PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 48 (62), 1990, 96–100

REPRESENTATIONS OF MONOTONIC FOURIER COEFFICIENTS IN TAUBERIAN L^1-CONVERGENCE CLASSES*

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Abstract. It is proved that for real even monotonic Fourier coefficients $\{\hat{f}(n)\}$ of functions in L^1, the Tauberian condition [1], $\overline{\lim}_n \sum_{|k|=n+1}^{[\lambda n]} |k|^{p-1}|\Delta \hat{f}(k)|^p$ finite, for $\lambda > 1$ and $p \in (1,2]$, is equivalent to the existence of a O-regularly varying sequence $\{R(n)\}$ such that $\hat{f}(n) = \sum_{k=n}^{\infty} \frac{(R(k)/R(k-1)-1)^{1/p}}{k^{1/q}}$, $1/p + 1/q = 1$.

The Tauberian approach to L^1 convergence problems of Fourier series and the L^p methods led to wide L^1-convergence classes. Let $f \in L^1(T)$, $T = \mathbf{R}/2\pi\mathbf{Z}$, and let $\hat{L}^1$ be the sequential dual of L^1, i.e. the sequence space of Fourier coefficients of functions in L^1. A subclass K of $\hat{L}^1$ is called L^1-convergence class if $\{\hat{f}(n)\} \in K$ implies that

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty, \tag{1}$$

is equivalent to

$$\hat{f}(n)\lg|n| = o(1), \qquad |n| \to \infty, \tag{2}$$

where $S_n(f) = S_n(f,t) = \sum_{|k|\le n} \hat{f}(k)e^{ikt}$, and where $\|\cdot\|$ denotes L^1-norm.

A very general L^1-convergence class is obtained by Č. V. Stanojević [**1**], and is defined as

$$\overline{\lim_n} \sum_{|k|=n+1}^{[\lambda n]} |k|^{p-1}|\Delta \hat{f}(k)|^p \quad \text{finite, for } \lambda > 1, \tag{3}$$

where $p \in (1,2]$. A special case of (3) is

$$\lim_{\lambda \to 1+0} \overline{\lim_n} \sum_{|k|=n+1}^{[\lambda n]} |k|^{p-1}|\Delta \hat{f}(k)|^p = 0, \tag{4}$$

AMS Subject Classification (1985): Primary 42 A 20, 42 A 32.

* Supported in parts by SMSU 1988 Summer Faculty Fellowship Award

found earlier in [**2**].

There is an important common example for conditions (3) and (4), i.e.

$$n\Delta\hat{f}(n) = O(1), \qquad n \to \infty. \tag{5}$$

A simple "limiting case" of (4) that follows from straight forward estimations, instead of L^p-techniques, is

$$\lim_{\lambda\to 1+0} \overline{\lim_{n}} \sum_{|k|=n+1}^{[\lambda n]} |\Delta\hat{f}(k)| \lg k = 0. \tag{6}$$

Notice that the analogous example to example (5) for condition (6) is of the form

$$n(\lg n)\Delta\hat{f}(n) = O(1), \qquad n \to \infty,$$

which indicates the crudeness of the condition (6).

The concept of sequential monotonicity in the complex plane has been introduced in [**3**, **4**, **5**] and studied in relation with conditions (3) and (4). In the real case the relation between monotonicity and the above Tauberian conditions has not been studied. In this paper I shall show that real monotonic even Fourier coefficients $\{\hat{f}(n)\}$, satisfying above Tauberian conditions, have a representation in terms of regularly varying sequences in the sense of Karamata [**6**]. For succinct formulations of my results I need the following definitions.

Definition 1. A sequence $\{R(n)\}$ of positive numbers is O-regularly varying if $\overline{\lim}_n R([\lambda n])/R(n)$ is finite, for $\lambda > 1$.

Definition 2. A sequence $\{R(n)\}$ of positive numbers is $*$-regularly varying if $\lim_{\lambda\to 1+0} \overline{\lim}_n R([\lambda n])/R(n) = 1$.

THEOREM 1. *Let $\{\hat{f}(n)\}$ be a real even monotonically decreasing sequence of Fourier coefficients of $f \in L^1(0,\pi)$. Then* (3) *holds if and only of there exists a nondecreasing O-regularly varying sequence $\{R(n)\}$ such that*

$$\hat{f}(n) = \sum_{k=n}^{\infty} \frac{\left(R(k)/R(k-1) - 1\right)^{1/p}}{k^{1/q}}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad n > n_0. \tag{7}$$

Proof. Assume that (7) is valid. Then

$$\begin{aligned}\Delta\hat{f}(n) &= \hat{f}(n) - \hat{f}(n+1) \\ &= \sum_{k=n}^{\infty} \frac{\left(R(k)/R(k-1) - 1\right)^{1/p}}{k^{1/q}} - \sum_{k=n+1}^{\infty} \frac{\left(R(k)/R(k-1) - 1\right)^{1/p}}{k^{1/q}} \\ &= \frac{\left(R(n)/R(n-1) - 1\right)^{1/p}}{n^{1/q}},\end{aligned}$$

or $n^{p-1}(\Delta \hat{f}(n))^p = R(n)/R(n-1) - 1$. Hence, for $\lambda > 1$,

$$\sum_{k=n+1}^{[\lambda n]} k^{p-1}(\Delta \hat{f}(k))^p = \sum_{k=n+1}^{[\lambda n]} \left(\frac{R(k)}{R(k-1)} - 1\right)$$
$$\le \prod_{k=n+1}^{[\lambda n]} \left(1 + \frac{R(k)}{R(k-1)} - 1\right) = \prod_{k=n+1}^{[\lambda n]} \frac{R(k)}{R(k-1)} = \frac{R([\lambda n])}{R(n)}.$$

Taking the limit superior of both sides of the last inequality we get

$$\overline{\lim_n} \sum_{k=n+1}^{[\lambda n]} k^{p-1}(\Delta \hat{f}(k))^p \le \overline{\lim_n} \frac{R([\lambda n])}{R(n)}.$$

Since the right hand side is finite for $\lambda > 1$, we have that (3) holds. Suppose that (3) holds. Define $R(n) = \prod_{k=1}^{n} \left[1 + k^{p-1}(\Delta \hat{f}(k))^p\right]$. Let $\lambda > 1$. Then from

$$\frac{R([\lambda n])}{R(n)} = \prod_{k=n+1}^{[\lambda n]} \left[1 + k^{p-1}(\Delta \hat{f}(k))^p\right] \le \exp\left(\sum_{k=n+1}^{[\lambda n]} k^{p-1}(\Delta \hat{f}(k))^p\right),$$

it follows that $\{R(n)\}$ is O-regularly varying, and that

$$\Delta \hat{f}(n) = \frac{\left(R(n)/R(n-1) - 1\right)^{1/p}}{n^{1/q}}.$$

Since $\{\hat{f}(n)\}$ is a null sequence it follows that the series

$$\sum_{k=1}^{\infty} \frac{\left(R(k)/R(k-1) - 1\right)^{1/p}}{k^{1/q}}$$

is convergent, and the representation (7) follows. This concludes the proof of Theorem 1.

THEOREM 2. *Let $\{\hat{f}(n)\}$ be a real even monotonically decreasing sequence of Fourier coefficients of $f \in L^1(0,\pi)$. Then (4) holds if and only if there exists a nondecreasing $*$-regularly varying sequence $\{R(n)\}$ such that*

$$\hat{f}(n) = \sum_{k=n}^{\infty} \frac{\left(R(k)/R(k-1) - 1\right)^{1/p}}{k^{1/q}}, \qquad \frac{1}{q} + \frac{1}{p} = 1, \quad n > n_0. \tag{10}$$

The proof of Theorem 2 as well as the proof of the next theorem follow the lines of the proof of Theorem 1.

THEOREM 3. *Let $\{\hat{f}(n)\}$ be a real even monotonically decreasing sequence of Fourier coefficients of $f \in L^1(0,\pi)$. Then (6) holds if and only if there exists a nondecreasing $*$-regularly varying sequence $\{R(n)\}$ such that*

$$\hat{f}(n) = \sum_{k=n}^{\infty} \frac{\left(R(k)/R(k-1) - 1\right)}{\lg k}, \qquad n > 1. \tag{11}$$

Notice that in all these theorems the building blocks of the representation of monotonic Fourier coefficients, satisfying Tauberian conditions, undergo certain restrictions due to the representation form of $\hat{f}(n)$ as the remainders of certain convergent series. For instance, $R(n) = n$ cannot serve as an appropriate sequence in Theorem 3, because the series

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)\lg k}$$

does not converge. However, $R(n) = \lg n$ is an admissible sequence in Theorem 3, for

$$\sum_{k=2}^{\infty} \frac{(\lg k/\lg(k-1) - 1)}{\lg k}$$

is a convergent series. Similar restrictions on the rate of growth of $\{R(n)\}$ occur in Theorem 1 and Theorem 2.

It is plain that Theorem 1 and Theorem 2 are valid for $p > 1$ and that all three theorems are valid for any monotonically decreasing null sequences.

The above results show that in spite of sophisticated condition (3) and (4), corresponding L^1-convergence classes do not include all monotonic Fourier coefficients. Moreover, Theorem 1 and Theorem 2 show that in the corresponding L^1-convergence classes we can have only those monotonic sequences which have the form (7) and (10). Concerning Theorem 3 it is quite evident that the L^1-convergence class defined by the condition (3) cannot contain all monotonic Fourier coefficients. It contains only those of the form (11).

From the above analysis it follows that the regularity conditions on the Fourier coefficients are much more delicate than the conditions obtained through robust L^p-methods (or the naive estimations such as in Theorem 3).

In conclusion, the work of Telyakovskii and Fomin [**7**], and the consequent substential generalization of it [**8**], demonstrate that regularity conditions on Fourier coefficients are independent of L^p-methods and point out a direction of research in the theory of L^1-convergence that is far from being foreclosed.

The author wishes to express profound gratitude to Professor D. Adamović for his advice and guidance.

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(Received 08 11 1989)