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APPROXIMATION OF CONTINUOUS FUNCTIONS BY MONOTONE SEQUENCES OF POLYNOMIALS WITH INTEGRAL COEFFICIENTS

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Abstract. The problem of approximation by polynomials with integral coefficients, is considered in several papers (e.g. [1–7], [19–24]). In [8–18] I have proved among other things that every $f \in C[a,b]$ can be uniformly approximated by two polynomial sequences $(Q_n)_n$ and $(P_n)_n$ such that $(Q_n)_n$ is monotonically increasing and $(P_n)_n$ is monotonically decreasing on $[a,b]$. The aim of this paper is to extend the ideas of [8–18] to the case of approximation by polynomials with integral coefficients.

1. Introduction. The problem of approximation by polynomials with integral coefficients is treated in several papers [1–7], [19–24]. Thus, for example, in [24] the following result is proved.

THEOREM 1.1. *Let $0 < \alpha < 1$ and let us denote by $C[-\alpha,\alpha] = \{f : [-\alpha,\alpha] \to \mathbf{R};\ f$ continuous on $[-\alpha,\alpha]\}$. Then $f \in C[-\alpha,\alpha]$ can be uniformly approximated by polynomials with integral coefficients iff $f(0)$ is an integral number.*

In several papers [8–18], I have proved, among other things, that every $f \in C[a,b]$ can be uniformly approximated by two polynomial sequences $(Q_n)_n$, $(P_n)_n$ such that $Q_n(x) < Q_{n+1}(x) < f(x) < P_{n+1}(x) < P_n(x)$, for all $x \in [a,b]$ and all $n \in \mathbf{N}$. In the present paper we will extend the ideas of [8–18] to the case of approximation by polynomials with integral coefficients.

2. Basic results. For $0 < \alpha < 1$, let us denote by $C_0^\infty[-\alpha,\alpha] = \{f \in C[-\alpha,\alpha] : f$ infinitely differentiable at 0 and $f^{(n)}(0) = 0,\ n = 0,1,\dots\}$.

Remark. It is easily seen that $C_0^\infty[-\alpha,\alpha]$ is a linear subspace of $C[-\alpha,\alpha]$. For example, if $f(x) = \exp(-1/x^2)$, $x \in [-\alpha,\alpha]$, $x \neq 0$ and $f(0) = 0$, then $f \in C_0^\infty[-\alpha,\alpha]$.

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THEOREM 2.1. *For every $f \in C_0^\infty[-\alpha, \alpha]$, there exist two sequences of polynomials with integral coefficients $(Q_n)_n$ and $(P_n)_n$ such that*

$$Q_n \xrightarrow{n} f, \quad P_n \xrightarrow{n} f \quad \textit{uniformly on } [-\alpha,\alpha],$$
$$Q_n(0) = P_n(0) = f(0) = 0, \quad Q_n(x) < Q_{n+1}(x) < f(x) < P_{n+1}(x) < P_n(x),$$

for all $x \in [-\alpha, \alpha]$, $x \neq 0$ and all $n \in \mathrm{N}$.

Proof. For $f \in C_0^\infty[-\alpha, \alpha]$ and $n \in \mathrm{N}$ fixed, take $F_n(x) = f(x)/x^{2n}$, $x \in [-\alpha, \alpha]$, $x \neq 0$ and $F_n(0) = 0$. Since $f \in C_0^\infty[-\alpha, \alpha]$, we have:

$$\lim_{x\to 0} F_n(x) = \lim_{x \to 0} \frac{f'(x)}{2n \cdot x^{2n-1}} = \ldots = \frac{f^{(2n)}(0)}{(2n)!} = 0,$$

therefore $F_n \in C[-\alpha, \alpha]$, $F_n(0) = 0$. Taking into account Tehorem 1.1, there exists a polynomial $R_n(x)$ with integral coefficients such that $|F_n(x) - R_n(x)| < 1$, for all $x \in [-\alpha, \alpha]$. Hence

$$|f(x) - x^{2n} \cdot R_n(x)| \leq x^{2n}, \qquad \text{for all } x \in [-\alpha, \alpha]. \tag{1}$$

Take $S_n(x) = x^{2n} \cdot R_n(x)$, $Q_n(x) = S_n(x) - 2K \cdot x^{2n}$ and $P_n(x) = S_n(x) + 2K \cdot x^{2n}$, where K is a fixed integral satisfying $K > 1/(1-\alpha^2)$. From (1) it is evident that $S_n \to f$ uniformly (when $n \to +\infty$) and therefore $Q_n \xrightarrow{n} f$, $P_n \xrightarrow{n} f$ uniformly on $[-\alpha, \alpha]$. Also, $Q_n(0) = P_n(0) = 0$, for all $n \in \mathrm{N}$. Then by (1) we obtain

$$\begin{aligned}|S_n(x) - S_{n+1}(x)| &\leq |S_n(x) - f(x)| + |f(x) - S_{n+1}(x)| \leq x^{2n} + x^{2n+1} \\ &\leq 2x^{2n} = 2 \cdot [1/(1-\alpha^2)] \cdot x^{2n} \cdot (1-\alpha^2) \\ &\leq 2K \cdot x^{2n}(1-\alpha^2) \leq 2K \cdot x^{2n}(1-x^2),\end{aligned}$$

for all $x \in [-\alpha, \alpha]$ and all $n = 1, 2, \ldots$, and therefore

$$Q_{n+1}(x) - Q_n(x) = S_{n+1}(x) - S_n(x) + 2Kx^{2n}(1-x^2) > 0,$$

for all $x \in [-\alpha, \alpha]$, $x \neq 0$ and all $n = 1, 2, \ldots$. Also,

$$P_n(x) - P_{n+1}(x) = S_n(x) - S_{n+1}(x) + 2K \cdot x^{2n}(1-x^2) > 0,$$

for all $x \in [-\alpha, \alpha]$, $x \neq 0$ and all $n = 1, 2, \ldots$. Finally, since $R_n(x)$ is a polynomial with integral coefficients, it is evident that $Q_n(x)$ and $P_n(x)$ are polynomials with integral coefficients.

Remark. This result does not remain valid for all $f \in C[-\alpha, \alpha]$ with $f(0)$ an integer. One such example is the following. Let $f : [-\alpha, \alpha] \to \mathbf{R}$ be defined by $f(x) = |x|$ for all $x \in [-\alpha, \alpha]$. Evidentily $f \in C[-\alpha, \alpha]$ and $f(0) = 0$ but f does not satisfy Theorem 2.1.

Indeed, if Theorem 2.1 held for the function f so defined, it would follow that there exists a sequence of polynomials with integral coefficients $(P_n)_n$, such that

$P_n \to f$ uniformaly on $[-\alpha, \alpha]$, $P_n(0) = 0$, $n = 1, 2, \ldots$, and $|x| < P_{n+1}(x) < P_n(x)$ for all $x \in [-\alpha, \alpha]$, $x \neq 0$ and $n = 1, 2, \ldots$. For $x > 0$, we obtain $|x|/x < P_n(x)/x$, wherefrom

$$\lim_{x \searrow 0} \frac{|x|}{x} = 1 \leq \lim_{x \searrow 0} \frac{P_n(x)}{x} = P_n'(0).$$

Also, for $x < 0$ we have $|x|/x > P_n(x)/x$, wherefrom

$$\lim_{x \nearrow 0} \frac{|x|}{x} = -1 \geq \lim_{x \nearrow 0} \frac{P_n(x)}{x} = P_n'(0),$$

contradiciting the previous inequality $1 \leq P_n'(0)$.

Analogously to Corollary 1.3 of [12] we have

THEOREM 2.2. *For every $f \in C_0^\infty[-\alpha, \alpha]$, there exists a sequence of polynomials with integral coefficients $(T_n)_n$, uniformly convergent to f on $[-\alpha, \alpha]$ and satisfying $T_n(0) = f(0) = 0$, $n = 1, 2, \ldots$,*

$$f(x) < T_{n+1}(x) < T_n(x), \qquad T_{n+2}(x) - 2T_{n+1}(x) + T_n(x) > 0,$$

for all $x \in [-\alpha, \alpha]$, $x \neq 0$ and all $n = 1, 2, \ldots$.

Proof. Let $P_n(x)$ be the polynomial defined in the proof of Theorem 2.1 and let $T_n(x) = P_n(x) + K \cdot A \cdot x^{2n}$, where A is an integral satisfying $A > 4/(1 - \alpha^2)$ (and K is defined in the proof of Theorem 2.1). Evidently $T_n(x)$ is a polynomial with integral coefficients, $T_n(0) = 0 = f(0)$, $n = 1, 2, \ldots$, and taking into account that $P_n(x) \searrow f(x)$ for $x \in [-\alpha, \alpha]$, $x \neq 0$, we obtain $T_n(x) \searrow f(x)$, $x \in [-\alpha, \alpha]$, $x \neq 0$. Then,

$$\begin{aligned} &T_{n+2}(x) - 2T_{n+1}(x) + T_n(x) = P_{n+2}(x) - 2P_{n+1}(x) + P_n(x) + KAx^{2n}(1 - x^2)^2 \\ &> P_{n+2}(x) - 2P_{n+1}(x) + P_n(x) + 4Kx^{2n}(1 - x^2)/(1 - \alpha^2) \\ &\geq P_{n+2}(x) - 2P_{n+1}(x) + P_n(x) + 4Kx^{2n}(1 - x^2), \end{aligned}$$

for all $x \in [-\alpha, \alpha]$, and all $n = 1, 2, \ldots$, (since $(1 - x^2)/(1 - \alpha^2) \geq 1$, for all $x \in [-\alpha, \alpha]$). Consequently

$$T_{n+2}(x) - 2T_{n+1}(x) + T_n(x) > P_{n+2}(x) - 2P_{n+1}(x) + P_n(x) + 4Kx^{2n}(1 - x^2), \quad (2)$$

for all $x \in [-\alpha, \alpha]$, $x \neq 0$ and all $n = 1, 2, \ldots$.

Now, since $(P_n)_n$ is monotonically decreasing we have

$$\begin{aligned} &|P_{n+2}(x) - P_{n+1}(x) + P_n(x) - P_{n+1}(x)| \\ &\qquad \leq \max\{|P_{n+2}(x) - P_{n+1}(x)|, |P_n(x) - P_{n+1}(x)|\}, \end{aligned}$$

where

$$\begin{aligned} &|P_n(x) - P_{n+1}(x)| = |S_n(x) - S_{n+1}(x) + 2Kx^{2n}(1 - x^2)| \\ &\leq |S_n(x) - S_{n+1}(x)| + 2Kx^{2n}(1 - x^2) < 4Kx^{2n}(1 - x^2), \end{aligned}$$

for all $x \in [-\alpha, \alpha]$, $x \neq 0$ (taking into account the proof of Theorem 2.1). Hence

$$|P_{n+2}(x) - P_{n+1}(x)| < 4Kx^{2(n+1)}(1-x^2) < 4Kx^{2n}(1-x^2), \quad x \in [-\alpha, \alpha],\ x \neq 0$$

and consequently $|P_{n+2}(x) - 2P_{n+1}(x) + P_n(x)| < 4Kx^{2n}(1-x^2)$, for all $x \in [-\alpha, \alpha]$, $x \neq 0$ and $n = 1, 2, \ldots$.

In conclusion, by (2), we have $T_{n+2}(x) - 2T_{n+1}(x) + T_n(x) > 0$, for all $x \in [-\alpha, \alpha]$, $x \neq 0$ and all $n \in \mathbf{N}$. This completes the proof of Theorem 2.2.

Remark. It is known that if $\alpha - \beta > 4$ and $f \in C[\beta, \alpha]$ is not a polynomial with integral coefficients, then f cannot be uniformly approximated on $[\beta, \alpha]$ by polynomials with integral coefficients (see [7]). But if $0 < \beta < \alpha < 1$, then it can be proved that Theorem 2.1 is valid for any $f \in C[\beta, \alpha]$. Indeed, for $f \in C[\beta, \alpha]$, let us define $f_1 : [-\alpha, \alpha] \to \mathbf{R}$ by $f_1(x) = f(x)$, $x \in [\beta, \alpha]$ and $f_1(x) = f(\beta)x/\beta$, $x \in [-\alpha, \beta)$. It is easily seen that $f_1 \in C[-\alpha, \alpha]$ and $f_1(0) = 0$.

Taking into account Theorem 1.1, there exists a polynomial sequence $R_n(x)$, with integral coefficients such that $|f_1(x) - R_n(x)| < \beta^{2n}(1-\alpha^2)$, for all $x \in [-\alpha, \alpha]$ and all $n = 1, 2, \ldots$. Since $\beta^{2n} \leq x^{2n}$ and $1 - \alpha^2 \leq 1 - x^2$ for all $x \in [\beta, \alpha]$, we obtain $|f(x) - R_n(x)| < x^{2n}(1-x^2)$, $x \in [\beta, \alpha]$, $n = 1, 2, \ldots$, wherefrom

$$\begin{aligned} |R_n(x) - R_{n+1}(x)| &\leq |R_n(x) - f(x)| + |f(x) - R_{n+1}(x)| \\ &< x^{2n}(1-x^2) + x^{2(n+1)}(1-x^2) < 2x^{2n}(1-x^2), \end{aligned}$$

for all $x \in [\beta, \alpha]$ and all $n = 1, 2, \ldots$.

Denoting $Q_n(x) = R_n(x) - 2x^{2n}$ and $P_n(x) = R_n(x) + 2x^{2n}$, it is easily seen that the conclusion of Theorem 2.1 is satisfied.

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