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CONVOLUTION IN COLOMBEAU'S SPACES OF GENERALIZED FUNCTIONS PART I. THE SPACE $\mathcal{G}_{\mathbf{a}}$ AND THE a-INTEGRAL

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Abstract. We investigate subspaces of Colombeau's generalized function space and the generalized integrals in these spaces. The obtained results enable us to study, in the second part of the paper, the generalized convolutions and the Fourier transform in these spaces.

0. Introduction

The Colombeau new generalized function space $\mathcal{G}$ [**3**] has a lot of advantages in applications. This stimulates investigations of various subspaces of that space. In this paper we investigate spaces $\mathcal{G}_{\boldsymbol{a}}$. The obtained results will be used for investigations of convolution in $\mathcal{G}_{\boldsymbol{a}}$ spaces.

In Section 1 we repeat Colombeau's definitions of spaces of generalized functions $\mathcal{G}$, generalized complex numbers $\overline{\mathbb{C}}$ and generalized tempered functions $\mathcal{G}_\tau$. In Section 2, we introduce the class of functions denoted by A and for every $\boldsymbol{a} \in A$ the space of generalized $\boldsymbol{a}$-functions $\mathcal{G}_{\boldsymbol{a}}$. For $\boldsymbol{t}(x) = x$, $x > x_0$, we get $\mathcal{G}_{\boldsymbol{t}} = \mathcal{G}_\tau$. In Section 3, we introduce the notion of $\boldsymbol{a}, \mu$-integral and of $\boldsymbol{a}$-integral.

1. Colombeau's definitions

As usual, $\mathbb{R}^n$ is Euclidean n-dimensional space, $\mathbb{C}$ is the set of complex numbers and $\mathbb{N}$ is the set of natural numbers $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\beta \in \mathbb{N}_0^n$, $\partial^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}$, where $|\beta| = \beta_1 + \dots + \beta_n$ and if $i \in \mathbb{N}_0^n$, then $i \le \beta$ means $i_k \le \beta_k$, $k = 1, \dots, n$. $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ are well known Schwartz test function spaces. For the properties of these spaces and other test function spaces and their duals we refer to [**7**]. Let us recall some definitions from [**3**].

$\mathcal{A}_q$ is the set of all functions $\phi \in \mathcal{D}(\mathbb{R}^n)$ which satisfy:

$$(1) \qquad \int \phi(t)\,dt = 1, \quad \int \phi(t) t^i\,dt = 0, \quad 1 \le |i| \le q, \quad \text{where } q \in \mathbb{N}.$$

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($\int$ means $\int_{\mathbb{R}^n}$).

Clearly $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$. Let $\phi_\varepsilon(\,\cdot\,) = \varepsilon^{-n}\phi(\,\cdot\,/\varepsilon)$, $\varepsilon > 0$; $\phi_\varepsilon \in \mathcal{A}_q$ iff $\phi \in \mathcal{A}_q$.

$\mathcal{E}[\mathbb{R}^n]$ is the set of all functions $G : \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C}$ which satisfy:

$$\text{For every } \phi \in \mathcal{A}_1,\ \ G(\phi, \cdot) \in C^\infty(\mathbb{R}^n). \tag{2}$$

$\mathcal{E}_M[\mathbb{R}^n]$ is the set of all functions $G \in \mathcal{E}[\mathbb{R}^n]$ which satisfy:

$$\left\{\begin{array}{l}\text{For every compact set } K \text{ and every } \beta \in \mathbb{N}_0^n \text{ there exists } N \in \mathbb{N} \text{ such that}\\ \text{for every } \phi \in \mathcal{A}_N \text{ there exist } \eta > 0 \text{ and } c > 0 \text{ such that } |\partial^\beta G(\phi_\varepsilon, x)| \le\\ c\varepsilon^{-N},\ x \in K,\ \varepsilon \in (0,\eta).\end{array}\right. \tag{3}$$

Denote by Γ the set of all increasing sequences tending to infinity. $\mathcal{N}[\mathbb{R}^n]$ is the set of all functions $G \in \mathcal{E}[\mathbb{R}^n]$ which satisfy:

$$\left\{\begin{array}{l}\text{For every compact set } K \text{ and every } \beta \in \mathbb{N}_0^n \text{ there exist } N \in \mathbb{N} \text{ and}\\ \alpha \in \Gamma \text{ such that for every } \phi \in \mathcal{A}_q \text{ and } q \ge N, \text{ there exist } \eta > 0 \text{ and}\\ c > 0 \text{ such that } |\partial^\beta G(\phi_\varepsilon, x)| \le c\varepsilon^{\alpha(q)-N},\ x \in K,\ \varepsilon \in (0,\eta).\end{array}\right. \tag{4}$$

$\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M[\mathbb{R}^n]/\mathcal{N}[\mathbb{R}^n]$ is the set of generalized functions. We shall omit the sign $\mathbb{R}^n$ when it does not cause misinterpretations. $\boldsymbol{G} \in \mathcal{G}$ is defined by its representative $G \in \mathcal{E}_M$.

If we use an open set Ω instead of $\mathbb{R}^n$ we have $\mathcal{G}(\Omega)$ instead of $\mathcal{G}$.

$\mathcal{E}_c$ is the set of all functions $Z : \mathcal{A}_1 \to \mathbb{C}$ which satisfy:

$$\left\{\begin{array}{l}\text{There exists } N \in \mathbb{N} \text{ such that for every } \phi \in \mathcal{A}_N \text{ there exist } \eta > 0 \text{ and}\\ c > 0 \text{ such that } |Z(\phi_\varepsilon)| \le c\varepsilon^{-N},\ \varepsilon \in (0,\eta).\end{array}\right. \tag{5}$$

$\mathcal{F}$ is the set of all functions $Z \in \mathcal{E}_c$ which satisfy:

$$\left\{\begin{array}{l}\text{There exist } N \in \mathbb{N} \text{ and } \alpha \in \Gamma \text{ such that for every } \phi \in \mathcal{A}_q,\ q \ge N, \text{ there}\\ \text{exist } \eta > 0,\ c > 0 \text{ such that } |Z(\phi_\varepsilon)| \le c\varepsilon^{\alpha(q)-N},\ \varepsilon \in (0,\eta).\end{array}\right. \tag{6}$$

$\overline{\mathbb{C}} = \mathcal{E}_c/\mathcal{F}$ is the set of generalized complex numbers. $\boldsymbol{Z} \in \overline{\mathbb{C}}$ is given by its representative $Z \in \mathcal{E}_c$.

The mapping $\mathrm{Cd} : \mathcal{D}' \to \mathcal{G}$, is defined in the following way. Let $g \in \mathcal{D}'$. Then $\mathrm{Cd}(g) = \boldsymbol{G}$, where $G(\phi_\varepsilon, x) = \langle g(t), \phi_\varepsilon(t-x)\rangle = g * \check{\phi}_\varepsilon(x)$, $x \in \mathbb{R}^n$, $\phi_\varepsilon \in \mathcal{A}_1$ ($\check{\phi}(x) = \phi(-x)$). This is an injective mapping.

The pointwise product, the sum and the derivative in $\mathcal{G}$ are naturally defined on the corresponding representatives. These definitions are correct in the sense that they do not depend on representatives from $\mathcal{E}_M$.

Let $\mathcal{H}$ be the set of all monotone functions $h : (0,1) \to (0,1)$ such that $\lim_{\varepsilon\to\infty} h(\varepsilon) = 0$, and let $\mathrm{diam}(\mathrm{supp}(\phi)) = 1$, $\phi \in \mathcal{A}_1$. For every $h \in \mathcal{H}$, $j \in \{1, \dots, n\}$ and $\boldsymbol{G} \in \mathcal{G}$ the h-regularized derivative $\overline{\partial}_j^h \boldsymbol{G}$ is defined by its representative

$$\begin{aligned}\big(\partial_j G(\phi_\varepsilon, \cdot\,) * \phi_{h(\varepsilon)}\big)(x) &= \int (\partial_j G)(\phi_\varepsilon, x - h(\varepsilon)y)\phi(y)\,dy\\ &= \frac{1}{h(\varepsilon)} \int G(\phi_\varepsilon, x - h(\varepsilon)y)\partial_j\phi(y)\,dy, \quad \varepsilon > 0.\end{aligned}$$

$\boldsymbol{Z} \in \overline{\mathbb{C}}$ is associated with $z \in \mathbb{C}$, denoted by $\boldsymbol{Z} \approx z$, if for its representative Z there exists $N \in \mathbb{N}$ such that $\lim_{\varepsilon\to 0} Z(\phi_\varepsilon) = z$, $\phi \in \mathcal{A}_q$, $\varepsilon > 0$, $q \geq N$. It is easy to see that this definition does not depend on representatives of $\boldsymbol{Z}$.

Let K be a compact set. The integral $\int_K \boldsymbol{G}(x)\,dx \in \overline{\mathbb{C}}$ is defined by its representative $\int_K G(\phi_\varepsilon, x)\,dx \in \mathcal{E}_c$, $\phi_\varepsilon \in \mathcal{A}_1$.

The support of a generalized function $\boldsymbol{G}$ is the complement of the largest open set $\mathcal{O}$ such that $\boldsymbol{G}_{|\mathcal{O}} = \mathbf{0}$ in $\mathcal{G}(\mathcal{O})$. Let $\boldsymbol{G}$ have a compact support. Then for any compact set K which contains $\operatorname{supp}(\boldsymbol{G})$ in his interior we define $\int \boldsymbol{G}(x)\,dx = \int_K \boldsymbol{G}(x)\,dx$.

A $\boldsymbol{G} \in \mathcal{G}$ is equal to zero in the sense of generalized distributions ($\boldsymbol{G} = \mathbf{0}\,$(g.d.)) if for every $\varphi \in \mathcal{D}$, $\int \boldsymbol{G}(x)\varphi(x)\,dx = \mathbf{0}$ (the representative of $\int \boldsymbol{G}(x)\varphi(x)\,dx$ is $\int G(\phi_\varepsilon, x)\varphi(x)\,dx$, $x \in \mathbb{R}^n$, $\phi \in \mathcal{A}_1$, $\varepsilon > 0$, and it belongs to $\mathcal{F}$). $\boldsymbol{G}_1 = \boldsymbol{G}_2\,$(g.d.) iff $\boldsymbol{G}_1 - \boldsymbol{G}_2 = \mathbf{0}\,$(g.d.)

$\boldsymbol{G} \in \mathcal{G}$ is associated with $g \in \mathcal{D}'$, $\boldsymbol{G} \approx g$, iff for every $\varphi \in \mathcal{D}$, $\int \boldsymbol{G}(x)\varphi(x)\,dx \approx \langle g, \varphi\rangle$. $\boldsymbol{G}_1 \approx \boldsymbol{G}_2$ iff $\boldsymbol{G}_1 - \boldsymbol{G}_2 \approx 0$ (zero distribution). If $\boldsymbol{G}_1 = \boldsymbol{G}_2\,$(g.d.), then $\boldsymbol{G}_1 \approx \boldsymbol{G}_2$ and $\operatorname{Cd}(g) \approx g$, $g \in \mathcal{D}'$.

$\mathcal{E}_\tau[\mathbb{R}^n]$ is the set of all functions $G \in \mathcal{E}[\mathbb{R}^n]$ which satisfy:

$$\left\{\begin{array}{l}\text{For every } \beta \in \mathbb{N}_0^n \text{ there exists } N \in \mathbb{N} \text{ such that for every } \phi \in \mathcal{A}_N \text{ there}\\ \text{exist } \eta > 0 \text{ and } c > 0 \text{ such that } |\partial^\beta G(\phi_\varepsilon, x)| \leq c\varepsilon^{-N}(1 + |x|^N),\ x \in \mathbb{R}^n,\\ \varepsilon \in (0, \eta).\end{array}\right. \tag{7}$$

$\mathcal{N}_\tau[\mathbb{R}^n]$ is the set of all functions $G \in \mathcal{E}[\mathbb{R}^n]$ which satisfy:

$$\left\{\begin{array}{l}\text{For every } \beta \in \mathbb{N}_0^n \text{ there exist } N \in \mathbb{N} \text{ and } \alpha \in \Gamma \text{ such that for every}\\ \phi \in \mathcal{A}_q,\ q \geq N, \text{ there exist } \eta > 0 \text{ and } c > 0 \text{ such that } |\partial^\beta G(\phi_\varepsilon, x)| \leq\\ c\varepsilon^{\alpha(q)-N}(1 + |x|^N),\ x \in \mathbb{R}^n,\ \varepsilon \in (0, \eta).\end{array}\right. \tag{8}$$

$\mathcal{G}_\tau(\mathbb{R}^n) = \mathcal{E}_\tau[\mathbb{R}^n]/\mathcal{N}_\tau[\mathbb{R}^n]$ is the set of generalized tempered functions.

For $\boldsymbol{G} \in \mathcal{G}_\tau$, $\int^\tau \boldsymbol{G}(x)\,dx$ is defined by its representative $\int G(\phi_\varepsilon, x)\hat{\phi}_\varepsilon(x)\,dx$, where $\hat{\phi}(x) = \int \phi(t)e^{-ixt}\,dt$ is the Fourier transform of ϕ, $x \in \mathbb{R}^n$, $\phi \in \mathcal{A}_1$. If the integration is made over a compact set K, instead of $\mathbb{R}^n$, then we get the definition of the integral $\int_K^\tau \boldsymbol{G}(x)\,dx$ by its representative $\int_K G(\phi_\varepsilon, x)\hat{\phi}_\varepsilon(x)\,dx$, $\phi_\varepsilon \in \mathcal{A}_1$, $\varepsilon > 0$. This is an element of $\overline{\mathbb{C}}$.

2. New definitions

Denote by $\boldsymbol{a}$ a function defined on an interval $[1_{\boldsymbol{a}}, \infty)$ such that it is continuous, nondecreasing,

$$\lim_{x\to\infty} \boldsymbol{a}(x) = \infty \quad \text{and} \quad \boldsymbol{a}(x) = \mathcal{O}(x), \quad x \to \infty. \tag{9}$$

The set of such functions is denoted by A. Notice that if $\boldsymbol{a} \in A$, then $\ln(\boldsymbol{a}) \in A$, as well. Let $\boldsymbol{a} \in A$ be fixed. We define $\Theta_{\boldsymbol{a}}$ as the set of all functions θ defined on the interval $[0, \infty)$, with the following properties:

$$\left\{\begin{array}{l}\theta \text{ is continuous, increasing, } \theta(0) = 1 \text{ and for every } p \geq 0 \text{ there exists a}\\ \gamma > 0 \text{ such that } \theta(p + \boldsymbol{a}(x)) = \mathcal{O}(x^\gamma),\ x \to \infty.\end{array}\right. \tag{10}$$

If we assume that (10) holds only for $p = 0$, then the corresponding set is denoted by $\overline{\Theta}_{\boldsymbol{a}}$. Clearly $\Theta_{\boldsymbol{a}} \subset \overline{\Theta}_{\boldsymbol{a}}$, and $\overline{\Theta}_{\boldsymbol{a}} \neq \varnothing$. One can easily prove the following proposition.

PROPOSITION 1. a) *If* θ_1, θ_2 *are in* $\Theta_{\boldsymbol{a}}$, *then* $\theta_1 + \theta_2$ *and* $\theta_1\theta_2$ *are in* $\Theta_{\boldsymbol{a}}$. b) *If* $\theta \in \overline{\Theta}_{\boldsymbol{a}}$, *then* $\theta \in \Theta_{\ln(\boldsymbol{a})}$. c) *If for* $\boldsymbol{a}_1, \boldsymbol{a}_2 \in A$ *there exists constant* $c > 1$ *such that* $\boldsymbol{a}_1(x) \geq c\boldsymbol{a}_2(x)$, *then* $\Theta_{\boldsymbol{a}_1} \subset \Theta_{\boldsymbol{a}_2}$ *and* $\overline{\Theta}_{\boldsymbol{a}_1} \subset \overline{\Theta}_{\boldsymbol{a}_2}$.

We define the set $\overset{\odot}{\Theta}_{\boldsymbol{a}}$ as follows:

$\overset{\odot}{\Theta}_{\boldsymbol{a}}$ is the set of all function θ from $\Theta_{\boldsymbol{a}}$ which satisfy:

$$\left\{ \begin{array}{l} \text{There are a constant } c > 1 \text{ and functions } \theta_1 \text{ and } \theta_2 \text{ in } \Theta_{\boldsymbol{a}} \text{ such that for} \\ x, y > x_0,\ \theta(x+y) \leq c\theta_1(x)\theta_2(y). \end{array} \right. \tag{11}$$

Let us notice that Proposition 1 a) and b) also holds for $\overline{\Theta}_{\boldsymbol{a}}$ and $\overset{\odot}{\Theta}_{\boldsymbol{a}}$.

PROPOSITION 2. a) *If* $\theta \in \Theta_{\boldsymbol{a}}$, *then* $\theta \in \overset{\odot}{\Theta}_{\ln(\boldsymbol{a})}$. b) *If for some* $\alpha > 1$ *and some* $m \in \mathbb{N}$, $\alpha\boldsymbol{a}(x) \leq \boldsymbol{a}(x^m)$, *then* $\overset{\odot}{\Theta}_{\boldsymbol{a}} = \Theta_{\boldsymbol{a}}$. c) *If for* $\boldsymbol{a}_1, \boldsymbol{a}_2 \in A$ *there exists constant* $c > 1$ *such that* $\boldsymbol{a}_1(x) \geq c\boldsymbol{a}_2(x)$, *then* $\overset{\odot}{\Theta}_{\boldsymbol{a}_1} \subset \overset{\odot}{\Theta}_{\boldsymbol{a}_2}$.

Proof. Since c) trivially follows we shall give the proof of a) and b).

a) We have

$$\theta(2\ln(\boldsymbol{a}(x))) = \theta(\ln(\boldsymbol{a}(x))^2) \leq \theta(\boldsymbol{a}(x)) = \mathcal{O}(x), \quad x \to \infty,$$

so, we get $\theta(2x) \in \Theta_{\ln(\boldsymbol{a})}$. If $\lim_{x\to\infty} \theta(x) \leq 1$, then there exists a $c > 0$ such that $\theta(x+y) \leq c\theta(2x)\theta(2y)$, for x, y large enough.

If $\lim_{x\to\infty} \theta(x) > 1$, then, for large enough x, y, we have $\theta(2x) \geq 1$, $\theta(2y) \geq 1$, and thus $\theta(x+y) \leq \theta(2x)\theta(2y)$. In both cases $\theta \in \overset{\odot}{\Theta}_{\ln(\boldsymbol{a})}$, because $\theta(2x) \in \Theta_{\ln(\boldsymbol{a})}$.

b) The given condition implies that for any $\theta \in \Theta_{\boldsymbol{a}}$, $\theta(2\,\cdot) \in \Theta_{\boldsymbol{a}}$ as well. As in a), the proof follows. □

Remarks. For $\boldsymbol{t}(x) = 1 + x$, we have $\overset{\odot}{\Theta}_{\boldsymbol{t}} = \Theta_{\boldsymbol{t}} = \overline{\Theta}_{\boldsymbol{t}}$. Propositions 1 c) and 2 c) imply that for every $\boldsymbol{a} \in A$, $\Theta_{\boldsymbol{a}}$ and $\overset{\odot}{\Theta}_{\boldsymbol{a}}$ are not empty. For $\boldsymbol{a}(x) = \ln(x+1) + 1$, we have also $\overset{\odot}{\Theta}_{\boldsymbol{a}} = \Theta_{\boldsymbol{a}} = \overline{\Theta}_{\boldsymbol{a}}$. For $\boldsymbol{a}(x) = \ln(\ln(x+1)+1)+1$, $\overset{\odot}{\Theta}_{\boldsymbol{a}}$ is not equal to $\Theta_{\boldsymbol{a}}$; neither in this case $\overline{\Theta}_{\boldsymbol{a}}$ is not equal to $\Theta_{\boldsymbol{a}}$. □

Let $\boldsymbol{a} \in A$. The space of generalized $\boldsymbol{a}$-functions is defined as follows: $\mathcal{E}_{\boldsymbol{a}}[\mathbb{R}^n]$ is the set of all functions $G \in \mathcal{E}[\mathbb{R}^n]$ which satisfy:

$$\left\{ \begin{array}{l} \text{For every } \beta \in \mathbb{N}_0^n \text{ there exist } N \in \mathbb{N} \text{ and } \theta \in \Theta_{\boldsymbol{a}} \text{ such that for every} \\ \phi \in \mathcal{A}_N \text{ there exist an } \eta > 0 \text{ and a } c > 0 \text{ such that } |\partial^\beta G(\phi_\varepsilon, x)| \leq \\ c\varepsilon^{-N}\theta(|x|),\ x \in \mathbb{R}^n,\ \varepsilon \in (0,\eta). \end{array} \right. \tag{12}$$

$\mathcal{N}_{\boldsymbol{a}}[\mathbb{R}^n]$ is the set of all functions $G \in \mathcal{E}[\mathbb{R}^n]$ which satisfy:

$$(13)\qquad \begin{cases} \text{for every } \beta \in \mathbb{N}_0^n \text{ there exist } N \in \mathbb{N},\ \theta \in \Theta_{\boldsymbol{a}} \text{ and } \alpha \in \Gamma \text{ such that} \\ \text{for every } \phi \in \mathcal{A}_q,\ q \geq N, \text{ there exist an } \eta > 0 \text{ and a } c > 0 \text{ such that} \\ |\partial^\beta G(\phi_\varepsilon, x)| \leq c\varepsilon^{\alpha(q)-N}\theta(|x|),\ x \in \mathbb{R}^n,\ \varepsilon \in (0,\eta). \end{cases}$$

PROPOSITION 3. *Let* $\boldsymbol{a} \in A$. *Then* $\mathcal{N}_{\boldsymbol{a}}$ *is an ideal of* $\mathcal{E}_{\boldsymbol{a}}$.

Proof. Let $G \in \mathcal{N}_{\boldsymbol{a}}$, and $H \in \mathcal{E}_{\boldsymbol{a}}$. We have to prove that $GH \in \mathcal{N}_{\boldsymbol{a}}$. For every $\beta \in \mathbb{N}_0^n$

$$\partial^\beta(GH) = \sum_{i+j=\beta} s_{ij}\partial^i G\,\partial^j H, \qquad \text{where } s_{ij} = \frac{\beta!}{i!\,j!}.$$

We adopt the notation for G and ∂^i , $i \leq \beta$ in (13) by using symbols with subindex $_{2,i}$ ($N_{2,i}$, $\theta_{2,i}$, $\dots$). We do the same for H and ∂^j in (12) by using symbols with subindex $_{1,j}$. Let

$$N = \max_{i+j=\beta}\{N_{1,j} + N_{2,i}\}, \qquad \alpha(q) = \max_{i\leq\beta}\alpha_i(q),$$
$$\theta = \sum_{i+j=\beta} s_{ij}\theta_{1,j}\theta_{2,i}, \qquad c = \sum_{i+j=\beta} c_{1,j}c_{2,i}.$$

For every $\phi \in \mathcal{A}_q$, $q \geq N$, let $\eta = \min_{i,j\leq\beta}\{\eta_{1,j},\eta_{2,i}\}$. Then for $x \in \mathbb{R}^n$, $\varepsilon \in (0,\eta)$, we have

$$|\partial^\beta(GH)(\phi_\varepsilon,x)| \leq \sum_{i+j=\beta} s_{ij}|\partial^i G(\phi_\varepsilon,x)|\,|\partial^j H(\phi_\varepsilon,x)|$$
$$\leq \sum_{i+j=\beta} s_{ij}c_{1,j}\theta_{1,j}(|x|)\varepsilon^{\alpha(q)-N_{1,j}}c_{2,i}\theta_{2,i}(|x|)\varepsilon^{-N_{2,i}} \leq c\theta(|x|)\varepsilon^{\alpha(q)-N}.$$

By Proposition 1 a) we have that $\theta \in \Theta_{\boldsymbol{a}}$ and the proof is complete. □

Now we define the space of generalized $\boldsymbol{a}$-functions by $\mathcal{G}_{\boldsymbol{a}} = \mathcal{E}_{\boldsymbol{a}}/\mathcal{N}_{\boldsymbol{a}}$. This is a vector space over $\overline{\mathbb{C}}$ and if $\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{a}}$, then for any $\beta \in \mathbb{N}_0^n$, $\partial^\beta\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{a}}$. By similar computations as in Proposition 3 we have that the pointwise product of two generalized functions from $\mathcal{G}_{\boldsymbol{a}}$ is again in $\mathcal{G}_{\boldsymbol{a}}$.

One can easily prove the following

PROPOSITION 4. *Let* $\boldsymbol{a}_1, \boldsymbol{a}_2 \in A$ *such that* $\boldsymbol{a}_1(x) \geq c\boldsymbol{a}_2(x)$, *where* $c > 1$ *is a suitable constant. Then*: a) $\mathcal{E}_{\boldsymbol{a}_1} \subset \mathcal{E}_{\boldsymbol{a}_2}$, $\mathcal{N}_{\boldsymbol{a}_1} \subset \mathcal{N}_{\boldsymbol{a}_2}$; b) *for every* $\boldsymbol{a} \in A$ *the inclusion mapping* $i : \mathcal{G}_{\boldsymbol{a}} \to \mathcal{G}$, $i(G + \mathcal{N}_{\boldsymbol{a}}) = G + \mathcal{N}$, $G \in \mathcal{E}_{\boldsymbol{a}}$ *is not injective*; c) *the inclusion mapping* $i : \mathcal{G}_{\boldsymbol{a}_1} \to \mathcal{G}_{\boldsymbol{a}_2}$, $i(G + \mathcal{N}_{\boldsymbol{a}_1}) = G + \mathcal{N}_{\boldsymbol{a}_2}$, $G \in \mathcal{E}_{\boldsymbol{a}_1}$ *is not injective in general case.*

Proof. Since a) is trivial, we shall prove b) and c).

b) One can prove that $\mathcal{E}_{\boldsymbol{t}} = \mathcal{E}_\tau$, $\mathcal{N}_t = \mathcal{N}_\tau$, so $\mathcal{G}_t = \mathcal{G}_\tau$. Proposition 4 a) implies $\mathcal{E}_{\boldsymbol{t}} \subset \mathcal{E}_{\boldsymbol{a}}$, $\mathcal{N}_{\boldsymbol{t}} \subset \mathcal{N}_{\boldsymbol{a}}$. Colombeau has proved in [**3**] that the mapping $i : \mathcal{G}_\tau \to \mathcal{G}$

is not injective. For the same function R, which Colombeau has constructed, we have: $R \in \mathcal{E}_\tau \cap \mathcal{N} \subset \mathcal{E}_{\boldsymbol{a}} \cap \mathcal{N}$ and $R \notin \mathcal{N}_{\boldsymbol{a}}$ for every $\boldsymbol{a} \in A$. So, neither the mapping $G + \mathcal{N}_{\boldsymbol{a}} \to G + \mathcal{N}$ is not injective.

c) Let $\boldsymbol{a}_1 = \boldsymbol{t}$ and $\boldsymbol{a}_2(x) = \ln(x+1)$. We shall construct a function $G(\phi_\varepsilon, x)$, $x \in \mathbb{R}$, $\phi \in \mathcal{A}_N$, $\varepsilon > 0$, such that for some $c > 0$

$$G(\phi_\varepsilon, x) \leq c\varepsilon^{-N}(1+x^2)^{1/2}, \qquad x \in \mathbb{R},\ \phi \in \mathcal{A}_N,\ \varepsilon > 0,$$
$$G(\phi_\varepsilon, x) \leq c\varepsilon^{N} e^x (1+x^2)^{1/2}, \qquad x \in \mathbb{R},\ \phi \in \mathcal{A}_N,\ \varepsilon > 0;$$

but there does not exist constants $N' \in \mathbb{N}$, $c > 0$, $\alpha \in \Gamma$ and $\theta \in \Theta_{\boldsymbol{t}}$ such that

$$G(\phi_\varepsilon, x) \leq c\varepsilon^{\alpha(q) - N'} \theta(|x|), \qquad x \in \mathbb{R},\ \phi \in \mathcal{A}_q,\ q > N',\ \varepsilon > 0.$$

This means that $G \in (\mathcal{E}_{\boldsymbol{t}} \cap \mathcal{N}_{\boldsymbol{a}_2}) \setminus \mathcal{N}_{\boldsymbol{t}}$ and that the inclusion mapping $i : \mathcal{G}_{\boldsymbol{a}_1} \to \mathcal{G}_{\boldsymbol{a}_2}$, $i(G + \mathcal{N}_{\boldsymbol{a}_1}) = G + \mathcal{N}_{\boldsymbol{a}_2}$, $G \in \mathcal{E}_{\boldsymbol{a}_1}$ is not injective.

Put

$$G(\phi_\varepsilon, x) = g(\phi_\varepsilon, x)\sigma(\phi_\varepsilon)\varepsilon^N (1+x^2)^{1/2}, \qquad x \in \mathbb{R}^n,\ \phi \in \mathcal{A}_N \setminus \mathcal{A}_{N+1},$$

where $\sigma(\phi_\varepsilon) = \sup\{|x| : x \in \operatorname{supp}(\phi_\varepsilon)\}$, $g(\phi_\varepsilon, x) = \phi_\varepsilon * f(\phi_\varepsilon, \cdot)(x)$, and where $f(\phi_\varepsilon, \cdot)$ is an even function defined by

$$f(\phi_\varepsilon, x) = \begin{cases} e^x, & x \leq -(N+1)\ln(\sigma(\phi_\varepsilon)) \\ \sigma(\phi_\varepsilon)^{-N-1}, & x > -(N+1)\ln(\sigma(\phi_\varepsilon)). \end{cases}$$

By a simple calculation, for $b = (N+1)\ln(\sigma(\phi_\varepsilon))$, $a = \sigma(\phi_\varepsilon)$ we get the following function:

$$g(\phi_\varepsilon, x) =$$
$$= \begin{cases} \sigma(\phi_\varepsilon)^{-N-1}, & a - b \leq x \\ e^x \int_{x+b}^{a} \phi_\varepsilon(t)\,dt + \sigma(\phi_\varepsilon)^{-N-1} \int_{-a}^{x+b} \phi_\varepsilon(t)\,dt, & -a - b \leq x < a - b \\ e^x \int_{-a}^{a} e^{-t}\phi_\varepsilon(t)\,dt, & 0 \leq x < -a - b \\ e^{-x} \int_{-a}^{a} e^{-t}\phi_\varepsilon(t)\,dt, & a + b \leq x < 0 \\ e^{-x} \int_{-a}^{x-b} \phi_\varepsilon(t)\,dt + \sigma(\phi_\varepsilon)^{-N-1} \int_{x-a}^{a} \phi_\varepsilon(t)\,dt, & -a + b \leq x < a + b \\ \sigma(\phi_\varepsilon)^{-N-1}, & x \leq -a + b. \square \end{cases}$$

If we use the set $\overset{\odot}{\Theta}_{\boldsymbol{a}}$, instead of $\Theta_{\boldsymbol{a}}$, then we get definitions of the sets $\overset{\odot}{\mathcal{E}}_{\boldsymbol{a}}$, $\overset{\odot}{\mathcal{N}}_{\boldsymbol{a}}$, $\overset{\odot}{\mathcal{G}}_{\boldsymbol{a}}$ instead $\mathcal{E}_{\boldsymbol{a}}$, $\mathcal{N}_{\boldsymbol{a}}$, $\mathcal{G}_{\boldsymbol{a}}$, which have the same properties. Since $\overset{\odot}{\mathcal{E}}_{\boldsymbol{a}} \subset \mathcal{E}_{\boldsymbol{a}}$, $\overset{\odot}{\mathcal{N}}_{\boldsymbol{a}} \subset \mathcal{N}_{\boldsymbol{a}}$, there is the inclusion mapping $i : \overset{\odot}{\mathcal{G}}_{\boldsymbol{a}} \to \mathcal{G}_{\boldsymbol{a}}$, $i(G + \overset{\odot}{\mathcal{N}}_{\boldsymbol{a}}) = G + \mathcal{N}_{\boldsymbol{a}}$, $G \in \overset{\odot}{\mathcal{E}}_{\boldsymbol{a}}$. The injectivity of this mapping is an open problem.

Proposition 5. *Let $g \in \mathcal{D}'$ be a distribution of finite order. There exists $\boldsymbol{a} \in A$ such that* $\mathrm{Cd}(g) \in \mathcal{G}_{\boldsymbol{a}}$. *Particularly, the space of distributions of finite order can be embedded into $\bigcup_{\boldsymbol{a} \in A} \mathcal{G}_{\boldsymbol{a}}$.*

Proof. There exist a continuous function f and $\alpha \in \mathbb{N}_0^n$ such that $g = \partial^\alpha f$. For $\beta \in \mathbb{N}_0^n$ we have $\partial^\beta \boldsymbol{G} = \mathrm{Cd}(\partial^\beta g) = \mathrm{Cd}(\partial^{\alpha+\beta} f)$, and

$$\begin{aligned}\partial^\beta G(\phi_\varepsilon, x) &= \partial^{\alpha+\beta} f * \check{\phi}_\varepsilon(x) = \int f(x+t)\partial^{\alpha+\beta}\phi_\varepsilon(t)\,dt \\ &= \varepsilon^{-(|\alpha|+|\beta|)} \int f(x+\varepsilon t)\partial^{\alpha+\beta}\phi(t)\,dt, \quad x \in \mathbb{R},\ \phi \in \mathcal{A}_1,\ \varepsilon > 0.\end{aligned}$$

We have

$$\left|\int f(x+\varepsilon t)\partial^{\alpha+\beta}\phi(t)\,dt\right| \le \int |f(x+\varepsilon t)|\,|\partial^{\alpha+\beta}\phi(t)|\,dt, \qquad \varepsilon > 0.$$

Let f_1 be an even, positive and continuous function on $\mathbb{R}$ increasing on $[0,\infty)$ such that $f_1(|t|) \ge |f(t)|$ and $f_1(|t|) \ge c|t|$ for $t \in \mathbb{R}^n$. Then

$$\int |f(x+\varepsilon t)|\,|\partial^{\alpha+\beta}\phi(t)|\,dt \le f_1(|x| + \varepsilon a_{\alpha,\beta}) \int |\partial^{\alpha+\beta}\phi(t)|\,dt, \quad x \in \mathbb{R}^n,\ \varepsilon > 0,$$

where $a_{\alpha,\beta} = \sup\{|t| : t \in \mathrm{supp}(\partial^{\alpha+\beta}\phi)\}$. Let $\theta(x) = f_1(x+1)$, $x \in [0,\infty)$ and $\boldsymbol{a} = \ln(\theta^{-1})$, where θ^{-1} is the inverse function for θ defined on the corresponding interval $(\theta(0),\infty)$. Clearly $\theta \in \Theta_{\boldsymbol{a}}$. Let $N = |\alpha| + |\beta|$, $\phi \in \mathcal{A}_N$, $c = \int |\partial^{\alpha+\beta}\phi(t)|\,dt$ and $\eta = 1/a_{\alpha,\beta}$. Then

$$|\partial^\beta G(\phi_\varepsilon, x)| \le c\theta(|x|)\varepsilon^{-N}, \qquad x \in \mathbb{R}^n, \quad \varepsilon \in (0,\eta).$$

This implies that $\mathrm{Cd}(g) \in \mathcal{G}_{\boldsymbol{a}}$ $\square$.

Let us show that the regularized derivative is well defined in the space $\mathcal{G}_{\boldsymbol{a}}$ for any $\boldsymbol{a} \in A$. Let $G \in \mathcal{E}_{\boldsymbol{a}}$, $j \in \{1,\dots,n\}$. By (12) we have

$$\begin{aligned}|\overline{\partial}_j^h G(\phi_\varepsilon, x)| &= \left|\int \partial_j G(\phi_\varepsilon, x - yh(\varepsilon))\phi(y)\,dy\right| \\ &\le \int |\phi(y)|\,dy \cdot \sup_{y \in \mathrm{supp}(\phi)} |\partial_j G(\phi_\varepsilon, x - yh(\varepsilon))| \\ &\le c_1 c\theta(|x| + |y|h(\varepsilon))\varepsilon^{-N} \le c_2\theta(|x|),\end{aligned}$$

where $c_2 = c_1 c \cdot \sup\{\theta(|y|h(\varepsilon)) : y \in \mathrm{supp}(\phi),\ \varepsilon \in (0,1)\}$. That means that $\overline{\partial}_j^h \in \mathcal{E}_{\boldsymbol{a}}$. The proof for $G \in \mathcal{N}_{\boldsymbol{a}}$ is similar.

3. $\boldsymbol{a}$-integrals

We define the unit net μ_ε, $\varepsilon > 0$, which corresponds to $\boldsymbol{a}$ as follows. This is a net in $\mathcal{D}$ such that

$$(14) \qquad \begin{cases} \text{(i)} & 0 \le \mu_\varepsilon(x) \le 1,\ x \in \mathbb{R}^n,\ \varepsilon > 0; \\ \text{(ii)} & \mu_\varepsilon(x) = \begin{cases} 1, & |x| < \boldsymbol{a}(b/\varepsilon) \\ 0, & |x| > \boldsymbol{a}(b/\varepsilon) + r,\ \varepsilon > 0; \end{cases} \\ & \text{where } b > 0,\ r > 0 \text{ are constants;} \\ \text{(iii)} & \text{for every } l \in \mathbb{N}_0^n \text{ there is } c_l \ge 0 \text{ such that} \\ & |\partial^l \mu_\varepsilon(x)| \le c_l,\ x \in \mathbb{R}^n,\ \varepsilon > 0. \end{cases}$$

Let B be a measurable set. For $\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{a}}$ we define $\int_B^{\boldsymbol{a},\mu} \boldsymbol{G}(x)\,dx$ as an element of $\overline{\mathbb{C}}$ by the representative

$$ {}_BY_{\mu,G}(\phi_\varepsilon) = \int_B G(\phi_\varepsilon, x)\mu_\varepsilon(x)\,dx, \qquad \phi_\varepsilon \in \mathcal{A}_1,\ \varepsilon > 0. \tag{15} $$

Let $G \in \mathcal{N}_{\boldsymbol{a}}$. Then

$$ \begin{aligned} |{}_BY_{\mu,G}(\phi_\varepsilon)| &\le \int_B |G(\phi_\varepsilon, x)|\,\mu_\varepsilon(x)\,dx \\ &\le \int_{\substack{|x|\le \boldsymbol{a}(b/\varepsilon+r)\\ B}} |G(\phi_\varepsilon, x)|\,dx \le \theta(\boldsymbol{a}(b/\varepsilon)+r)\varepsilon^{\alpha(q)-N} \le c\varepsilon^{\alpha(q)-N_1}, \quad \varepsilon > 0, \end{aligned} $$

where we used the notation from (13). Similarly, one can prove that if $G \in \mathcal{E}_{\boldsymbol{a}}$, then ${}_BY_{\mu,G}(\phi_\varepsilon) \in \mathcal{E}_c$. So, the definition is correct. If for every unit net μ_ε, $\varepsilon > 0$, ${}_BY_{\mu_\varepsilon,G}(\phi_\varepsilon) \approx c \in \mathbb{C}$, then we say that there exists the associated $\boldsymbol{a}$-integral of $\boldsymbol{G}$ and write $\int_B^{\boldsymbol{a}} \boldsymbol{G}(x)\,dx \approx c$. If we have ${}_BY_{\mu^1,G}(\phi_\varepsilon) - {}_BY_{\mu^2,G}(\phi_\varepsilon) \in \mathcal{F}$ for every two unit nets μ^1, μ^2, then we define the $\boldsymbol{a}$-integral of $\boldsymbol{G}$ over B by

$$ \int_B^{\boldsymbol{a}} \boldsymbol{G}(x)\,dx = \mathcal{F} + {}_BY_{\mu^1,G}(\phi_\varepsilon). $$

PROPOSITION 6. *Let $g \in \mathcal{D}'_{L^1}$. Then $\boldsymbol{G} = \mathrm{Cd}(g) \in \mathcal{G}_{\boldsymbol{a}}$ for every $\boldsymbol{a} \in A$. Moreover, there exists the associated $\boldsymbol{a}$-integral of $\boldsymbol{G}$.*

Proof. There exists $m \in \mathbb{N}$ such that

$$ g = \sum_{|i|\le m} \partial^i g_i, \qquad \text{where } g_i \text{ are } L^1 \text{ functions.} \tag{16} $$

Let us put

$$ \bar{g}_i(x) = \int_0^{x_n} dt_n \cdot \ldots \cdot \int_0^{x_1} g_i(t_1,\ldots,t_n)\,dt_1, \qquad x \in \mathbb{R}^n,\ |i| \le m. $$

We have $|\bar{g}_i(x)| \le \int |g_i(t)|\,dt = M_i$, and $\partial^{i_1+1,\ldots,i_n+1}\bar{g}_i(x) = \partial^i g(x)$, $x \in \mathbb{R}^n$, $|i| \le m$.

This implies that g is a finite sum of distributional derivatives of bounded functions and since bounded functions are in $\mathcal{G}_{\boldsymbol{a}}$, for any $\boldsymbol{a} \in A$, the same follows for g, as well. By using the representation (16) we have

$$ G(\phi_\varepsilon, x) = \sum_{|i|\le m} (g_i * \partial^i \check{\phi}_\varepsilon)(x), \qquad x \in \mathbb{R}^n,\ \phi \in \mathcal{A}_1,\ \varepsilon > 0. $$

For $|i| > 0$ and a unit net μ_ε, $\varepsilon > 0$, we have

$$ \begin{aligned} I_{i,\varepsilon} &= \int_{\mathbb{R}^n} (g_i * \partial^i \check{\phi}_\varepsilon)(x)\mu_\varepsilon(x)\,dx = (-1)^{|i|}\int_{\mathbb{R}^n} (g_i * \check{\phi}_\varepsilon)(x)\partial^i\mu_\varepsilon(x)\,dx \\ &= (-1)^{|i|}\int_{\mathbb{R}^n} g_i(x)(\phi_\varepsilon * \partial^i\mu_\varepsilon)(x)\,dx, \quad \varepsilon > 0. \end{aligned} $$

Clearly, for every $x \in \mathbb{R}^n$, $g_i(x)(\phi_\varepsilon * \partial^i \mu_\varepsilon)(x) \to 0$, as $\varepsilon \to 0$. We have

$$|g_i(x)(\phi_\varepsilon * \partial^i \mu_\varepsilon)(x)| \le |g_i(x)| \left| \int \phi_\varepsilon(t) \partial^i \mu_\varepsilon(x-t)\, dt \right|$$
$$\le c_i |g_i(x)| \int |\phi_\varepsilon(t)|\, dt \le c c_i |g_i(x)|, \qquad x \in \mathbb{R}^n,\ \varepsilon > 0.$$

Thus, by applying Lebesgue's dominated convergence theorem, for $m \ge |i| > 0$, we have $I_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$. By a similar calculation, for $i = 0$ we have $I_{0,\varepsilon} \to \int g_0(x)\, dx$, as $\varepsilon \to 0$. This implies

$$\int^{\boldsymbol{a},\mu} \boldsymbol{G}(x)\, dx \approx \int g_0(x)\, dx, \qquad \text{i.e.} \qquad \int^{\boldsymbol{a}} \boldsymbol{G}(x)\, dx \approx \int g_0(x)\, dx. \square$$

Remark. If $g \in L^1$ and for every $b > 0$, $r > 0$, there exist $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that for every $\phi \in \mathcal{A}_q$, $q \ge N$, there exist an $\eta > 0$ and a $c > 0$ such that

$$\left| \int_{|x| \le \boldsymbol{a}(b/\varepsilon)+r} (\check{\phi}_\varepsilon * g)(x)\, dx \right| \le c\varepsilon^{\alpha(q)-N}, \qquad 0 < \varepsilon < \eta,$$

then there exists the $\boldsymbol{a}$-integral $\int^{\boldsymbol{a}} \mathrm{Cd}(g)\, dx$.

The following example will show that it is not possible to define $\boldsymbol{a}, \mu$-integral in $\mathcal{G}$ in the way described above.

Example. Let μ_ε be a unit net. There are constants $r_1 < r$ and $c > 0$ such that $\mu_\varepsilon(x) \ge c$ for $|x| \le \boldsymbol{a}(b/\varepsilon) + r_1$, $\varepsilon > 0$. Define $G \in \mathcal{E}_M$ by

$$G(\phi_\varepsilon, x) = \begin{cases} 0, & |x| \le \boldsymbol{a}(b/\varepsilon), \\ e^{\boldsymbol{a}^{-1}(x)}, & r_1/2 + \boldsymbol{a}(b/\varepsilon) \le |x| \le r_1 + \boldsymbol{a}(b/\varepsilon), \end{cases} \qquad \phi \in \mathcal{A}_1,\ \varepsilon > 0.$$

One can prove that $G \in \mathcal{N}$. Since $\int^{\boldsymbol{a},\mu} G(x)\, dx \ge (r_1/2)e^{b/\varepsilon}$, $\varepsilon > 0$, it follows that $\int^{\boldsymbol{a},\mu} G(x)\, dx \notin \mathcal{F}$. $\square$

Let $\varphi \in \mathcal{S}$ and $\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{t}}$. We define $\langle \boldsymbol{G}, \varphi \rangle = \int \boldsymbol{G}(x)\varphi(x)\, dx$ by its representative

$$I(\phi_\varepsilon) = \int_{\mathbb{R}^n} G(\phi_\varepsilon, x)\varphi(x)\, dx, \qquad \phi \in \mathcal{A}_1,\ \varepsilon > 0. \tag{17}$$

PROPOSITION 7. a) *The definition of* $\langle \boldsymbol{G}, \varphi \rangle$ *given by* (17) *is correct.* b) *For every* $\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{t}}$, *every unit net* μ_ε *and every* $\varphi \in \mathcal{S}$, *we have*

$$\int^{\boldsymbol{t},\mu} \boldsymbol{G}(x)\varphi(x)\, dx = \int \boldsymbol{G}(x)\varphi(x)\, dx, \qquad \int^{\boldsymbol{t},\mu} \boldsymbol{G}(x)\varphi(x)\, dx = \int^{\tau} \boldsymbol{G}(x)\varphi(x)\, dx.$$

Proof. a) Let $G \in \mathcal{N}_{\boldsymbol{t}}$. By using the notation from (13), this means that for every $\phi \in \mathcal{A}_q$, $|G(\phi_\varepsilon, x)| \le c_1(1+|x|)^\gamma \varepsilon^{\alpha(q)-N}$, $x \in \mathbb{R}^n$, $0 < \varepsilon < \eta$. For every $p \in \mathbb{N}$ there is a $c_2 > 0$ such that $|\varphi(x)| \le c_2(1+|x|)^{-p}$, $x \in \mathbb{R}^n$, and by putting $p = [\gamma] + n + 1$, with suitable $c > 0$, we have

$$|I(\phi_\varepsilon)| \le \int |G(\phi_\varepsilon, x)|\, |\varphi(x)|\, dx \le c\varepsilon^{\alpha(q)-N}, \qquad 0 < \varepsilon < \eta,\ \phi \in \mathcal{A}_q,\ q > N.$$

So, $I \in \mathcal{F}$. Similarly, one can prove that $G \in \mathcal{E}_{\boldsymbol{t}}$ implies $I \in \mathcal{E}_c$.

b) Adopting the notation from (12) and (13) we obtain

$$\begin{aligned}|Z(\phi_\varepsilon)| &= \left|\int G(\phi_\varepsilon, x)\varphi(x)\,dx - \int G(\phi_\varepsilon, x)\phi(x)\mu_\varepsilon(x)\,dx\right| \\ &\le \int_{|x|>b/\varepsilon+r} |G(\phi_\varepsilon, x)|\,|\varphi(x)|\,dx \\ &\le \varepsilon^{-N}\int_{|x|>b/\varepsilon+r} c_1(1+|x|)^\gamma c_2(1+|x|)^{-p}\,dx \le c\varepsilon^{q-N},\end{aligned}$$

where $p = \gamma + n + 1 + q$. This implies $Z \in \mathcal{F}$. Similarly,

$$\begin{aligned}&\left|\int G(\phi_\varepsilon, x)\varphi(x)\,dx - \int G(\phi_\varepsilon, x)\varphi(x)\hat{\phi}_\varepsilon(x)\,dx\right| \\ &\le \int |G(\phi_\varepsilon, x)\varphi(x)|\,|1-\hat{\phi}_\varepsilon(x)|\,dx \le \int c\varepsilon^{-N}\varepsilon^q|x|^{-p}\,dx \le c\varepsilon^{q-N}. \square\end{aligned}$$

We say that $\boldsymbol{G}_1 = \boldsymbol{G}_2$ (g.t.d) (equal in the sense of generalized tempered distributions) if for every $\varphi \in \mathcal{S}$, $\langle \boldsymbol{G}_1, \varphi\rangle = \langle \boldsymbol{G}_2, \varphi\rangle$ (in $\overline{\mathbb{C}}$).

PROPOSITION 8. *Let $\boldsymbol{G} \in \mathcal{G}_{\boldsymbol{t}}$, $g \in \mathcal{S}'$ and $\boldsymbol{G} \approx g$. Then*

$$\lim_{\varepsilon\to 0}\int G(\phi_\varepsilon, x)\varphi(x)\,dx = \langle g, \varphi\rangle, \qquad \varphi \in \mathcal{S}.$$

Proof. Let $\varphi \in \mathcal{S}$, $\phi \in \mathcal{A}_N$. There exists a $\gamma > 0$ such that $|G(\phi_\varepsilon, x)| \le c_1|x|^\gamma$. Let $\delta > 0$. Chose $\psi \in \mathcal{D}$ such that

$$|\langle g, \varphi\rangle - \langle g, \psi\rangle| < \delta, \qquad |\varphi(x) - \psi(x)| < c\delta/(1+|x|)^p, \qquad x \in \mathbb{R}^n,$$

where $p > \gamma + 1$ and $c^{-1} = c_1\int |x|^\gamma(1+|x|)^{-p}\,dx$. There exists an $\eta > 0$ such that

$$\left|\int G(\phi_\varepsilon, x)\psi(x)\,dx - \langle g, \psi\rangle\right| < \delta, \qquad \text{for } 0 < \varepsilon < \eta.$$

Thus, we obtain

$$\begin{aligned}&\left|\int G(\phi_\varepsilon, x)\varphi(x)\,dx - \langle g, \varphi\rangle\right| \\ &\le \left|\int G(\phi_\varepsilon, x)\left(\varphi(x) - \psi(x)\right)dx\right| + \left|\int G(\phi_\varepsilon, x)\psi(x)\,dx - \langle g, \psi\rangle\right| + |\langle g, \psi\rangle - \langle g, \varphi\rangle| \\ &< \delta + \delta + \delta = 3\delta, \qquad \text{for } 0 < \varepsilon < \eta. \square\end{aligned}$$

REFERENCES

The list of references is at the end of Part II of this paper.

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