PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 32 (46), 1982, pp. 3—4

ON THE DIFFERENCE OF THE NUMBER OF ZEROS AND POLES OF THE MEMBERS OF CONVERGENT SEQUENCES OF MEROMORPHIC FUNCTIONS

Alexander Abian

Abstract. Under rather general assumptions, it is shown here that the difference of the number of zeros and the number of poles inside a simple closed curve, which with its interior lies in a region G, is the same for almost every member of a convergent sequence of functions meromorphic on G. This generalizes the corresponding theorem of Hurwitz concerning the number of zeros of the members of convergent sequences of analytic functions.

In what follows, the domain of every function is a subset of the complex z-plane and every function is complex-valued. Also, by *almost every* we mean *every, with the exception of at most a finite number.*

As usual, by a *region* we mean an open connected subset of the z-plane.

Theorem. *Let C be a simple closed curve lying with its interior in a region G. Let $(f_n)_{n\in\omega}$ be a sequence of functions f_n such that every f_n is meromorphic on G and (counting the multiplicities) has Z_n many zeros and P_n many poles inside C.*

Let P be the set of the poles (in G) of all the members of $(f_n)_{n\in\omega}$.

Let f be a function continuous on $G-P$ such that:

$$\lim_n f_n(z)=f(z) \qquad \text{for every} \qquad z\in(G-P) \tag{1}$$

If f vanishes at no point of C and if no point of C is an accumulation point of P then Z_n-P_n is the same for almost every $n\in\omega$.

Proof. Since no point of C is an accumulation point of P and since every f_n is meromorphic on G, there exists an open set A such that $C\subseteq A\subseteq (G-P)$ and such that f_n is analytic on A for almost every $n\in\omega$. Moreover, f is continuous on A since f is continuous on $G-P$ and $A\subseteq(G-P)$. But then, since C is a compact subset of A, from (1) in view of [2, p. 141] it follows that $(f_n)_{n\in\omega}$ converges uniformly on C to f and since f is continuous on C and vanishes at no point of C we see that

$$f_n(z)\neq 0 \quad \text{with} \quad z\in C \quad \text{for almost every} \quad n\in\omega. \tag{2}$$

AMS (MOS) subject classifications (1970): primary 30 A 68
Key words and phrases: zeros, poles, sequences of meromorphic functions.
This work is partly supported by Iowa State University SHRI.

From (2), in view of the fact (as mentioned above) that almost every f_n is analytic on A it follows that (meromorphic on G function) f has no zero and no pole located on C for almost every $n \in \omega$. Hence, by [2, p. 19] we have:

$$\frac{1}{2\pi i} \int_C \frac{f_n'(z)}{f_n(z)} dz = Z_n - P_n = \text{an integer, for almost every } n \in \omega \tag{3}$$

As mentioned above, almost every f_n iz analytic on A and since f is continuous on A, from (1) in view of [2, p. 147] it follows that f is analytic on A and in fact:

$$\lim_n f_n' = f' \quad \text{on} \quad A \tag{4}$$

which, again by [2, p. 141] implies that $(f_n')_{n\in\omega}$ (just as $(f_n)_{n\in\omega}$) converges uniformly on C. But then from (2) it follows that $(f_n'/f_n)_{n\in\omega}$ converges uniformly on C to f'/f. Thus, by [2, p. 71] we have

$$\lim_n \int_C \frac{f_n'(z)}{f_n(z)} dz = \int_C \frac{f'(z)}{f(z)} dz \tag{5}$$

which by (3) implies that the integer-valued sequence $(Z_n - P_n)_{n\in\omega}$ is convergent and therefore $Z_n - P_n$ is the same for almost every $n \in \omega$, as desired.

Remark 1. In our Theorem we assumed that the limit function f is continuous on $G-P$. Had we assumed that f is continuous on G then from [2, p. 152] it would follow that f is meromorphic on G and then (5) and (3) would imply that $Z_n - P_n = Z_f - P_f$ for almost every $n \in \omega$ where Z_f and P_f would respectively be the number of zeros and the number of poles of f inside C.

Remark 2. From Remark 1 it follows that our Theorem implies Hurwitz's theorem [1] which (employing our notations introduced above) states: *if f_n is analytic on G for every $n \in \omega$ and if $(f_n)_{n\in\omega}$ converges on G to a function f continuous on G and nonvanishing at any point of C then $Z_n = Z_f$ for almost every $n \in \omega$.*

The author thanks Prof. Elgin H. Johnston for helpful discussions.

REFERENCES

[1] A. Abian, *Hurwitz's theorem implies Rouché's theorem*, J. Math. Anal. Appl. **61** (1977), 113—115.

[2] J. B. Conway, *Functions of one complex variable*, Springer-Verlag, Berlin, New York, 1975.

Department of Mathematics
Iowa State University
Ames, Iowa 50011
U.S.A.

(Received 15 02 1980)