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COMPLETENESS THEOREM FOR A MONADIC LOGIC WITH BOTH FIRST-ORDER AND PROBABILITY QUANTIFIERS

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Abstract. We prove a completeness theorem for a logic with both probability and first-order quantifiers in the case when the basic language contains only unary relation symbols.

Let $\mathcal{A} \subseteq HC$ be an admissible set which contains infinite ordinals and let L be a nonempty $\mathcal{A}$-recursive language which contains only unary relation symbols; HC denotes, as usual, the set of hereditarily countable sets.

Definition 1. The set of formulas of $(L_{\omega P\exists})_{\mathcal{A}P}$ is the least set such that: (i) each atomic formula of first-order logic without equality symbol is a formula of $(L_{\omega P\exists})_{\mathcal{A}P}$; (ii) if φ is a formula, then $\neg\varphi$ is a formula; (iii) if $\Phi \in \mathcal{A}$ is a set of formulas, then $\wedge\Phi$ is a formula; (iv) if φ is a finite formula, then $(\exists v_n)\varphi$ is a formula; (v) if φ is a formula and $r \in \mathcal{A} \cap [0,1]$, then $(P\mathbf{x} \geq r)\varphi$ is a formula.

Abbreviations $(P\mathbf{x} \leq r)$, $(P\mathbf{x} = r)$ and $(\forall v_n)$ are introduced as usual.

Definition 2. A probability structure for L is a structure $(\mathfrak{A}, \mu)$ where $\mathfrak{A}$ is a first-order structure for L (with universe A), and μ is a σ-additive probability measure on A such that each relation of $\mathfrak{A}$ is σ-measurable.

We can define in the usual way satisfaction relation in a probability structure; here μ^n denotes the n-fold product of μ's.

Thus: $(\mathfrak{A}, \mu) \models (P\mathbf{x} \geq r)\varphi(\mathbf{x}, \mathbf{a})$ iff $\mu^n\{\mathbf{b} \in A^n \mid (\mathfrak{A}, \mu) \models \varphi(\mathbf{b}, \mathbf{a})\} \geq r$.

The axioms for $(L_{\omega P\exists})_{\mathcal{A}P}$ are the axioms A1–A6 and B1–B6 from [**K**] with the usual first-order axioms. The rules of inference are the rules R1–R3 from [**K**] with the usual first-order generalization added.

SOUNDNESS THEOREM. *If the set Φ of sentences of $(L_{\omega P\exists})_{\mathcal{A}P}$ has a model, then it is consistent.*

LEMMA 1. *Each $(L_{\omega P\exists})_{\mathcal{A}P}$ sentence is $(L_{\omega P\exists})_{\mathcal{A}P}$-equivalent to a σ-Boolean combination of finite sentences.*

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Proof. The proof can be obtained in the similar way as the proof of the Normal Form Theorem from [**H2**]. So we omit it.

The notion of a weak structure $(\mathfrak{A}, \mu_n)_{n\in\omega}$ can be introduced as in [**H1**].

LEMMA 2. *A sentence of $(L_{\omega P\exists})_{\mathcal{A}P}$ is consistent if and only if it has a weak model in which each theorem of $(L_{\omega P\exists})_{\mathcal{A}P}$ is true.*

Proof. Hoover's modification of Henkin's argument (see [**H1**]) would work.

LEMMA 3. *Let $(\mathfrak{A}, \mu_n)_{n\in\omega}$ be a weak structure, $\varphi(\mathbf{x}, \mathbf{y})$ a finite $(L_{\omega P\exists})_{HC\,P}$-formula and $\mathbf{b} \in A^m$. Then there is a quantifier free formula $\Phi(\mathbf{x})$ such that: $(\mathfrak{A}, \mu_n)_{n\in\omega} \models (\forall\mathbf{x})\big(\varphi(\mathbf{x}, \mathbf{b}) \iff \Phi(\mathbf{x})\big)$.*

Proof. We use induction on the complexity of φ. If φ is atomic the statement is trivial. The inductive step when φ is a propositional combination of formulas of smaller rank is also trivial. Suppose now that φ is of the form $(P\mathbf{z} \geq r)\psi(\mathbf{x}, \mathbf{z})$. By the inductive assumption we may assume that ψ is a finite quantifier free formula. Further, suppose that $\mathbf{x}$ is $(x_0, x_1, \ldots, x_n)$ and that all relational symbols which occur in ψ are $R_0, R_1, \ldots, R_k$. Now define: $\Gamma(v) = \big\{\bigwedge\{R_i^{f(i)}(v) \mid 0 \leq i \leq k\} \bigm| f \in 2^{k+1}\big\}$.

Let $\Sigma(\mathbf{x})$ be the set of all formulas of the form

$$\bigwedge\big\{\Phi_i(x_i) \mid 0 \leq i \leq n\big\}\, \Phi_i(x_i) \in \Gamma(x_i)$$

for which there exists $a_0, a_1, \ldots, a_n \in \mathcal{A}$ with: $(\mathfrak{A}, \mu_n)_{n\in\omega} \models \Phi_i(a_i)$ for $0 \leq i \leq n$, and $(\mathfrak{A}, \mu_n)_{n\in\omega} \models (P\mathbf{z} \geq r)\psi(\mathbf{a}, \mathbf{z})$. Finally let $\Phi(\mathbf{x})$ be the formula $\vee\Sigma(\mathbf{x})$. It is straightforward to check that the following holds:

$$(\mathfrak{A}, \mu_n)_{n\in\omega} \models (\forall\mathbf{x})\big(\varphi(\mathbf{x}, \mathbf{b}) \iff \Phi(\mathbf{x})\big).$$

The case when φ is of the form $(\exists\mathbf{z})\psi(\mathbf{x}, \mathbf{z})$ can be dealt with in the same way as the previous one, so the claim of the lemma is established.

COROLLARY 1. *Let $(\mathfrak{A}, \mu_n)_{n\in\omega}$ be a weak probability structure.*

(a) *If $B \subseteq A^n$ is definable by a finite formula, with parameters from A, then B is ${}^n\mu_1$ measurable; here by ${}^n\mu_1$ we denote the finitely additive n-product of μ_1's.*

(b) *If $B \subseteq A^n$ is definable by a formula, possible infinite with parameters from A, and μ_n is σ-additive then B is μ_1^n-measurable.*

Thus, the corollary allows us to identify $(\mathfrak{A}, \mu_1)$ with $(\mathfrak{A}, \mu_n)_{n\in\omega}$ when only finite formulas are considered.

COROLLARY 2. *Let $(\mathfrak{A}, \mu_n)_{n\in\omega}$ be a weak probability structure. Then for every finite $(L_{\omega P\exists})_{HC\,P}$-formula $\varphi(\mathbf{x}, \mathbf{y})$ with parameters from A, the set $\big\{{}^n\mu_1\{\mathbf{b} \in A^n \mid (\mathfrak{A}, \mu_1) \models \varphi(\mathbf{b}, \mathbf{a})\} \mid \mathbf{a} \in A^m\big\}$ is finite.*

COMPLETENESS THEOREM. *A sentence φ of $(L_{\omega P\exists})_{\mathcal{A}P}$ is consistent if and only if it has a probability model.*

Proof. The nontrivial part is to prove that $\nvdash \varphi$ implies $\nvDash \varphi$, so suppose $\nvdash \varphi$. By Lemma 2 there is a weak structure $(\mathfrak{A}, \mu_n)_{n\in\omega}$ which is a model for $\neg\varphi$ and every axiom. By Lemma 1 it is enough to find a probability structure $(\mathfrak{B}, \nu)$ which is a model for all finite $(L_{\omega P\exists})_{\mathcal{A}P}$ sentences which hold in $(\mathfrak{A}, \mu_n)_{n\in\omega}$. To do that we will use Rašković's method from [**R**]. Let $K = L \cup C$ $((K_{\omega P\exists})_{\mathcal{A}P})$ be the language (logic) introduced in Hoover's construction [**H1**], where C is a countable set of new constant symbols and $C \in \mathcal{A}$.

Now, we introduce a language M with three sorts of variables. Let $X, Y, Z, \dots$ be variables for sets, $x_0, x_1, \dots$ variables for urelements and $r, s, \dots$ variables for reals from $[0,1]$. We suppose that predicates of our language are $E_n(x_0, x_1, \dots, x_{n-1}, X)$ for $n \geq 1$ (with a canonical meaning $(x_0, x_1, \dots, x_{n-1}) \in X$) and $\mu(X, r)$ (with a meaning $\mu(X) = r$). For each finite $(K_{\omega P\exists})_{HC\,P}$-formula we have a constant symbol A_φ for a set, for each real number $r \in [0,1]$ a constant symbol $\mathbf{r}$, and a set D of new constant symbols of the cardinality of the continuum. Functional symbols are $+$ and $\cdot$ for reals.

Let T be the first order theory with the following list of axioms:

(1) $(\forall X) \bigwedge_{n<m} \neg(\exists \mathbf{x}, \mathbf{y})(E_m(\mathbf{x}, \mathbf{y}, X) \wedge E_n(\mathbf{x}, X))$, where $\{\mathbf{x}\} \cap \{\mathbf{y}\} = \varnothing$.

(2) Axioms of extensionality: $(\forall \mathbf{x})(E_n(\mathbf{x}, X) \iff E_n(\mathbf{x}, Y)) \iff X = Y$.

(3) Axioms of satisfaction:

(a) $(\forall \mathbf{x})\big(E_n(\mathbf{x}, A_\varphi) \iff \bigwedge_{\psi\in\Phi} E_n(\mathbf{x}, A_\varphi)\big)$ for φ is $\wedge\Phi$, Φ finite;

(b) $(\forall \mathbf{x})\big(E_n(\mathbf{x}, A_\varphi) \iff \neg E_n(\mathbf{x}, A_\psi)\big)$ for φ in $\neg\psi$.

(c) $(\forall \mathbf{x})(E_n(\mathbf{x}, A_\varphi) \iff (\exists \mathbf{y}) E_n(\mathbf{x}, \mathbf{y}, A_\psi))$ for φ is $(\exists y)\psi$;

(d) $(\forall \mathbf{x})\big(E_n(\mathbf{x}, A_\varphi) \iff (\exists_1 X)(\mu(X, r_1^\varphi) \vee \mu(X, r_2^\varphi) \vee \dots \vee \mu(X, r_n^\varphi) \wedge (\forall \mathbf{y})(E_{n+m}(\mathbf{x}, \mathbf{y}, A_\psi) \iff E_m(\mathbf{y}, X)))\big)$ for φ is $(P\mathbf{x} \geq r)\psi$ where $r_1^\varphi, r_2^\varphi, \dots, r_k^\varphi$ are all reals from the set

$$\big\{{}^n\mu_1\{\mathbf{b} \in A^n \mid (\mathfrak{A}, \mu_1) \vDash \psi(\mathbf{b}, \mathbf{a})\} \bigm| \mathbf{a} \in A^m\big\} \qquad (*)$$

(4) Axioms of additivity:

(a) $(\forall X)(\exists_1 r)\mu(X, r)$

(b) $(\forall X)(\forall Y)\big(\neg(\exists \mathbf{x})(E_n(\mathbf{x}, X) \wedge E_n(\mathbf{x}, Y)) \implies (\exists Z)((\exists \mathbf{x})E_n(\mathbf{x}, Z) \wedge (\forall \mathbf{x})((E_n(\mathbf{x}, Z) \iff (E_n(\mathbf{x}, X) \vee E_n(\mathbf{x}, Y)) \wedge \mu(Z, r+s)))\big)$ for $n \in \omega$.

(5) Axioms which are transformations of finite axioms of $(K_{\omega P\exists})_{HC\,P}$: $(\forall \mathbf{x})E_n(\mathbf{x}, A_\varphi)$ where φ is a finite axiom.

(6) Sets of axioms which ensures σ-additivity of extended measure:

$$\big\{E_n(\mathbf{d}, A_\varphi)\big\} \cup \big\{\neg E_n(\mathbf{d}, A_{\varphi_m}) \mid m \in \omega\big\}$$

where $\{\varphi_m \mid m \in \omega\}$ is a sequence of finite formulas, $\mathbf{d}$ is a tuple of different constant symbols from D and all such tuples for a different sequences of formulas are pairwise disjoint, $\big\{\{\mathbf{a} \in A^n \mid (\mathfrak{A}, \mu) \vDash \varphi_m(\mathbf{a})\} \bigm| m \in \omega\big\}$ is a monotone increasing sequence of subsets of A^n, $(\mathfrak{A}, \mu) \vDash (\forall \mathbf{x})(\varphi_m(\mathbf{x}) \implies \varphi(\mathbf{x}))$ and

$$\begin{aligned} &\mu\big(\{\mathbf{a} \in A^n \bigm| (\mathfrak{A}, \mu) \vDash \varphi(\mathbf{a})\}\big) \\ &\quad > \sup\big\{\mu\big(\{\mathbf{a} \in A^n \mid (\mathfrak{A}, \mu_1) \vDash \varphi_m(\mathbf{a})\}\big)\big\} \qquad m \in \omega \end{aligned} \qquad (**)$$

(7) Axioms of a field (for real numbers) with a diagram for $+$ and $\cdot$.

Let a standard structure for the first order logic for M be the structure $\mathfrak{M} = \big(M, B, F, E_n^{\mathfrak{M}}, \mu^{\mathfrak{M}}, +, \cdot, d^{\mathfrak{M}}, A_\varphi^{\mathfrak{M}}, r\big)_{n\geq 1,\, \varphi\in S,\, r\in F,\, d\in D}$ (for short, $\mathfrak{M} = (M, B, F, A_\varphi)_S$), where $B \subseteq \bigcup_{n\geq 1} \mathcal{P}(M^n)$, $F = F' \cap [0,1]$, $F' \subseteq R$ a field, $E_n^{\mathfrak{M}} \subseteq M^n \times B$, $\mu^{\mathfrak{M}} : B \to F$, $+, \cdot : F^2 \to F$, $d^{\mathfrak{M}} \in M$, $A_\varphi^{\mathfrak{M}} \in B$ and $S \subseteq \big\{\varphi \in (L_{\omega P\exists})_{HC\,P} \mid \varphi \text{ is finite}\big\}$.

We claim that T is consistent. To prove the claim it is enough, by compactness, to show that all finite subtheories of T are consistent.

First, note that a weak structure can be transformed to a standard structure by taking:

$$A_\varphi^{\mathfrak{M}} = \big\{\mathbf{a} \in M^n \mid (\mathfrak{A}, \mu) \vDash \varphi(\mathbf{a})\big\}, \quad B = \big\{A_\varphi \mid \varphi \in (K_{\omega P\exists})_{HC\,P} \text{ is finite}\big\},$$

and arbitrarily interpeting constants from D, we may get a model for a fixed finite subtheory of T.

Let T' be a finite subtheory of T and let φ, $\{\varphi_n \mid n \in \omega\}$ be as in the axiom 6. Pick some $m \in \omega$ such that $\neg E_n(\mathbf{d}, A_{\varphi_k}) \in T'$ for all $k \geq m$. By $(**)$ we may choose $\mathbf{d}^{\mathfrak{M}} \in \big\{\mathbf{a} \in M^n \mid (\mathfrak{A}, \mu_1) \vDash \varphi(\mathbf{a})\big\} \setminus \bigcup_{i<m} \big\{\mathbf{a} \in M^n \mid (\mathfrak{A}, \mu_1) \vDash \varphi_i(\mathbf{a})\big\}$. Thus we get a model for T'.

Since every finite subtheory $T' \subseteq T$ has a model, by compactness, we conclude that T has a model, say $\mathfrak{M}$. Now we can transform our model $\mathfrak{M}$ to a probability structure with a first order part $\mathfrak{B}$. For a relational symbol R of the language L we define relation $R^{\mathfrak{B}} = \{x \in M \mid E_1^{\mathfrak{M}}(x)\}$, and a finitely additive measure $\overline{\mu}$ on the ring $\{A_\varphi \mid \varphi \text{ is finite}\}$ with: $\overline{\mu}(A_\varphi) = r$ iff $\mu(A_\varphi, r)$ holds in $\mathfrak{M} = (M, \dots)$.

Note that axiom $3d$ ensures $\overline{\mu}$ to map $\{A_\varphi \mid \varphi \text{ is finite}\}$ into the reals. Axiom 6 allows us to apply Karatheodory's Theorem to the measure $(\{A_\varphi \mid \varphi \text{ is finite}\}, \overline{\mu})$. Thus $\overline{\mu}$ can be extended to a σ-additive measure ν on the σ-ring which extends $\{A_\varphi \mid \varphi \text{ is finite}\}$. Let ν be the σ-additive extension of $\overline{\mu}$. It is straightforward to check that $(\mathfrak{B}, \nu)$ is a probability structure which satisfies the same finite $(L_{\omega P\exists})_{AP}$ sentences as $(\mathfrak{A}, \mu_1)$ does. That finishes a proof of the theorem.

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