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# RIGID ARONSZAJN TREES[1)]

by

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Abstract. We shall construct $2\aleph_1$ mutually nonisomorphic totally rigid Aronszajn trees. So, in particular, we get a positive answer to a Jech problem.

1. *Introduction and definitions.* We work in *ZFC* set theory, and adopt the usual notations and conventions. Terminology on trees is as in [Ku 1] and [Je 2].

A *tree* is any (partially) ordered set $(T, \leqslant_T)$, such that for every $x \in T$ the set $(\cdot, x) := \{y \in T \mid y <_T x\}$ is well ordered. The *height* $\gamma(x)$ of a point $x \in T$ is the order type of the set $(\cdot, x)$. Let $\alpha$ be any ordinal, then the $\alpha^{th}$ *level* of the tree is defined by $R_\alpha T = \{x \in T \mid \gamma(x) = \alpha\}$.

Let $\gamma T = \min\{\alpha \mid R_\alpha T = \varnothing\}$; $\gamma T$ is the height of the tree $T$ (we shall often write $T$ instead of $(T, \leqslant_T)$). With $\|$ (more precisely $\|_T$) we shall indicate the relation of noncomparability in the tree $T$. A totally $\leqslant_T$-ordered subset of $T$ is called a *chain* of the tree $T$. A maximal chain is called *branch*; $\alpha$*-branch* is any branch whose order type equals $\alpha$. A set $A \subseteq T$ is called *antichain* of the tree $T$ provided $x \| y$, for every $x, y \in A$, $x \neq y$. A set $X \subseteq T$ is an *initial part* of $T$, if $(\cdot, x) \subseteq X$, for every $x \in X$. A *node* of the tree $T$ is every equivalence class wrt the relation $\sim$ defined by: $x \sim y$ iff $(\cdot, x) = (\cdot, y)$. For every $x \in T$ let $\{y \in T \mid y \geqslant_T x\}$ be denoted by $T^x$; it will be considered also as a tree wrt the order $\leqslant_T \cap (T^x)^2$.

Let $A \subseteq T$ be any subset, Let $\mathcal{F} = \{(T(x), \leqslant_x) \mid x \in A\}$ be any family of pairwise disjoint trees, so that $T \cap T(x) = \varnothing$, for every $x \in A$. We define the *colateral intercalation* (see "*intercalation latérale*" [Ku 1] p. 101), yielding a new tree $S = Lt_A(T, \mathcal{F})$ so that $S = T \cup \{T(x) \mid x \in A\}$ where $\leqslant_S$ is defined by:

(i) $\leqslant_S \upharpoonright T^2 = \leq_T$ and $\leqslant_S \upharpoonright T^2(x) = \leqslant_x$, for every $x \in A$;

(ii) $(\cdot, x]_T <_S T(x)$ and $(S - (\cdot, x]_T) \|_S T(x)$, for every $x \in A$.

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1) The main result of this paper wes presented 1978:01:06 and is contained in § 7 pp. 71—79 of the author's Master Thesis *Drveta* [Trees] Beograd 1978 pp. 106, and which was defended on the University of Belgrade in 1978. I wish to express my gratitude to Professor Đ. Kurepa for supervising this research.

Let us remark that $T$ is an initial part of $S$ and that $T$ is a subtree of the tree $S$.

*A normal* $\omega_1$*-tree* is defined as in [Je 2] p. 58. Let ${}^{\omega_1}\omega = \bigcup \{{}^{\alpha}\omega \mid \alpha < \omega_1\}$ be the set of all sequences $s$ whose domain is a countable ordinal and whose values are the natural numbers. Let $l(s) = \mathrm{dom}(s)$. Any normal $\omega_1$-tree can be represented by a set $T \subseteq {}^{\omega_1}\omega$ (ordered by the natural ordering of ${}^{\omega_1}\omega$), which satisfies the following conditions:

(1) for each $s \in T$ and each $n \in \omega$, $s^\frown n \in T$,

(2) for each $s \in T$ and each $\alpha < \omega_1$, there is $t \in T$ such that $l(t) = \alpha$ and $t \subseteq s$ or $t \supseteq s$,

(3) for each $\alpha < \omega_1$, $T \cap {}^{\alpha}\omega$ is at most countable.

An *isomorphism* of two trees is a one-to-one mapping of $T_1$ onto $T_2$ which preserves the partial ordering. An *automorphism* of $T$ is an isomorphism of $T$ onto itself. A tree $T$ is called *rigid* if the only automorphism of $T$ is the identity. $T$ is said to be *totally rigid* if for no $x \neq y \in T$, $T^x$ and $T^y$ are isomorphic. By $\sigma(T)$ we denote the cardinality of the set of all automorphisms of $T$ ([Je 2] p. 59). A *Suslin tree* is any normal $\omega_1$-tree with no uncountable antichain; an *Aronszajn tree* is any normal $\omega_1$-tree with no uncountable chain; a *Kurepa tree* is any normal $\omega_1$-tree having at least $\aleph_2$ $\omega_1$-branches.

2. *Known results and a Problem.* Jech [Je 2] has proved that if $T$ is a normal $\omega_1$-tree, then $\sigma(T)$ is either finite, or $2^{\aleph_0} \leqslant \sigma(T) \leqslant 2^{\aleph_1}$ and $\sigma(T)^{\aleph_0} = \sigma(T)$. He has also proved that if $T$ has no Suslin subtree then either $\sigma(T)$ is finite or $\sigma(T) = 2^{\aleph_0}$ or $\sigma(T) = 2^{\aleph_1}$ and that it is consistent that there is a Suslin tree $T$ with $\sigma(T)$ of arbitrary prescribed cardinality $k$ betwen $2^{\aleph_0}$ and $2^{\aleph_1}$ provided $k^{\aleph_0} = k$.

Aronszajn, Kurepa and Suslin trees are main kinds of normal $\omega_1$-trees. The following two theorems give a summary of the known results on these trees with respect to their existence, isomorphism and automorphism.

Theorem 2.1.

1. (Aronszajn) There exists an Aronszajn tree.
2. (Gaifman-Specker) There exist $2^{\aleph_1}$ non-isomorphic Aronszajn trees.
3. (Jech, Tennernbaum) It is consistent that a Suslin tree exists.
4. (Jech) It is consistent that there are $2^{\aleph_1}$ non-isomorphic Suslin trees.
5. (Lévy, Rowbottom, Stewart) It is consistent that a Kurepa tree exists.
6. (Jech) It is consistent that there are $2^{\aleph_1}$ non-isomorphic Kurepa trees.

A proof of 1 is in [Ku 1] p. 96,; a proof of 2 is in [Ga Sp]; proofs of 3 and 5 could be found, e.g., in [Je 1]; proofs of 4 and 6 are in [Je 2]. Statements 3, 4, 5 and 6 hold also in $L$ according to Jensen, Jech Solovay and Jech, respectively.

Theorem 2.2. It is consistent that:

1. (Jensen) There exists a rigid Suslin tree:
2. (Jensen) There exists a Suslin tree $T$, such that $\sigma(T)=2^{\aleph_0}$.
3. (Jensen) There exists a Suslin tree $T$, such that $\sigma(T)=2^{\aleph_1}$.
4. (Jech) There exists a rigid Kurepa tree.
5. (Jech) There exists a Kurepa tree $T$, such that $\sigma(T)=2^{\aleph_0}$.
6. (Jech) There exists a Kurepa tree $T$, such that $\sigma(T)=2^{\aleph_1}$.

For proofs of 1, 2, 3 see [DeJo]; for proofs of 4, 5, 6 see [Je 2]. Moreover, all these statements hold in $L$.

After the results in Theorem 2.2. Jech (see [Je 2, Problem] p. 70) raises the question as to whether in $ZFC$ there exists a rigid normal $\omega_1$-tree. Here we shall prove that the answer to this problem is affirmative. Namely, we shall construct $2^{\aleph_1}$ mutually non-isomorphic totally rigid Aronszajn trees. The main result was announced in [To].

3. *Rigid Aronszajn trees.* We recall some definitions from [Ga Sp]. Let s be a sequence and $X$ a set of ordinals, then $s\,|\,X=(s_{\alpha_0},\ldots,s_{\alpha_\beta},\ldots)\,\alpha_{\beta<l(s)}$, where $X=\{\alpha_0,\ldots,\alpha_\beta,\ldots\}$, $\alpha_0<\cdots<\alpha_\beta<\cdots<$ and $s=(s_\alpha)_{\alpha<l(s)}\cdot s\,|\,\overline{X}$ is defined by $s\,|\,(l(s)-X)$. If $s\,|\,X=s'$ and $s\,|\,\overline{X}=s''$ then we write $s=s'*_X s''$. If $S'$ and $S''$ are sets of sequences then we put $S'*_X S''=\{s'*_X s''\,|\,s'\in S'$ and $s''\in S''\}$.

The following properties folow easily (see [Ga Sp] p. 4.):

— If $s,\ t\in S'*_X S''$ then $s\subseteq t$ iff $s\,|\,X\subseteq t\,|\,X$ and $s\,|\,\overline{X}\subseteq t\,|\,\overline{X}$.

— If $S'$ and $S''$ are normal $\omega_1$-trees satisfying (1) — (3) from § 1 then $S'*_X S''$ is also such a tree.

— If $S'$ and $S''$ are normal (sequential) $\omega_1$-trees where $S'$ is Aronszajn and $X\subseteq\omega_1$ has a power $\aleph_1$, then $S'*_X S''$ is Aronszajn tree.

Let $S_0=\{s\in{}^{\omega_1\!\!\smile}\omega\,|\,\{\alpha\,|\,s_\alpha\neq 0\}$ is finite$\}$. It is clear that $S_0$ is a normal $\omega_1$-tree of sequences (i.e. that it satisfies (1) — (3) from § 1). The following Lemma (whose proof uses the *pressing down Lemma* (*PDL*): If $f$ is pressing down on a stationary set (in $\omega_1$), it is constant on a stationary set) was first proved in [Ku 2].

Lemma 3.1. Every uncountable initial part of the tree $S_0$ contains an $\omega_1$-branch.

Proof: Let $U\subseteq S_0$ be an initial part of $S_0$. For each $\alpha<\omega_1$, $\lim(\alpha)$, let $s_\alpha\in U$ be arbitrary. For each $\alpha<\omega_1$, $\lim(\alpha)$, define $f(\alpha)=$ the largest $\beta<\alpha$ such that $s_\alpha(\beta)\neq 0$, or else $f(\alpha)=0$ if no such $\beta$ exists. Then $f\colon\{\alpha<\omega_1\,|\,\lim(\alpha)\to\omega_1$ is regressive, so by *PDL* we can find stationary set $C\subseteq\{\alpha<\omega_1\,|\,\lim(\alpha)\}$ such that $f''(C)=\{\beta_0\}$ for some fixed $\beta_0$. It follows immediately (using again *PDL*) that there must be a stationary $D\subseteq C$ such that $s_\alpha\upharpoonright(\beta_0+1)=s_\beta\upharpoonright(\beta_0+1)$ for every $\alpha,\ \beta\in D$, i.e. $\alpha,\ \beta\in D$ and $\alpha<\beta$ implies $s_\alpha\subseteq s_\beta$. Hence $\{s_\alpha\,|\,\alpha\in D\}$ determines an $\omega_1$-branch of $U$.

Up to the end of this section $S \subseteq {}^{\omega_1}\omega$ will be a fixed Aronszajn tree of sequences (i.e it will satisfy (1) — (3) from § 1). Following [Ga Sp] for every subset $X \subseteq \omega_1$ we define $T(X) = S *_X S_0$. Let us mention also a well known fact that $\alpha = \omega^\beta$ has the property that $\gamma < \alpha$ implies $\gamma + \alpha = \alpha$.

Lemma 3.2. Let $X, Y \subseteq \{\alpha < \omega_1 \mid \alpha = \omega^\beta$ for some $\beta\}$ be disjoint and uncountable and let $x \in T(X)$ and $y \in T(Y)$. Then $T(X)^X$ and $T(Y)^Y$ have no isomorphic uncountable initial parts.

Proof: Let us assume the contrary, i.e. that there exist $x \in T(X)$ and $y \in T(Y)$, such that $T(X)^x$ and $T(Y)^y$ have isomorphic uncountable initial parts $A$ and $B$, respectively. Assume also that $|A \cap R_\delta T(X)| > 1$ or $|B \cap R_\delta T(Y)| > 1$ implies $\delta > l(x)$, $l(y)$.

Let $A_0 = \{s \mid \overline{X} \mid s \in A\}$; then $A_0 \cup \{s \in S_0 \mid s \subseteq x \mid \overline{X}\}$ is an uncountable initial portion of $S_0$ According to Lemma 3.1. there exists an $\omega_1$-branch $a \subseteq A_0 \cup \cup \{s \in S_0 \mid s \subseteq x \mid \overline{X}\}$. Let $A_1 = \{s \in A \mid s \mid \overline{X} \in a\}$.

We can easily check that $A_1$ is an uncountable initial part of $T(X)^x$ and that every node $N$ of $(A_1, \subseteq)$ is of power 1 whenever the height of $N$ (in $T(X)$) is $\delta + 1$ and $\delta \notin X$.

Since $A_1$ is an uncountable initial portion of $T(X)^x$ and $A$ and $B$ are isomorphic, there exists an uncountable initial part $B_1$ of the tree $T(Y)^y$, so that $A_1$ and $B_1$ are isomorphic. Let $B_2 = \{s \mid \overline{Y} \mid s \in B_1\}$. Then $B_2 \cup \{s \in S_0 \mid s \subseteq y \mid \overline{Y}\}$ is an uncountable initial part of $S_0$. According to Lemma 3.1. there exists an $\omega_1$-branch $b \subseteq B_2 \cup \{s \in S_0 \mid s \subseteq y \mid \overline{Y}\}$. Let $B_3 = \{s \in B_1 \mid s \mid \overline{Y} \in b\}$, then $B_3$ is an uncountable initial part of $T(X)^y$. Let us prove that $B_3$ is totally ordered, contradicting the fact that $T(Y)^y$ is an Aronszajn tree.

Let us assume the contrary, i.e. that there exist $s, t \in B_3$ so that neither $s \subseteq t$ nor $t \subseteq s$. Let $\beta < l(s)$, $l(t)$ be the least ordinal with property $s_\beta \neq t_\beta$ and let $u = s \restriction (\beta + 1)$ and $v = t \restriction (\beta + 1)$. Let $u'$ and $v'$ be elements of $A_1$, such that $h(u') = u$ and $h(v') = v$, where $h: A_1 \to B_1$ is an isomorphism. This means that there is an ordinal $\delta < \omega_1$, so that $u', v' \in A_1 \cap R_{\delta+1} T(X)$, $u' \restriction \delta = v' \restriction \delta$ and $u' \neq v'$. According to the property of the set $A_1$, $\delta \in X$ must hold. According to the above assumption $\delta > l(x)$, $l(y)$, which, according to $\delta \in X \subseteq \{\alpha < \omega_1 \mid \alpha = \omega^\beta$ for some $\beta\}$, means that the height of $u'$ and $v'$ in the tree $T(X)^x$ equals $\delta + 1$. So, the height of $u$ and $v$ in the tree $T(Y)^y$ equals $\delta + 1$, which, according to $\delta = \omega^\alpha$, for some $\alpha$ and $\delta > 1(y)$, means that $1(u) = 1(v) = \delta + 1$. So, we have two points $u \neq v \in B_3$ with properties $l(u) = l(v) = \delta + 1$, $u \restriction \delta = v \restriction \delta$ and $\delta \notin Y$ (for $\delta \in X$ and $X \cap Y = \varnothing$).

Since $\delta \notin Y$, $u \mid Y = v \mid Y$, and since $u \mid \overline{Y}$ and $v \mid \overline{Y}$ are elements of $b$ of the same height it follows that $u \mid \overline{Y} = v \mid \overline{Y}$. This means that $u = v$, contrary to the fact that $u \neq v$. So, $B_3$ has no incomparable elements, what was to be proved.

Theorem 3.3. *There exist $2^{\aleph_1}$ mutually non-isomorphic totally rigid Aronszajn trees.*

Proof: Let $\{X_\delta \mid \delta < \omega_1\}$ be a disjoint collection of uncountable subsets of $\{\alpha < \omega_1 \mid \alpha = \omega^\beta$ for some $\beta\}$. Let $\mathcal{T} = \{T(X_\delta) \mid \delta < \omega_1\}$ be the corresponding family of Aronszajn trees. Considering trees isomorphic to them we assume that $T(X_\delta) \cap T(X_{\delta'}) = \varnothing$, for $\delta < \delta' < \omega_1$.

We shall construct inductively a sequence $T_\alpha$, $\alpha<\omega_1$ of Aronszajn trees, so that $T_\alpha$ is an initial part of $T_\beta$, for every $\alpha<\beta<\omega_1$.

Let $T_0=T(X_0)$. To every $x\in R_0T_0$ (however, there is only one) we correspond a tree $T(x)\in\mathcal{T}-\{T_0\}$, so that this correspodence is one-to-one and minimal. Let $\mathcal{T}_1=\{T(x)\mid x\in R_0\,T_0\}$. Let

$$T_1=Lt_{R_0T_0}(T_0,\ \mathcal{T}_1),$$

be a tree obtained by the colateral intercalation of $T(x)$ in place of $x\in T_0$, for every $x\in R_0T_0$ (see § 1). It is clear that $T_1$ is an Aronszajn tree and that $T_0$ is its initial part.

Let us assume that $\alpha<\omega_1$ and that $T_\beta$, $\beta<\alpha$ have been constructed already. If $\alpha$ is a limit ordinal, then we put

$$T_\alpha=\bigcup\{T_\beta\mid\beta<\alpha\}\,(\leqslant_{T_\alpha}=\bigcup\{\leqslant_{T_\beta}\mid\beta<\alpha\}).$$

Then $T_\alpha$ is Aronszajn tree and $T_\beta$ is an initial part of $T_\alpha$, for every $\beta<\alpha$.

Let $\alpha=\beta+1$. For every $x\in R_\beta T_\beta$ let us choose some $T(x)\in\mathcal{T}$, which we haven't used before, so that this correspondence would be one-to-one and minimal. Let

$$T_\alpha=Lt_{R_\beta T_\beta}(T_\beta,\ \mathcal{T}_\alpha),$$

where $\mathcal{T}_\alpha=\{T(x)\mid x\in R_\beta T_\beta$. We omit the simple proof that $T_\alpha$ is an Aronszajn tree and that $T_\beta$ (and $T_\delta$, $\delta<\beta$) is an initial part of $T_\alpha$. Let

$$T\omega_1=\bigcup\{T_\alpha\mid\alpha<\omega_1\}\,(\leqslant\,=\,\leqslant_{T\omega_1}=\bigcup\{\leqslant T_\alpha\mid\alpha<\omega_1\}).$$

Then $T_\alpha$ is an initial part of $T_{\omega_1}$, for every $\alpha<\omega_1$ according to the same fact for $T_\beta$ and $T_{\beta'}$, $\beta<\beta'<\omega_1$.

By construction it is easy to show that $R_\beta T_\beta=R_\beta T_\alpha$, for every $\beta<\alpha<\omega_1$, whence $R_\alpha T_{\omega_1}=R_\alpha T_\alpha$, for every $\alpha<\omega_1$, which means that $T_{\omega_1}$ is a normal $\omega_1$-tree (see § 1). Let us prove that $T_{\omega_1}$ is an Aronszajn tree. Let us assume the contrary, i.e. that $T_{\omega_1}$ has an $\omega_1$-branch $b$. Let $x_\alpha\in b$ be the unique element of $b\cap R_\alpha T_{\omega_1}$, for every $\alpha<\omega_1$. If $\alpha<\omega_1$ is a limit ordinal then $x_\alpha\in R_\alpha T_{\omega_1}=R_\alpha T_\alpha$ and there exists $\beta<\alpha$, so that $x_\alpha\in T_\beta$. Let $h(\alpha)=\min\{\beta\mid x_\alpha\in T_\beta\}$. So, $h:\{\alpha<\omega_1\mid\lim(\alpha)\}\to\omega_1$ is regressive. So, (using *PDL*) there exists stationary $C\subseteq\{\alpha<\omega_1\mid\lim(\alpha)\}$ and $\beta<\omega_1$, so that $h''(C)=\{\beta\}$. This means that $x_\alpha\in T_\beta$, for every $\alpha\in C$. So, $\{x_\alpha\mid\alpha\in C\}$ is an uncountable chain of $T_\beta$, which is absurd since $T_\beta$ is an Aronszajn tree. So, $T_{\omega_1}$ is an Aronszajn tree.

Let us prove now that $T_{\omega_1}$ is totally rigid. Let us assume the contrary, i.e. that there exist $x\neq y\in T_{\omega_1}$, so that $T^x_{\omega_1}$ and $T^y_{\omega_1}$ are isomorphic. We may assume $x\parallel y$. Let $\pi:T^x_{\omega_1}\to T^y_{\omega_1}$ be an isomorphism.

Let $\beta=\gamma(x)$, then the tree $T(x)\in\mathcal{T}_{\beta+1}$ is colaterally intercalated in the place of $x\in R_\beta T_\beta$. This means that $T(x)\subseteq T^x_{\beta+1}\subseteq T^x_{\omega_1}$ and that $U=\{x\}\cup T(x)$ is an initial part of the tree $T^x_{\omega_1}$. Hence $\pi''(U)$ is an initial part of the tree $T^y_{\omega_1}$. For every limit ordinal $\alpha<\omega_1$, $\alpha>\gamma(y)$ we choose $y_\alpha\in\pi''(U)\cap R_\alpha T_{\omega_1}$. Since $R_\alpha T_{\omega_1}=R_\alpha T_\alpha$, we know that $y_\alpha\in R_\alpha T_\alpha$. According to the definition of $T_\alpha$ for limit $\alpha<\omega_1$ we know that there exists a least $\beta<\alpha$, such that $y_\alpha \in T_\beta$. Moreover, in case $\beta>0$, $\beta=\delta+1$ for some $\delta$. If this last case holds, then from

$$T_\beta = Lt_{R_\delta T_\delta}(T_\delta, \mathcal{T}_\beta)$$

we know that there exists a unique $z_\alpha \in R_\delta T_\delta$, so that $y_\alpha \in T(z_\alpha)$. Let $\beta$ be denoted by $i(\alpha)$. By this we have a regressive mapping $i:\{\alpha<\omega_1 \mid \lim(\alpha) \wedge \wedge \alpha>\gamma(y)\} \to \omega_1$. So, there exists a stationary $D \subseteq \{\alpha<\omega_1 \mid \lim(\alpha) \wedge \alpha>\gamma(y)\}$ and $\beta_1<\omega_1$, so that $i''(D)=\{\beta_1\}$. Moreover, we assume that in case $\beta_1>0$, $z_\alpha = z \in R_{\delta_1} T_{\delta_1}$, for every $\alpha \in D$, where $\beta_1=\delta_1+1$. According to the definition of the mapping $i$ we have that $\{y_\alpha \mid \alpha \in D\} \subseteq T_{\beta_1}$, where in case $\beta_1>0$ we have $\{y_\alpha \mid \alpha \in D\} \subseteq T(z)$.

Let us consider the case $\beta_1>0$. We can assume that there exists a $v \in T(z)$, so that $y<v\leqslant y_\alpha$, for every $\alpha \in D$.

Let $V=\{v' \mid y\leqslant v'\leqslant v\} \cup T(z)^v$ and let $B=V \cap \pi''(U)$, then $B$ is an uncountable initial part of $T^y_{\omega_1}$, for it contains $\{y \mid \alpha \in D\}$. Let $u \in T(x)$ be such that $\pi(u)=v$ and let $A=T(x)^u \cap \pi^{-1}(B)$, then $A$ and $B-\{v' \mid y\leqslant v'<v\}$ are isomorphic uncountable initial parts of the trees $T(x)^u$ and $T(z)^v$. However, this contradiicts Lemma 3.2., for $x\neq z$ (since $x \| y$ and $z\leqslant y$ or $y\leqslant z$) implies that $T(x)=T(X_\delta)$ and $T(z)=T(X_{\delta'})$ for $\delta\neq\delta'$. In case $\beta_1=0$ we get a contradiction similarly. This finishes the proof that $T_{\omega_1}$ is a totally rigid Aronszajn tree.

The tree $T_{\omega_1}$ could be constructed in the above manner without using all the elements of $\mathcal{T}$, but only the elements of some uncoutable subfamily $\{T(X_\delta) \mid \delta \in F\}$, $F \subseteq \omega_1$, $|F|=\aleph_1$. Let us denote by $T^F_{\omega_1}$ the tree so obtained. Let $F$ and $G$ be arbitrary uncoutable and different subsetts of $\omega_1$. Let us prove that the trees $T^F_{\omega_1}$, $T^G_{\omega_1}$ are nonisomorphic. Let, for example $F-G\neq\varnothing$ and let $\delta \in F-G$. The tree. $T(X_\delta)$ has been used in the construction of the tree $T^F_{\omega_1}$, which means that there exist $\beta<\omega_1$ and $x \in R_\beta T^F_\beta$, so that $T(x)= =T(X_\delta) \subseteq T^F_{\beta+1} \subseteq T^F_{\omega_1}$. If we assume the contrary, i.e. that there exists an isomorphism $\pi: T^F_{\omega_1} \to T^G_{\omega_1}$, then, repeating the arguments from above, we can find $T(z) \subseteq T^G_{\omega_1}$, $z' \in T(z)$ and $x' \in T(x)$, so that $T(x)^{x'}$ and $T(z)^{z'}$ have isomorphic uncountable initial parts which is in contradiction with Lemma 3.2,, since $T(z)=T(X_{\delta'})$ for some $\delta'\neq\delta$ ($\delta' \in G \not\ni \delta$). This proves that

$$\{T^F_{\omega_1} \mid F \subseteq \omega_1 \text{ and } |F|=\aleph_1\}$$

is a family of power $2^{\aleph_1}$ of mutually nonisomorphic totally rigid Aronszajn trees, which finishes the proof of Theorem 3.3.

Remark. Obviously, the Theorem 3.3. is transferrable for the case of other regular cardinals $k$ for which there exist $k$-Aronszajn trees.

To the above Jech's problem we could give an answer in an other direction also. Let us indicate some definitions. If $T$ is an $\omega_1$-tree then for $C \subseteq \omega_1$, $T \upharpoonright C = \bigcup\{R_\alpha T \mid \alpha \in C\}$ is a tree under the restriction of the partial ordering of $T$. A function $f: T \to T'$ is an *embedding* of the tree $T$ into the tree $T'$ iff $f$ is one-to-one and order preserving $x<y \Leftrightarrow f(x)<' f(y)$. Using some extensions of arguments of the proof of Theorem 3.3 we can prove the following.

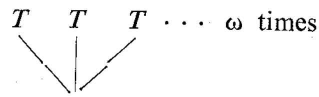
Theorem: *There exists a family* $T_\alpha$, $\alpha < 2^{\aleph_1}$ *of* $\omega_1$*-trees such that: If* $C$ *is a closed unbounded subset of* $\omega_1$, $f$ *an embedding of* $T_\alpha \upharpoonright C$ *into* $T_\beta \upharpoonright C$ *then* $\alpha = \beta$ *and* $f$ *is trivial.*

The following theorem is a final supplement to Theorem 2.2.

Theorem 3.4.

1. *There exists a rigid Aranszajn tree.*
2. *There exists an Aronszajn tree* $T$, *such that* $\sigma(T) = 2^{\aleph_0}$.
3. *There exists an Aronszajn tree* $T$, *such that* $\sigma(T) = 2^{\aleph_1}$.

Proof: Statement 1 follows from the Theorem 3.3. For the proof of 3 see [Je 2] p. 70. Consider the following tree $T'$:




where $T$ is some rigid Aronszajn tree. Obviously, $\sigma(T') = 2^{\aleph_0}$. This proves 2.

## REFERENCES


[De Jo] K. J. Devlin and H. Johnsbraten, *The Souslin Problem*, Springer Lecture Notes 405 (1974).

[Ga Sp] H. Gaifman and E. Specker, *Isomorphism Types of Trees*, Proc. Amer Math. Soc. 15 (1964), 1—7.

[Je 1] T. J. Jech, *Trees*, Journal Symb. Logic 36 (1971), 1—14.

[Je 2] T. J. Jech, *Automorphisms of* $\omega_1$*-trees*, Trans. Amer. Math. Soc. 173 (1972), 57—70.

[Ku 1] Đ. Kurepa, *Ensembles ordonnés et ramifiés*, Publ. Math. Univ. Belgrade 4 (1935) 1—138.

[Ku 2] Đ. Kurepa, *On regressing functions*, Zeitschrift für Math. Logik und Grundlagen der Math. 4 (1958), 148—157.

[To] S. Todorčević, *Some results in set theory. I*, Notices of the Amer. Math. Soc. 26: 4 (1979), A-390.



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