PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 47 (61), 1990, 137–140

A FIXED POINT THEOREM IN BANACH SPACE

Nadim A. Assad

Abstract. A fixed point theorem is proved for continuous mappings from a nonempty compact subset K, of a Banach space X, into X, and which satisfies contractive condition (2) and property (a) below.

The following result was established in [**2**]: Let X be a Banach space, K a nonempty closed subset of X. Let $T : K \to X$ satisfy the following contractive condition on K: There exists a constant h, $0 < h < 1$ such that, for each $x, y \in K$,

$$d(Tx,Ty) \le h \max\{d(x,y)/2, d(x,Tx), d(y,Ty), [d(x,Ty) + d(y,Tx)]/q\}, \tag{1}$$

where q is any real number satisfying $q \geq 1+2h$. Suppose that T has the additional property:

$$\text{for each } x \in \partial K, \text{ the boundary of } K, Tx \in K. \tag{a}$$

Then T has a unique fixed point.

In this paper, we show that if we require T to be continuous and K compact, then we may replace condition (1) on T by the following: For all $x, y \in K$, $x \neq y$,

$$d(Tx,Ty) < \max\{d(x,y)/2, d(x,Tx), d(y,Ty), [d(x,Ty) + d(y,Tx)]/q\}, \tag{2}$$

where $q \geq 3$, and still conclude that T has a unique fixed point. Actually, the condition (2) is obtained from (1) by putting $h = 1$, and by replacing the inequality by a strict inequality.

In the proof of the following theorem we shall use the fact that, if $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ such that $d(x,z) + d(z,y) = d(x,y)$.

THEOREM. *Let X be a Banach space, K a nonempty compact subset of X, $T : K \to X$ a continuous mapping satisfying* (2) *on K. If T has property* (a)*, then T has a unique fixed point in K.*

Proof. Let $x_0 \in K$. We shall construct two sequences $\{x_n\}$, $\{x_n^1\}$ as follows. Define $x_1^1 = Tx_0$. If $x_1^1 \in K$, set $x_1 = x_1^1$. If $x_1^1 \notin K$, choose $x_1 \in \partial K$ so that

---

AMS Subject Classification (1985): Primary 54 H 25, Secondary 47 H 10

$d(x_0, x_1) + d(x_1, x_1^1) = d(x_0, x_1^1)$. Let $x_2^1 = Tx_1$. If $x_2^1 \in K$, set $x_2 = x_2^1$. If not, choose $x_2 \in \partial K$ so that $d(x_1, x_2) + d(x_2, x_2^1) = d(x_1, x_2^1)$. Continuing in this manner, we obtain $\{x_n\}$, $\{x_n^1\}$ satisfying:

(i) $x_{n+1}^1 = Tx_n$,

(ii) $x_n = x_n^1$ if $x_n^1 \in K$, and

(iii) $x_n \in \partial K$ and $d(x_{n-1}, x_n) + d(x_n, x_n^1) = d(x_{n-1}, x_n^1)$, if $x_n^1 \notin K$.

Let $P = \{x_i \in \{x_n\} : x_i = x_i^1\}$ and $Q = \{x_i \in \{x_n\} : x_i \neq x_i^1\}$. Note that if $x_n \in Q$, then x_{n-1} and x_{n+1} belong to P by condition (a).

Putting $G_n = d(x_n, x_{n+1})$, we may assume that for $n = 0, 1, 2, \ldots$, $G_n > 0$; for otherwise, i.e. if $G_n = 0$ for some n, it follows that $x_n = x_{n+1}$. Now if $x_n \in \partial K$, then $x_{n+1}^1 \in K$ or $x_{n+1} = x_{n+1}^1 = Tx_n$, and thus $x_n = Tx_n$, or x_n is a fixed point of T. On the other hand, if $x_n \notin \partial K$, then $x_{n+1}^1 \in K$ and we conclude again that x_n is a fixed point of T, because in this case, if $x_{n+1}^1 \notin K$, we get that $x_{n+1} \in \partial K$ while $x_n \notin \partial K$ and thus we cannot have $x_n = x_{n+1}$.

By using the same argument presented in the proof of the theorem of Rhoades [**2**], with a slight modification that consists of applying condition (2) on T instead of (1), we reach an estimate for G_n, $n \geq 2$, in each of the following three cases:

Case I. $x_n, x_{n+1} \in P$: we have $G_n < G_{n-1}$.

Case II. $x_n \in P$, $x_{n+1} \in Q$: we have $G_n < G_{n-1}$.

Case III. $x_n \in Q$, $x_{n+1} \in P$: since $x_n \in Q$ and is a convex linear combination of x_{n-1} and x_n^1, it follows that

$$G_n \leq d(x_n^1, x_{n+1}), \qquad \text{or} \tag{3}$$

$$G_n \leq d(x_{n-1}, x_{n+1}). \tag{4}$$

If (3) occurs, we get:

$$G_n < d(x_{n-1}, x_n^1) < G_{n-2}. \tag{5}$$

On the other hand, if (4) occurs, we get that $G_n < G_{n-2}$. Therefore in all cases we have:

$$G_n < G_{n-1} \quad \text{or} \quad G_n < G_{n-2}. \tag{6}$$

Following the proof of Theorem 4.1 in [**1**], we may assume that $\{x_n\}$ has one of the following three properties:

(P_1) $\{x_n\}$ has a subsequence $\{x_{n(k)}\}$ such that for $k = 1, 2, 3, \ldots, x_{n(k)+1}$ and $x_{n(k)+2} \in P$.

Otherwise, eventually $\{x_n\}$ cannot have two consecutive points in P, i.e., we may assume that for $n = 1, 2, 3, \ldots, x_{2n} \in Q$. It follows by Case III that

$$\{G_{2n}\} \text{ is a decreasing sequence of real numbers,} \tag{7}$$

and in this case, we may assume that either $\{x_{2n}\}$ has a subsequence $\{x_{n(k)}\}$ satisfying the following property:

$$G_{n(k)} \leq d(x_{n(k)}^1, x_{n(k)+1}), \quad \text{and thus} \tag{8}$$

(P_2) $\{x_n\}$ has a subsequence $x_{n(k)} \subset Q$ satisfying (8), or

(P_3) there exists a positive integer N such that for every $n \geq N$, $x_{2n} \in Q$ and $d(x_{2n+2}, Tx_{2n+2}) \leq d(x_{2n+1}, Tx_{2n+2})$.

If $\{x_n\}$ has property (P_1), then assuming $x_{n(k)} \to z$ it is easy to see by (6) and cases I and II that $G_{n(k+1)} \leq d(x^1_{n(k)+1}, x^1_{n(k)+2}) < G_{n(k)}$; as $k \to \infty$ and by continuity of T, we obtain that $d(z, Tz) = d(Tz, T^2z)$. Similarly, if $\{x_n\}$ has property (P_2), by compactness of K, we assume that $x_{n(k)-2} \to z$, and by (5) we conclude that $G_{n(k)} \leq d(x^1_{n(k)-1}, x^1_{n(k)}) < G_{n(k)-2}$. Also here as $k \to \infty$, we apply (7) to get that $d(z, Tz) = d(Tz, T^2z)$. Finally, if $\{x_n\}$ has property (P_3), by compactness of $K, \{x_{2n}\}$ has a subsequence $\{x_{n(k)}\}$ such that $x_{n(k)} \to z$ and $x_{n(k)+2} \to u$. We claim that $u = z$. We first observe by (7) and by the continuity of T that we have:

$$\lim G_{n(k)} = d(z, Tz) = d(u, Tu) = \lim G_{n(k)+2}. \tag{9}$$

Moreover, $d(Tx_{n(k)}, x_{n(k)+2}) \leq d(Tx_{n(k)}, x^1_{n(k)+2}) \leq G_{n(k)}$ and, as $k \to \infty$, we get:

$$d(u, Tz) \leq d(z, Tz). \tag{10}$$

On the other hand, by (P_3) we have $G_{n(k)+2} \leq d(Tx_{n(k)}, Tx_{n(k)+2})$ and as $k \to \infty$, we obtain:

$$d(u, Tu) \leq d(Tz, Tu). \tag{11}$$

If $u \neq z$, then by (9), (10) and (11), we observe that

$$\begin{aligned} d(z, Tz) = d(u, Tu) &\leq d(Tz, Tu) \\ &< \max\{d(z, u)/2, d(z, Tz), d(u, Tu), [d(z, Tu) + d(u, Tz)]/q\} \\ &\leq \max\{d(z, u)/2, d(z, Tz), [d(z, Tu) + d(z, Tz)]/3\}. \end{aligned} \tag{12}$$

Noting that $d(z, u)/2 \leq [d(z, Tz) + d(Tz, u)]/2 \leq d(z, Tz)$ and that $[d(z, Tu) + d(u, Tz)]/3 \leq [d(z, Tz) + d(Tz, Tu) + d(u, Tz)]/3 \leq d(Tz, Tu)$, we see that (12) leads into a contradiction. Therefore $u = z$. Finally, note that:

$$G_{n(k)} - d(x_{n(k)}, x_{n(k)+2}) \leq G_{n(k)+1} \leq d(x^1_{n(k)+1}, x^1_{n(k)+2}) \leq G_{n(k)}. \tag{13}$$

Therefore $\lim d(x^1_{n(k)+1}, x^1_{n(k)+2}) = \lim G_{n(k)}$, i.e., $d(Tz, T^2z) = d(z, Tz)$. Now if $z \neq Tz$, then

$$\begin{aligned} &d(z, Tz) = d(Tz, T^2z) \\ &\quad < \max\{d(z, Tz)/2, d(z, Tz), d(Tz, T^2z), d(z, T^2z)/3\} = d(z, Tz) \end{aligned}$$

(because $d(z, T^2z)/3 \leq [d(z, Tz) + d(Tz, T^2z)]/3 = (2/3)d(z, Tz)$) which is inadmissible. Therefore z is a fixed point of T. If v is also a fixed point of T, then:

$$d(z, v) = d(Tz, Tv) < \max\{d(z, v)/2, [d(z, Tv) + d(v, Tz)]/3\},$$
$$\text{i.e.,} \qquad d(z, v) < (2/3)d(z, v),$$

contradiction. Thus the fixed point is unique and the proof is completed.

The theorem generalizes the following result.

COROLLARY 4.1 [**1**]. *Let X be a Banach space and K a nonempty compact subset of X. Let $T: K \to X$ be a continuous mapping such that $Tx \in K$ for every $x \in \partial K$. Suppose that for all distinct x, y in K, the inequality*

$$d(Tx, Ty) < \{d(x, Tx) + d(y, Ty)\}/2 \tag{14}$$

holds. Then T has a unique fixed point.

Acknowledgement. The author takes this opportunity to express his sincere thanks to Professor B. E. Rhoades for providing him with the reprints of his papers.

REFERENCES

[**1**] N. A. Assad, *On some nonself mappings in Banach spaces*, Math. Japonica **33** (1988), 501–515.

[**2**] B. E. Rhoades, *A fixed point theorem for some nonself mappings*, Math. Japonica **23** (1978), 457–459.

Department of Mathematics
Birzeit University
Birzeit, P.O. Box 14
Birzeit, West Bank
via Israel

(Received 17 11 1987)
(Revised 08 02 1990)