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COMMON TRANSVERSALS OF FINITE FAMILIES

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Let $\mathcal{F}$ denote a family ot t families of sets, each of the t families consisting of n finite, nonempty, but not necessarily distinct, sets. The simbol X will denote the union of all of sets contained in all of the t families. Thus $\mathcal{F}=(F_1, F_2, \ldots, F_t)$, where for each j, $1 \leqslant j \leqslant t$, $F_j=(F_j(1), F_j(2), \ldots, F_j(n))$ is a sequence of n finite nonempty, but not necessarily distinct, sets, and $X=\cup\{F_j(i): 1 \leqslant j \leqslant t,\ 1 \leqslant i \leqslant n\}$.

Recall that the set T is a *transversal* (called also *system of distinct representatives* or SDR) of the family F_j if there is a bijection $\varphi: T \rightarrow \{1, 2, \ldots, n\}$ such that $x \in F_j(\varphi(x))$ for all $x \in T$. The set T is a *common transvesal* of $F_1, F_2, \ldots, F_t$ if T is simultaneously a transversal of each F_j, $1 \leqslant j \leqslant t$.

There are many results concerning common transversals of two families of sets (see [1]), but very little is known on the existence of common transversal of more than two families. The only exeption is a sufficent condition of the existence of common transversals of a family $\mathcal{F}$ wchich *separates points* of X (family $\mathcal{F}$ separates points of X if, letting $F_j(0)=X \setminus \cup F_j$, $1 \leqslant j \leqslant t$, $|\cap\{F_j(a_j): 1 \leqslant j \leqslant t\}| \leqslant 1$, for every t-tuple $(a_1, a_2, \ldots, a_t)$, where $0 \leqslant a_j \leqslant n$, $1 \leqslant j \leqslant t$) see [2] and [3]).

In this note we give a necessary and sufficient condition for the existence of common transversals in terms of incidence matrices of families of sets.

If $\mathcal{F}$ is a finite family of finite sets and $X=\cup \mathcal{F}$, then, evidently, there is no transversal if $|X|<n$. So the condition $|X| \geqslant n$ we shall call the *minimal necessary condition.*

Also, for a sequence of families of sets, $F_j(i)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant t$ the minimal necessary condition for a transversal (resp. a common transversal) is $|X|=m \geqslant n$, where $X=\bigcup\limits_j \bigcup\limits_i F_j(i)$.

It the sequel we assume that the collection of families $F_j(i)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant t$, satisfies the minimal necessary condition.

Let M be a matrix and let the entries of M be 0 and 1. We call such a matrix a $\{0, 1\}$ — matrix. To every family $\mathcal{F}$ of finite sets $F_1, F_2, \ldots, F_n$, each subset of a set $X=\{x_1, x_2, \ldots, x_m\}$ is associated a $\{0, 1\}$ — matrix $M=[a_{ij}]$, such that $a_{ij}=1$, if $x_i \in F_j$ and $a_{ij}=0$, if $x_i \notin F_j$. The matrix M, so defined, is called the *incidence matrix* of the family $\mathcal{F}$.

A *line* of a matrix designates a row or a column of the matrix. A *place* is the position of an element in a matrix (fixed by the specification of its row and column) Thus, for example, we can speak of „the places occupied by non-zero elements".

A set S of elements (or, more precisely, of places) is said to be *incident* with a set L of lines if each line in L contains at least one element of S.

A set S of elements is said to be *independent* (or *scattered*) if no two elements of S lie on a same line.

Since we consider only those finite families of subsets of a finite set $X=\{x_1, x_2, \ldots, x_m\}$ which satisfy the minimal necessary condition, all considered $\{0, 1\}$ — matrices, of size m by n, satisfy the condition $m \geqslant n$.

An operation on matrices. Let $\mathcal{M}$ be a set of m by n $\{0, 1\}$ — matrices, and t a natural number. For $M_1, M_2, \ldots, M_t \in \mathcal{M}$, $M_k=[a_{ij}^{(k)}]$, for $k \in \{1, 2, \ldots, t\}$, we put

$$M_1 * M_2 * \cdots * M_t = M$$

where $M=[c_{ij}]$, is a m by n matrix defined by:

$$c_{ij}=a_{ij}^{(1)} \cdot a_{ij}^{(2)} \cdot \cdots \cdot a_{ij}^{(t)}.$$

Similarly, if $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t$ are nonempty sets of $\{0, 1\}$ — m by n matrices, then $\mathcal{M}_1 * \mathcal{M}_2 * \cdots * \mathcal{M}_t$ (or, more briefly $* \mathcal{M}_t$) whill denote the set of m by n matrices M, defined by $M=M_1 * M_2 * \cdots * M_t$, with $M_1 \in \mathcal{M}_1$, $M_2 \in \mathcal{M}_1, \ldots, M_t \in \mathcal{M}_t$.

If M is a m by n, matrix, then $P(M)$ will denote the set of all matrices obtained from M by permuting columns.

A finite family $(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t)$ of m by n $\{0, 1\}$ — matrices is said to be *coveringly complete* if the set $\mathcal{M}' = * \mathcal{M}_t$ contains at least one matrix that cannot be covered by less than n lines (that is, the minimal number of lines which contain all 1's of the matrix is n).

Theorem. *$F_j=(F_j(1), F_j(2), \ldots, F_j(n))$, $1 \leqslant j \leqslant t$, be t families of subsets of a finite set $X=\{x_1, x_2, \ldots, x_m\}$ $(m \geqslant n)$ with incidence matrices A_j. There exists a common transversal for all those families if and only if the family $(P(A_1), P(A_2), \ldots, P(A_t))$ of matrices is coveringly complete.*

Proof. Let $(F_j : j=1, 2, \ldots, t)$ has transversal $T=\{y_1, y_2, \ldots, y_n\} \subset X$. If F_j' is a numeration of sets in F_j such that $y_i \in F_j'(i)$ and A_j' corresponding incidence matrix, then $A_j' \in P(A_j)$. Since $y_i \in F_j'(i)$ for every $j=1, 2, \ldots, t$, the matrix $A_1' * A_2' * \cdots * A_t'$ has 1's on the main diagonal, and hence, one needs not less than n lines (n column, for example, suffices) to cover all 1's. The condition of the theorem is fulfilled.

Conversely, let the condition of theorem be fulfilled. We now need the following theorem due to König (see, for example, [1], p. 188, or [4]).

Theorem *K*. Let A be a $\{0,1\}$ — matrix of size m by n. The minimal number of lines in A that contains all of the l's in A is equal to the maximal number of independent l's in A.

According to the theorem K and the condition of the theorem, there exist exactly n independent l's. Let $c_{i_1 1}, c_{i_2 2}, \ldots, c_{i_n n}$ be the places of those l's. Then, for every $j \in \{1, 2, \ldots, t\}$ $a_{i_1 1}^{(j)}, a_{i_2 2}^{(j)}, \ldots, a_{i_n n}^{(j)}$ they are places of independent l's of the matrix A_j'. The set $T = \{x_n, \ldots, x_m\}$ is a transversal of the family F_j', for every prescribed j, since $x_k \in F_j'(i_k)$, because $a_{i_k k}^{(j)} = 1$, and so T is transversal of F_j obtained from F_j' by prenumeration of sets in F_j'. T is the common transversal for all families F_j, $j = 1, 2, \ldots, t$.

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