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DEFECT AND RADICALS OF Δ-ENDOMORPHISM NEAR-RINGS

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In this note we consider some properties of the defect of Δ-endomorphism near-rings. It is shown that, if for all $f \in \mathrm{End}_\Delta(G)$ the restriction of f to Δ is an endomorphism of the subgroup $(\Delta, +)$, then $\mathcal{D}^2=\{0\}$. Let G be a finite direct sum of minimal E_Δ-invariant subgroups $B_i\,(i\in I)$ and $\Delta \subseteq B_i$ for some $i \in I$. We show, if every $f\in \mathrm{End}_\Delta(G)$ is of the form $f=t+\delta$ $(t\in \mathrm{End}(G),\ \delta \in \mathcal{D})$, where the restriction of f to Δ is an endomorphism of the subgroup $(\Delta, +)$, then the radicals J_2, J_1, J_0 and nil radical η of the near-ring $E_\Delta(G)$ are equal to the defect $\mathcal{D}$. This generalizes a results of *M.* Johnson ([3], Prop. 3).

Let $M_0(G)$ be a set of all zero preserving mappings of the group $(G, +)$ into itself and let Δ be a fully invariant subgroup of G. The mapping $f\in M_0(G)$ with $(\Delta)f\subseteq\Delta$ is called Δ-endomorphism of the group $(G, +)$ if for all $x, y\in G$ there exists $d\in\Delta$ such that

$$(x+y)f=(x)f+(y)f+d.$$

If $\Delta=\{0\}$, then Δ-endomorphism f becomes an endomorphism of $(G, +)$.

Denote by $\mathrm{End}_\Delta(G)$ the multiplicative semigroup of all Δ-endomorphisms of $(G, +)$ for which every fully invariant subgroup of $(G, +)$ is invariant. We say that these subgroups are E_Δ-invariant. Let $E_\Delta(G)$ be the near-ring whose additive group is generated by $\mathrm{End}_\Delta(G)$. The normal subgroup $\mathcal{D}$ of the group $(E_\Delta(G),+)$ generated by the set

$$\{\delta:\delta=-(ht+ft)+(h+f)\,t,\quad h,f\in E_\Delta(G),\quad t\in \mathrm{End}_\Delta(G)\}$$

is a defect of distributivity of the near-ring $E_\Delta(G)$. It is shown in [2] that the defect of every near-ring R is an ideal of R. The defect $\mathcal{D}$ depends upon the choice of the normal subgroup Δ. It is clear that $\mathcal{D}\subseteq(G,\Delta)_0$, where $(G,\Delta)_0$ is the set of all zero preserving mappings $f: G\to\Delta$.

Let G be a group and $B\subseteq G$. We define the right annihilator of B by $A(B)=\{f\in E_\Delta(G):(b)f=0,\ \text{for all } b\in B\}$. If B is any E_Δ-invariant subgroup of $(G, +)$, then $A(B)$ is an ideal of $E_\Delta(G)$, that is, an annihilator ideal. The radical $J_2(E_\Delta(G))$ of $E_\Delta(G)$ is the intersection of all annihilating ideals of the mi nimal E_Δ-invariant subgroups. The nil radical $\eta(E_\Delta(G))$ of $E_\Delta(G)$ is the sum of all nil ideals of $E_\Delta(G)$. For definitions of the radicals J_1 and J_0 see [4].

Theorem 1. *Let Δ be a fully invariant subgroup of the group* $(G, +)$. *If for all* $t\in End_\Delta(G)$ *the restrction of t to Δ is an endomorphism of* $(\Delta +)$, *then* $\mathcal{D}^2=\{0\}$.

Proof. If $\delta\in\mathcal{D}$ and $x\in G$, then $(x)\delta=d_1\in\Delta$. For all δ, $\delta'\in\mathcal{D}$ and $x\in G$, where $\delta'=\sum_i(f_i+\theta_i-f_i)$ and $\theta_i=-(h_it_i+g_it_i)+(h_i+g_i)t_i$ $(h_i, g_i\in E_\Delta(G)$, $t_i\in \mathrm{End}_\Delta(G))$, we have

$$(x)\delta\delta'=(d_1)\delta'=(d_1)\sum_i(f_i+\theta_i-f_i).$$

But, $(d_1)\theta_i=0$, because the restriction of t_i to Δ is an endomorphism of $(\Delta,+)$. Thus, $(x)\delta\delta'=0$, i. e. $\mathcal{D}^2=\{0\}$.

Proposition 2. *Let Δ be a fully invariant subgroup of the group $(G,+)$. If for all $t\in End_\Delta(G)$ the restriction of t to Δ is an endomorphism of $(\Delta,+)$ then $E_\Delta(G)/\eta(E_\Delta(G))$ is a distributively generated near-ring.*

Proof. Using Theorem 1 it follows that the defect $\mathcal{D}$ is nilpotent. The nil radical of the near-ring R contains all the nilpotent ideals of R ([4], 5.66 Summary). Hence $\mathcal{D}\subseteq\eta(E_\Delta(G))$ and by Corollary of the Theorem 2.6 of [2] it follows that $E_\Delta(G)/\eta(E_\Delta(G))$ is a distributively generated near-ring.

Theorem 3. *Let G be a direct sum of a non-empty collection of minimal E_Δ-invariant subgroups B_i $(i\in B_i)$* and *$\Delta\subseteq B_i$ for some $i\in I$. If for all $t\in End_\Delta(G)$ the restriction of t to Δ is an endomorphism of the subgroup $(\Delta, +)$, then $\eta(E_\Delta(G))=\{0\}$,* if and only if *$\Delta=\{0\}$.*

Proof. If $\eta(E_\Delta(G))=\{0\}$, then by Theorem 1 it follows that $\mathcal{D}\subseteq\eta(E_\Delta(G))$ i.e. $\mathcal{D}=\{0\}$. Hence $\Delta=\{0\}$, for if $\Delta\neq\{0\}$, then it must be that $\mathcal{D}\neq\{0\}$. Namely, the mapping $f\in(G,\Delta)_0$ with

$$(x)f=\begin{cases}0, & \text{if } x\in G\setminus\Delta\\ x, & \text{if } x\in\Delta\end{cases}$$

is a nonzero Δ-endomorphism, but is not an endomorphism of the group $(G, +)$ Conversely, if $\Delta=\{0\}$, then $E_\Delta(G)=E(G)$. Hence, by Proposition 3 of [3], it follows that $\eta(E_\Delta(G))=\{0\}$ and the theorem is proved.

If Δ is a fully inveriant subgroup of $(G, +)$, then the mapping $f\in M_0(G)$ of the form $f=t+\delta$, $(t\in End(G),\ \delta\in\mathcal{D})$ is a Δ-endomorphism of $(G, +)$. Indeed, $(\Delta)f\subseteq\Delta$ because Δ is a fully invariant subgroup of $(G, +)$. Also, for all $x, y\in G$ there exists $d'\in\Delta$ such that

$$(x+y)f=(x+y)(t+\delta)=(x+y)t+(x+y)\delta=(x)t+(y)t+d',$$

where $(x+y)\delta=d'\in\Delta$. Since $\mathcal{D}\subseteq(G, \Delta)_0$, we have

$$(x+y)(t+\delta)=(x)(t+\delta)-(x)\delta+(y)(t+\delta)-(y)\delta+d'$$

$$(x+y)(t+\delta)=(x)(t+\delta)+(y)(t+\delta)+d,\quad (d\in\Delta),$$

i.e. the mapping $f=t+\delta$ is a Δ-endomorphism.

Theorem 4. *Let G be a finite direct sum of a non-empty collection of minimal E_Δ-invariant subgroups B_i $(i\in I)$* and *$\Delta\subseteq B_i$ for some $i\in I$. If every $f\in End_\Delta(G)$ is of the form $f=t+\delta$ $(t\in End(G),\ \delta\in\mathcal{D})$ and the restriction of f to Δ is an endomorphism of the subgroup $(\Delta, +)$, then the radicals J_2, J_1, J_0 and η are equal to the defect $\mathcal{D}$ of the near-ring $E_\Delta(G)$.*

P r o o f. We first prove that $J_2(E_\Delta(G)) = \mathcal{D}$. By Theorem 1 it follows that $\mathcal{D} \subseteq \eta(E_\Delta(G))$, i.e. $\mathcal{D} \subseteq J_2(E_\Delta(G))$. If $f \in J_2(E_\Delta(G))$, where $f = t + \delta$, $(t \in End(G)$, $\delta \in \mathcal{D})$, then $t \in J_2(E_\Delta(G))$, i.e. $t \in \bigcap_i A(B_i)$, because $\delta \in J_2(E_\Delta(G))$. For all $x \in G$, where $x = b_1 + \cdots + b_n$, $(b_i \in B_i,\ i = 1, \ldots, n)$, we have

$$(x)f = (b_1 + \cdots + b_n)(t + \delta)$$
$$(x)f = (b_1)t + \cdots + (b_n)t + (b_1 + \cdots + b_n)\delta$$
$$(x)f = (b_1 + \cdots + b_n)\delta$$

because $t \in \bigcap_i A(B_i)$. Thus, $(x)f = (x)\delta$, i.e. $J_2(E_\Delta(G)) \subseteq \mathcal{D}$. Hence, $J_2(E_\Delta(G)) = \mathcal{D}$. Since the defect $\mathcal{D}$, by Theorem 1, is nilpotent, it follows that the radical $J_2(E_\Delta(G))$ is nilpotent too. By using the fact that the above mentioned radicals contain all nilpotent ideals ([4], 5.66 Summary) we obtain

$$J_2(E_\Delta(G)) = J_1(E_\Delta(G)) = J_0(E_\Delta(G)) = \eta(E_\Delta(G)) = \mathcal{D}.$$

The previous theorem generalizes a result of *M*. Johnson ([3], Prop. 3). Namely, if $\Delta = \{0\}$, then all these radicals coincide with the zero ideal.

T h e o r e m 5. *Let* Δ *be a minimal fully invariant subgroup of the finite group* $(G, +)$. *If the interesection of all nilpotent* $E_\Delta(G)$*-subgroups contains the defect* $\mathcal{D}$, *then*

$$J_2(E_\Delta(G)) = J_1(E_\Delta(G)) = J_0(E_\Delta(G)) = \eta(E_\Delta(D)).$$

P r o o f. Let $\mathcal{N}$ be a nilpotent $E_\Delta(G)$-subgroup. By assumption $\mathcal{D} \subseteq \mathcal{N}$. Since the near-ring $E_\Delta(G)$ has the identity, the right ideal W generated by $\mathcal{N}$ has the elements of the form ([1], Prop. 1.2)

$$w = \sum_i (f_i \pm h_i n_i - f_i), \quad (f_i,\ h_i \in E_\Delta(G),\ n_i \in \mathcal{N}).$$

It was proved (see [1], Prop. 3.10) that W is a nil ideal. Thus, $\mathcal{N} \subseteq \eta(E_\Delta G))$. By using the Corollary of Theorem 2.6 of [2] it follows that $E_\Delta(G)/\eta(E_\Delta(G))$ is a distributively generated near-ring which contains no nonzero nilptotent $E_\Delta(G)$--subgroups. The remaining part of the proof is similar to the proof of Theorem 3.11 in [1].

The previous theorem generalizes a result of *M*. Johnson ([3], T16).

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