T, $k \geqslant 1$, and a symmetric $(0,2)$-tensor field A on M, we denote by the $(0,k)$-tensor $A \cdot T$ and the $(0, k+2)$-tensor fields $R \cdot T$ and $Q(A, T)$, respectively. For the definition of these tensors, we refer to [**19**] (see also [**5**], [**17**] or [**27**]). Setting $T = R$, $T = C$ or $T = S$ and $A = g$ or $A = S$ in the above formulas , we obtain the tensors: $S \cdot R$, $S \cdot C$, $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$, $R \cdot S$, $C \cdot S$, $Q(g, R)$, $Q(g, C)$, $Q(g, S)$, $Q(S, R)$, and $Q(S, C)$. The tensors $C \cdot R$, $C \cdot C$ and $C \cdot S$ are defined in the same manner as the tensors $R \cdot R$ and $R \cdot S$, respectively. We note that

$$g \wedge Q(A, g) = Q(A, G), \tag{3}$$

where the $(0,4)$-tensor G is defined by $G = \frac{1}{2} g \wedge g$.

A semi-Riemannian manifold (M, g) is said to be *pseudosymmetric* [**15**] if at every point of M we have:

($*_1$) the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

This is equivalent to $R \cdot R = L_R Q(g, R)$ on $U_R = \{x \in M \mid R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. Evidently, every semi-Riemannian semisymmetric manifold is pseudosymmetric. There exist pseudosymmetric manifolds which are nonsemisymmetric and a review of results on pseudosymmetric manifolds is given in [**15**] and [**40**]. A review of recent results on semisymmetric manifolds is presented in [**20**].

It is easy to see that if $(*_1)$ holds on a semi-Riemannian manifold (M, g), then at every point of M we have:

$(*_2)$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The converse statement is not true [**13**]. A semi-Riemannian manifold (M, g) is called *Ricci-pseudosymmetric* if at every point of M the condition $(*_2)$ is fulfilled. The condition $(*_2)$ is equivalent to $R \cdot S = L_S Q(g, S)$ on $U_S = \{x \in M \mid S \neq \frac{\kappa}{n} g \text{ at } x\}$, where L_S is some function on U_S. Examples of compact and non-Einstein Ricci-pseudosymmetric manifolds which are nonpseudosymmetric were found in [**30**] and [**33**]. For instance, in [**33**, Theorem 1] it was shown that the Cartan hypersurfaces have that property. We recall that the Cartan hypersurface in the sphere $S^{n+1}(c)$ is a compact minimal hypersurface with constant principal curvatures $-(3c)^{1/2}, 0, (3c)^{1/2}$ of the same multiplicity. It is known that the Cartan hypersurfaces are tubes of constant radius over the standard Veronese embeddings $i : \mathbb{F}P^2 \to S^{3d+1}(c) \to \mathbb{E}^{3d+2}$, $d = 1, 2, 4, 8$, of the projective plane $\mathbb{F}P^2$ in the sphere $S^{3d+1}(c)$ in a Euclidean space $\mathbb{E}^{3d+2}$, where $\mathbb{F} = \mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (quaternions) or $\mathbb{O}$ (Cayley numbers), respectively [**6**]. Every Cartan hypersurface satisfies the following [**33**, Proposition 1]

$$R \cdot S = \frac{\tau}{n(n+1)} Q(g, S), \tag{4}$$

where τ is the scalar curvature of the ambient space. In addition, the Cartan hypersurface in $S^4(c)$ is a nonsemisymmetric pseudosymmetric manifold satisfying $R \cdot R = \frac{\tau}{12} Q(g, R)$ [**32**, Example 2]. We remark also that every Ricci-semisymmetric manifold is Ricci-pseudosemisymmetric. Ricci-semisymmetric manifolds were investigated by several authors (see e.g., [**2**], [**13**], [**14**], [**35**] and [**36**]).

It is known that at every point of a hypersurface M of $N_s^{n+1}(c)$, $n \geqslant 4$, we have [**15**, Section 5.5], [**31**]:

$(*_3)$ the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent.

Precisely, on M we have

$$R \cdot R - Q(S, R) = -\frac{(n-2)\tau}{n(n+1)} Q(g, C), \tag{5}$$

where τ is the scalar curvature of the ambient space. In particular, if the ambient space is a semi-Euclidean space, then (5) reduces to

$$R \cdot R = Q(S, R). \tag{6}$$

Warped products satisfying $(*_3)$ were investigated in [**8**] and [**11**]. Every quasi-Einstein conformally flat manifold is a pseudosymmetric manifold satisfying (6) [**15**, Section 6.3]. Note also that every pseudosymmetric Einstein manifold satisfies $(*_3)$. Pseudosymmetric manifolds satisfying $(*_3)$ were investigated in [**21**].

Semi-Riemannian manifolds fulfilling $(*)$, $(*_1)$, $(*_2)$, $(*_3)$ or other conditions of this kind (see e.g., [**38**]) are called *manifolds of pseudosymmetry type* [**5**], [**15**], [**40**]. Recently, a review of results on pseudosymmetry type manifolds was presented in [**5**].

Let $(\overline{M},\overline{g})$ and $(\widetilde{N},\widetilde{g})$, $\dim\overline{M} = p$, $\dim\widetilde{N} = n-p$, $1 \leqslant p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\overline{U};x^a\}$ and $\{\widetilde{V};y^\alpha\}$, respectively. Let F be a positive smooth function on $\overline{M}$. The warped product $\overline{M}\times_F\widetilde{N}$ of $(\overline{M},\overline{g})$ and $(\widetilde{N},\widetilde{g})$ is the product manifold $\overline{M}\times\widetilde{N}$ with the metric $g = \overline{g}\times_F\widetilde{g}$ defined by $\overline{g}\times_F\widetilde{g} = \pi_1^*\overline{g} + (F\circ\pi_1)\pi_2^*\widetilde{g}$, where $\pi_1 : \overline{M}\times\widetilde{N}\to\overline{M}$ and $\pi_2 : \overline{M}\times\widetilde{N}\to\widetilde{N}$ are the natural projections on $\overline{M}$ and $\widetilde{N}$, respectively. Let $\{\overline{U}\times\widetilde{V};x^1,\dots,x^p,x^{p+1} = y^1,\dots,x^n = y^{n-p}\}$ be a product chart for $\overline{M}\times\widetilde{N}$. The local components of the metric $g = \overline{g}\times_F\widetilde{g}$ with respect to this chart are $g_{rs} = \overline{g}_{ab}$ if $r = a$ and $s = b$, $g_{rs} = F\widetilde{g}_{\alpha\beta}$ if $r = \alpha$ and $s = \beta$, and $g_{rs} = 0$ otherwise, where $a,b,c,\ldots \in \{1,\dots,p\}$, $\alpha,\beta,\gamma,\dots \in \{p+1,\dots,n\}$ and $r,s,t,\ldots \in \{1,2,\dots,n\}$. We will mark by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$). The local components Γ^r_{st} of the Levi-Civita connection ∇ of $\overline{M}\times_F\widetilde{N}$ are the following

$$\Gamma^a_{bc} = \overline{\Gamma}^a_{bc},\quad \Gamma^\alpha_{\beta\gamma} = \widetilde{\Gamma}^\alpha_{\beta\gamma},\quad \Gamma^a_{\alpha\beta} = -\frac{1}{2}\overline{g}^{ab}F_b\widetilde{g}_{\alpha\beta},\quad \Gamma^\alpha_{a\beta} = \frac{1}{2F}F_a\delta^\alpha_\beta,$$

$$\Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0,\quad F_a = \partial_a F = \frac{\partial F}{\partial x^a},\quad \partial_a = \frac{\partial}{\partial x^a}. \tag{7}$$

The local components $R_{rstu} = g_{rw}R^w_{stu} = g_{rw}(\partial_u\Gamma^w_{st} - \partial_t\Gamma^w_{su} + \Gamma^v_{st}\Gamma^w_{vu} - \Gamma^v_{su}\Gamma^w_{vt})$, $\partial_u = \frac{\partial}{\partial x^u}$, of the Riemann–Christoffel curvature tensor R and the local components S_{ts} of the Ricci tensor S of the warped product $\overline{M}\times_F\widetilde{N}$, which may not vanish identically, are the following (e.g., see [**14**], [**16**]):

$$(a)\ \ R_{abcd} = \overline{R}_{abcd},\qquad (b)\ \ R_{\alpha ab\beta} = -\frac{1}{2}T_{ab}\widetilde{g}_{\alpha\beta}, \tag{8}$$

$$R_{\alpha\beta\gamma\delta} = F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4}\widetilde{G}_{\alpha\beta\gamma\delta} = F(Z(\widetilde{R})_{\alpha\beta\gamma\delta} + \psi\widetilde{G}_{\alpha\beta\gamma\delta}), \tag{9}$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F}T_{ab},\quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \frac{1}{2}\Big(\operatorname{tr}(T) + (n-p-1)\frac{\Delta_1 F}{2F}\Big)\widetilde{g}_{\alpha\beta}, \tag{10}$$

where

$$Z(\widetilde{R})_{\alpha\beta\gamma\delta} = \widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\widetilde{\kappa}}{(n-p)(n-p-1)}\widetilde{G}_{\alpha\beta\gamma\delta},\quad \psi = \frac{\widetilde{\kappa}}{(n-p)(n-p-1)} - \frac{\Delta_1 F}{4F},$$

$$T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F}F_aF_b,\quad \operatorname{tr}(T) = \operatorname{tr}_{\overline{g}}(T) = \overline{g}^{ab}T_{ab},\quad \Delta_1 F = \Delta_{1\overline{g}}F = \overline{g}^{ab}F_aF_b,$$

and T is the $(0,2)$-tensor with the local components T_{ab}. The scalar curvature κ of $\overline{M}\times_F\widetilde{N}$ is expressed by

$$\kappa = \overline{\kappa} + \frac{1}{F}\widetilde{\kappa} - \frac{n-p}{F}\Big(\operatorname{tr}(T) + (n-p-1)\frac{\Delta_1 F}{4F}\Big).$$

3. Ricci-pseudosymmetric hypersurfaces

Let M, $n \geqslant 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold (N, g^N). We denote by g the metric tensor induced on M from g^N. Further, we denote by ∇ and ∇^N the Levi–Civita connections corresponding to the metric tensors g and g^N, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = g^N(\xi,\xi) = \pm 1$. We can write the *Gauss* *formula* and the *Weingarten formula* of (M, g) in (N, g^N) in the following form: $\nabla^N_X Y = \nabla_X Y + \varepsilon H(X,Y)\xi$ and $\nabla_X \xi = -\mathcal{A}X$, respectively, where X, Y are vector fields tangent to M, H is the *second fundamental tensor* of (M, g) in (N, g^N), $\mathcal{A}$ is the *shape operator* and $H^k(X,Y) = g(\mathcal{A}^k X, Y)$, $\mathrm{tr}_g(H^k) = \mathrm{tr}_g(\mathcal{A}^k)$, $k \geqslant 1$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by R and R^N the Riemann–Christoffel curvature tensors of (M, g) and (N, g^N), respectively. The *Gauss equation* of (M, g) in (N, g^N) has the form

$$R(X_1, \dots, X_4) = R^N(X_1, \dots, X_4) + \frac{\varepsilon}{2}(H \wedge H)(X_1, \dots, X_4), \tag{11}$$

where $X_1, \dots, X_4$ are vector fields tangent to M. Let $x^r = x^r(y^k)$ be the local parametric expression of (M, g) in (N, g^N), where y^k and x^r are local coordinates of M and N, respectively, and $h, i, j, k \in \{1, 2, \dots, n\}$ and $p, r, t, u \in \{1, 2, \dots, n+1\}$. Now (11) yields

$$R_{hijk} = R^N_{prtu} B_h^{\ p} B_i^{\ r} B_j^{\ t} B_k^{\ u} + \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}), \qquad B_k^{\ r} = \frac{\partial x^r}{\partial y^k}, \tag{12}$$

where R^N_{prtu}, R_{hijk} and H_{hk} are the local components of the tensors R^N, R and H, respectively. If M is a hypersurface of $N^{n+1}_s(c)$, $n \geqslant 4$, then (12) turns into

$$R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\tau}{n(n+1)} G_{hijk}, \tag{13}$$

where τ is the scalar curvature of the ambient space and G_{hijk} are the local components of the tensor $G = \frac{1}{2} g \wedge g$. Contracting (13) with g^{ij} and g^{hk} we obtain

$$S_{hk} = \varepsilon(\mathrm{tr}_g(H)H_{hk} - H^2_{hk}) + \frac{(n-1)\tau}{n(n+1)} g_{hk}, \tag{14}$$

$$\kappa = \varepsilon((\mathrm{tr}_g(H))^2 - \mathrm{tr}_g(H^2)) + \frac{(n-1)\tau}{n+1}, \tag{15}$$

respectively, where S_{hk} are the local components of the Ricci tensor S and κ is the scalar curvature of (M, g). Using (14) and Theorem 4.1 of [**31**] we can deduce that $U_H \subset U_C \cap U_S \subset M$, where the subset $U_C \subset M$, $n \geqslant 4$, is defined by $U_C = \{x \in M : C \neq 0 \text{ at } x\}$. We note that in the case when M is a hypersurface of $\mathbb{E}^{n+1}_s$, $n \geqslant 4$, (13) reduces to

$$R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}). \tag{16}$$

For a symmetric $(0,2)$-tensor A and a $(0,4)$-tensor T we define the $(0,6)$-tensor $U(A,T)$ by

$$\begin{aligned} U(A,T)&(X_1, \dots, X_4; X, Y) \\ &= -T((X \wedge_A Y)X_1, X_2, X_3, X_4) + T((X \wedge_A Y)X_2, X_1, X_3, X_4) \\ &\quad - T((X \wedge_A Y)X_3, X_4, X_1, X_2) + T((X \wedge_A Y)X_4, X_3, X_1, X_2). \end{aligned}$$

LEMMA 3.1. *Let (M, g), $n \geqslant 4$, be a semi-Riemannian manifold.*
(i) If A is a symmetric $(0,2)$-tensor and T a generalized curvature tensor on M,

then on M we have $U(A,T) = Q(A,T)$.
(ii) If A and B are symmetric $(0,2)$-tensors on M and the tensor T is defined by

$$\begin{aligned} T(X_1,X_2,X_3,X_4) &= B((X_4 \wedge_A X_3)X_2, X_1) \\ &= A(X_2,X_3)B(X_1,X_4) - A(X_2,X_4)B(X_1,X_3), \end{aligned}$$

then on M we have $U(A,T) = -\frac{1}{2}Q(B, A \wedge A)$.
(iii) If A and B are symmetric $(0,2)$-tensors on M and the tensor T is defined by

$$\begin{aligned} T(X_1,X_2,X_3,X_4) &= A((X_4 \wedge_B X_3)X_2, X_1) \\ &= A(X_1,X_4)B(X_2,X_3) - A(X_1,X_3)B(X_2,X_4), \end{aligned}$$

then on M we have $U(A,T) = 0$.

PROOF. Our assertions are immediate consequences of the definitions of the given tensors. □

PROPOSITION 3.1. *Let (M,g), $n \geqslant 4$, be a semi-Riemannian manifold. If T is the $(0,4)$-tensor on M defined by $T(X,X_2,X_3,X_4) = R(\mathcal{S}X,X_2,X_3,X_4)$ satisfies*

$$\begin{aligned} R(\mathcal{S}X,X_2,X_3,X_4) &= a_1 R(X,X_2,X_3,X_4) + a_2 S((X_4 \wedge_g X_3)X_2, X) \\ &\quad + a_3 g((X_4 \wedge_S X_3)X_2, X) + a_4 G(X,X_2,X_3,X_4), \end{aligned} \tag{17}$$

then on M we have

$$S^2 = (a_1 + (n-1)a_2 - a_3)S + (\kappa a_3 + (n-1)a_4)g, \tag{18}$$

$$R \cdot S = (a_2 - a_3)Q(g,S), \tag{19}$$

$$C \cdot S = \frac{1}{n-2}\Big(\frac{\kappa}{n-1} - a_1 - a_2 - (n-3)a_3\Big)Q(g,S), \tag{20}$$

$$S \cdot R = -4a_1 R - 2(a_2 + a_3) g \wedge S - 4a_4 G, \tag{21}$$

$$U(g,T) = a_1 Q(g,R) - a_2 Q(S,G), \tag{22}$$

$$C \cdot R - R \cdot C = -\frac{1}{n-2}Q(S,R) + \frac{1}{n-2}\Big(\frac{\kappa}{n-1} - a_1\Big)Q(g,R) + \frac{a_3}{n-2}Q(S,G). \tag{23}$$

PROOF. (18), (19), (20) and (21) are immediate consequences of (17). From (17), in view of Lemma 3.1, we obtain also (22). Further, on M we have [**37**]

$$C \cdot R = R \cdot R - \frac{1}{n-2}Q(S,R) + \frac{\kappa}{(n-2)(n-1)}Q(g,R) - \frac{1}{n-2}Q(g,T),$$

which by the identity

$$R \cdot C = R \cdot R - \frac{1}{n-2} g \wedge (R \cdot S), \tag{24}$$

turns into

$$\begin{aligned} C \cdot R &= R \cdot C + \frac{1}{n-2} g \wedge (R \cdot S) - \frac{1}{n-2}Q(S,R) \\ &\quad + \frac{\kappa}{(n-2)(n-1)}Q(g,R) - \frac{1}{n-2}Q(g,T). \end{aligned} \tag{25}$$

Applying to this (3), (17), (19) and Lemma 3.1 we obtain (23), which completes the proof. □

THEOREM 3.1. *Let M be a hypersurface of $N_s^{n+1}(c)$, $n \geqslant 4$. If the condition*

$$H^3 = \mathrm{tr}_g(H)H^2 + \lambda H, \tag{26}$$

is satisfied on M, then on M we have: (4), (17),

$$S^2 = \Big(\mu + \frac{(n-1)\tau}{n(n+1)}\Big)S - \frac{(n-1)\mu\tau}{n(n+1)}g, \tag{27}$$

$$C \cdot S = \frac{1}{n-2}\Big(\frac{\kappa}{n-1} - \mu - \frac{\tau}{n(n+1)}\Big)Q(g,S), \tag{28}$$

$$S \cdot R = -4\mu R - \frac{2\tau}{n(n+1)} g \wedge S + \frac{4\mu\tau}{n(n+1)}G, \tag{29}$$

$$C \cdot R - R \cdot C = -\frac{1}{n-2}Q(S,R) + \frac{1}{n-2}\Big(\frac{\kappa}{n-1} - \mu\Big)Q(g,R), \tag{30}$$

where λ and μ are some functions on M and

$$a_1 = \mu, \quad a_2 = \frac{\tau}{n(n+1)}, \quad a_3 = 0, \quad a_4 = -\frac{\mu\tau}{n(n+1)}, \quad \mu = \frac{(n-1)\tau}{n(n+1)} - \varepsilon\lambda. \tag{31}$$

PROOF. Transvecting (14) with $H_j^{\ h} = g^{hs}H_{js}$ and using (26) we obtain

$$H_j^{\ r} S_{rk} = \mu H_{jk}. \tag{32}$$

Next, transvecting (13) with $S_m^h = g^{hs}S_{ms}$ and using (32) we get

$$S_m^{\ r} R_{rijk} = \varepsilon\mu(H_{ij}H_{mk} - H_{ik}H_{mj}) + \frac{\tau}{n(n+1)}(g_{ij}S_{mk} - g_{ik}S_{mj}). \tag{33}$$

This, by symmetrization in m, i, yields (4). Further, (33), by (13), turns into

$$S_m^{\ r} R_{rijk} = \mu R_{mijk} + \frac{\tau}{n(n+1)}(g_{ij}S_{mk} - g_{ik}S_{mj}) - \frac{\mu\tau}{n(n+1)}G_{mijk},$$

which, evidently, gives (17) and (31). Now, Proposition 3.1 completes the proof. □

As an immediate consequence of (14), (32) and Theorem 3.1 we have

COROLLARY 3.1. [**27**, Theorem 3.1] *On every Einstein hypersurface M of $N_s^{n+1}(c)$, $n \geqslant 4$, we have $R \cdot C - C \cdot R = \frac{\kappa}{(n-1)n}Q(g,R)$.*

COROLLARY 3.2. *The Einstein hypersurfaces considered in* [**34**] *satisfy $C \cdot R = -\frac{\kappa}{(n-1)n}Q(g,R)$.*

THEOREM 3.2. *Let M be a Ricci-pseudosymmetric hypersurface of $N_s^{n+1}(c)$, $n \geqslant 4$. Then* (4), (17) *and* (27)–(31) *hold on $U_H \subset M$.*

PROOF. From Theorem 3.1 of [**9**] it follows that (26) holds on U_H. Now Theorem 3.1 completes the proof. □

Theorem 3.2, together with Proposition 3.2 and Theorem 3.1 of [**12**], implies

THEOREM 3.3. *Let M be a nonsemisymmetric Ricci-semisymmetric hypersurface of $\mathbb{E}_s^{n+1}$, $n \geqslant 5$. Then*

$$\begin{aligned} &(a) \quad H^3 = \mathrm{tr}_g(H)H^2 + \lambda H, \\ &(b) \quad H(\mathcal{S}X_1, X_2) = -\varepsilon\lambda H(X_1, X_2), \\ &(c) \quad R(\mathcal{S}X_1, X_2, X_3, X_4) = -\varepsilon\lambda R(X_1, X_2, X_3, X_4), \end{aligned} \tag{34}$$

on the subset $U_H \subset M$, where λ is some function on U_H.

4. Ricci-semisymmetric warped product hypersurfaces

In [**18**] examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces were given. In construction of these examples the following results were applied.

THEOREM 4.1. [**18**, Proposition 4.1] *Let $(\overline{M}, \overline{g})$, $\dim \overline{M} = p \geqslant 2$, be a semi-Riemannian manifold defined in Example* 3.1 *of* [**18**] *and let $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n-p \geqslant 1$, be a semi-Riemannian manifold isometric to a hypersurface of $N_s^{n-p+1}(c)$. Let $\overline{M} \times_F \widetilde{N}$ be the warped product of $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$ with F and c_0 are defined by* (12) *and* (13) *of* [**18**], *respectively, and let*

$$c_0 = \frac{\tau}{(n-p)(n-p+1)},$$

where τ is the scalar curvature of $N_s^{n-p+1}(c)$. Then we have
(i) $\overline{M} \times_F \widetilde{N}$ can be realized locally as a hypersurface of $\mathbb{E}_s^{n+1}$.
(ii) If $(\widetilde{N}, \widetilde{g})$, $n-p \geqslant 4$, is a semisymmetric Einstein manifold not of constant curvature, then $\overline{M} \times_F \widetilde{N}$ is a nonsemisymmetric Ricci-semisymmetric manifold which can be locally realized as a hypersurface of $\mathbb{E}_s^{n+1}$.
(iii) If $(\widetilde{N}, \widetilde{g})$, $U_{\widetilde{S}} = \widetilde{N}$, $n-p \geqslant 4$, is a non-Einstein Ricci-pseudosymmetric manifold satisfying $\widetilde{R} \cdot \widetilde{R} = L_{\widetilde{S}} Q(\widetilde{g}, \widetilde{S})$ on $U_{\widetilde{S}}$, with $L_{\widetilde{S}} = \frac{\tau}{(n-p)(n-p+1)}$, then $\overline{M} \times_F \widetilde{N}$ is a nonsemisymmetric Ricci-semisymmetric manifold, which can be locally realized as a hypersurface of $\mathbb{E}_s^{n+1}$.

With respect to the above theorem, we have the following converse statement.

THEOREM 4.2. *Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geqslant 5$, and let g be the metric induced on M from the metric tensor of $\mathbb{E}_s^{n+1}$. Let $U \subset U_H \subset M$ be an open submanifold of M such that $(U, g) = \overline{M} \times_F \widetilde{N}$, where $(\overline{M}, \overline{g})$, $p = \dim \overline{M} \geqslant 1$ and $(\widetilde{N}, \widetilde{g})$, $n-p = \dim \widetilde{N} \geqslant 4$, are some semi-Riemannian manifolds and F is the warping function. Let x be a point of U at which the tensors $R \cdot R$ and $Z(\widetilde{R})$ are nonzero and let $V \subset U$ be a coordinate neighbourhood of x such that the tensors $R \cdot R$ and $Z(\widetilde{R})$ are nonzero at every point of V.*
(i) The following relations are fulfilled on V

$$\begin{aligned} &(a) \quad \overline{R}_{abcd} = 0, \quad (b) \quad T_{ad} = 0, \quad (c) \quad \frac{\Delta_1 F}{4F} = c_0 = \text{const}, \\ &(d) \quad \kappa = \frac{1}{F}(\widetilde{\kappa} - (n-p)(n-p-1)c_0). \end{aligned} \tag{35}$$

(ii) The local components of the curvature tensor R and the Ricci tensor S of (U, g) and the second fundamental tensor H of U in M which may not vanish identically on V are the following

$$(36) \quad (a) \quad R_{\alpha\beta\gamma\delta} = \varepsilon F(\widetilde{H}_{\alpha\delta}\widetilde{H}_{\beta\gamma} - \widetilde{H}_{\alpha\gamma}\widetilde{H}_{\beta\delta}), \qquad (b) \quad \widetilde{H}_{\alpha\delta} = \widetilde{H}_{\alpha\delta}(x^{p+1}, \ldots, x^n),$$

$$(37) \qquad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - (n-p-1)c_0\widetilde{g}_{\alpha\beta},$$

$$(38) \qquad (a) \quad H_{\alpha\delta} = \sqrt{F}\widetilde{H}_{\alpha\delta}, \qquad (b) \quad \widetilde{\nabla}_\alpha\widetilde{H}_{\beta\delta} = \widetilde{\nabla}_\beta\widetilde{H}_{\alpha\delta}.$$

(iii) The following relation is satisfied on V

$$(39) \qquad (\widetilde{R}\cdot\widetilde{R})_{\alpha\beta\gamma\delta\epsilon\mu} - Q(\widetilde{S}, \widetilde{R})_{\alpha\beta\gamma\delta\epsilon\mu} = -(n-p-2)c_0 Q(\widetilde{g}, \widetilde{C})_{\alpha\beta\gamma\delta\epsilon\mu}.$$

(iv) If M is a Ricci-semisymmetric hypersurface, then on V we have

$$(40) \quad \widetilde{S}_\alpha{}^\mu\widetilde{R}_{\mu\beta\gamma\delta} = ((n-p-1)c_0 - \varepsilon\lambda F)(\widetilde{R}_{\alpha\beta\gamma\delta} - c_0\widetilde{G}_{\alpha\beta\gamma\delta}) + c_0(\widetilde{g}_{\beta\gamma}\widetilde{S}_{\alpha\delta} - \widetilde{g}_{\beta\delta}\widetilde{S}_{\alpha\gamma}),$$

where λ is defined by (34)(c).

Proof. By making use of (8), (9), (10) and (16) we obtain on V the following relations

$$(41) \qquad \overline{R}_{abcd} = R_{abcd} = \varepsilon(H_{ad}H_{bc} - H_{ac}H_{bd}),$$

$$(42) \qquad -\frac{1}{2}T_{ad}\widetilde{g}_{\alpha\beta} = R_{a\alpha\beta d} = \varepsilon(H_{ad}H_{\alpha\beta} - H_{a\beta}H_{\alpha d}),$$

$$(43) \qquad F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4}\widetilde{G}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} = \varepsilon(H_{\alpha\delta}H_{\beta\gamma} - H_{\alpha\gamma}H_{\beta\delta}),$$

$$(44) \qquad 0 = R_{a\alpha\beta\gamma} = \varepsilon(H_{a\gamma}H_{\alpha\beta} - H_{a\beta}H_{\alpha\gamma}),$$

$$(45) \qquad S_{ad} = \overline{S}_{ad} - \frac{n-p}{2F}T_{ab},$$

$$(46) \qquad S_{\alpha\delta} = \widetilde{S}_{\alpha\delta} - \frac{1}{2}\Big(\operatorname{tr}(T) + (n-p-1)\frac{\Delta_1 F}{2F}\Big)\widetilde{g}_{\alpha\delta}.$$

We note that if all components of the form $H_{\alpha\delta}$ vanish at a point $y \in V$, then from (43) it follows that the tensor $Z(\widetilde{R})$ vanishes at y, a contradiction. Thus, at every point of V at least one of the local components $H_{\alpha\delta}$ must be nonzero. Therefore, from (44) we can deduce that $H_{a\gamma} = 0$ at every point of V. Now (42) reduces to

$$(47) \qquad -\frac{1}{2}T_{ad}\widetilde{g}_{\alpha\beta} = \varepsilon H_{ad}H_{\alpha\beta},$$

whence $T_{ad} = \rho H_{ad}$ and $\rho = -\frac{2\varepsilon}{n-p}\widetilde{g}^{\alpha\beta}H_{\alpha\beta}$. Thus (47) turns into $H_{ad}(H_{\alpha\beta} + \frac{\varepsilon\rho}{2}\widetilde{g}_{\alpha\beta}) = 0$. If at least one of the local components H_{ad} is nonzero at a point $y \in V$, then $H_{\alpha\beta} = -\frac{\varepsilon\rho}{2}\widetilde{g}_{\alpha\beta}$ at y, whence, by (43), at y we have $Z(\widetilde{R}) = 0$, a contradiction. Thus all the components of H of the form H_{ad} must vanish at every point of V. It means that (41) reduces on V to (35)(a), whence

$$(48) \qquad \overline{S}_{ad} = 0.$$

On the other hand, from Proposition 3.2 of [8] it follows that $S_{ad} + \frac{1}{2F}T_{ad} = 0$. Applying this in (45) we obtain $\overline{S}_{ad} - \frac{n-p-1}{2F}T_{ad} = 0$, which, by (48), turns into (35)(b). Since the tensor H is a Codazzi tensor, we have $\nabla_a H_{\beta\gamma} = \nabla_\beta H_{a\gamma}$ and

$\nabla_\alpha H_{\beta\gamma} = \nabla_\beta H_{\alpha\gamma}$. From these relations, by making use of (7), we obtain (36)(b) and (38). Further, (43) and (38)(a) yield (35)(c). Now (46) turns into (35)(d). (39) is an immediate consequence of Proposition 3.3 of [**8**] and (38)(a) and (38)(b). Finally, using (9), (34)(c), (35)(b), (35)(c) and (37), we can check that (40) holds on V. Our theorem is thus proved. □

COROLLARY 4.1. *On the manifold* $(V, \widetilde{g})$, *defined in Theorem* 4.2, *the following relations are satisfied:* (17)–(23) *and* $a_1 = \mu = (n-p-1)c_0 - \varepsilon\lambda F$, $a_2 = c_0$, $a_3 = 0$ *and* $a_4 = -c_0((n-p-1)c_0 - \varepsilon\lambda F)$.

We finish this section with the following

THEOREM 4.3. *On every Cartan hypersurface* M *of* $S^{n+1}(c)$, $n = 6, 12$ *or* 24, *we have:* (4), (17), *and*

$$a_1 = \mu = \frac{(n-4)\tau}{n(n+1)}, \qquad a_2 = \frac{\tau}{n(n+1)}, \qquad a_3 = 0, \qquad a_4 = -\frac{(n-4)\tau}{n^2(n+1)^2}, \tag{49}$$

$$\kappa = \frac{(n-3)\tau}{n+1}, \tag{50}$$

$$S^2 = \frac{(2n-5)\tau}{n(n+1)} S - \frac{(n-1)(n-4)\tau}{n^2(n+1)^2} g, \tag{51}$$

$$C \cdot S = \frac{(n-3)\tau}{(n-2)(n-1)n(n+1)} Q(g, S), \tag{52}$$

$$S \cdot R = -\frac{4(n-4)\tau}{n(n+1)} R - \frac{2\tau}{n(n+1)} g \wedge S + \frac{4(n-4)\tau^2}{n^2(n+1)^2} G, \tag{53}$$

$$R \cdot C = Q(S, R) - \frac{(n-2)\tau}{n(n+1)} Q(g, R) - \frac{(n-3)\tau}{(n-2)n(n+1)} Q(S, G), \tag{54}$$

$$C \cdot R = \frac{n-3}{n-2} Q(S, R) - \frac{(n-3)\tau}{(n-1)(n+1)} Q(g, R) - \frac{(n-3)\tau}{(n-2)n(n+1)} Q(S, G), \tag{55}$$

$$R \cdot C - C \cdot R = \frac{1}{n-2} Q(S, R) - \frac{2\tau}{(n-1)n(n+1)} Q(g, R), \tag{56}$$

(57)

$$C \cdot C = \frac{n-3}{n-2} Q(S, R) - \frac{(n-3)\tau}{(n-1)(n+1)} Q(g, R) - \frac{(n-3)(n^2-n-3)\tau}{(n-2)^2 n(n+1)^2} Q(S, G).$$

PROOF. First of all, we note that $U_H = M$. Let ρ be the positive principal curvature of M. From the properties of the Cartan hypersurfaces it follows that on M we have $\rho^2 = 3c = \frac{3\tau}{n(n+1)}$, $\mathrm{tr}_g(H) = 0$ and $\mathrm{tr}_g(H^2) = \frac{2\tau}{n+1}$. Applying these relations to (15) we obtain (50). It is clear that $H^3 = \rho^2 H$ on M. Thus, in view of Theorem 3.2, the relations (17), (49), (51), (52) and (53) are fulfilled on M. Further, (3), (5), (4) and (24) yield (54). Next, applying (17) and (54) to (25) we obtain (55). Finally, (57) is an immediate consequence of the identity $C \cdot C = C \cdot R + \frac{\kappa}{(n-2)^2 n(n+1)} Q(S, G)$ and (50) and (55). Our theorem is thus proved. □

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