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ON MAPPINGS WITH A CONTRACTIVE ITERATE

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Concept of a quasi-contraction mapping was introduced in [3] as a self-mapping T on a metric space M which satisfies a condition equivalent to the following one condition:

$$\text{(A)} \qquad d(Tx,\ Ty) \leqslant q \operatorname{diam}(\{Tx,\ x,\ y,\ Ty\})$$

for some $q<1$ and all x, y in M. These mappings are extended and investigated in [1]–[13].

In this paper we consider mappings of a metric space M into itself which satisfy the following condition:

$$\text{(B)} \quad d(T^k x,\ T^k y) \leqslant q \operatorname{diam}(\{T^k x,\ T^{k-1} x, \ldots,\ Tx,\ x,\ y,\ Ty, \ldots,\ T^{k-1} y,\ T^k y\})$$

for some $q<1$ and a positive integer k; $x, y \in M$.

If $M=[0, 1]$ and T: $M \to M$ is defined by $T(x)=\frac{x}{2}$; if $x \neq 0$ and $T(0)=1$, then T satisfies (B) with $q=\frac{1}{2}$ and $k=2$, as $d(T^2 0,\ T^2 x)=\frac{1}{2}-\frac{x}{4}<\frac{1}{2}\cdot 1=\frac{1}{2} d(0,\ T0)$ and $d(T^2 y,\ T^2 x)=\frac{1}{4} d(y,\ x)$ for $x \cdot y \neq 0$. However T is not any of mappings considered in [1]–[13].

1. We first establish the following result.

Theorem 1. *Let (M, d) be a complete metric space and $T: M \to M$ be a selfmapping which satisfies (B) and a functional $D(x)=d(x,\ Tx)$ is such that for every sequence (x_n) in M, $x_n \to x_0 \in M$ implies $\limsup_{n\to\infty} D(x_n) \geqslant D(x_0)$. Then T has a unique fixed point, say u, and $u=\lim_{n\to\infty} T^n x$ for every $x \in M$.*

Proof. Let $x \in M$ be arbitrary. We shall show that

$$\operatorname{diam}(\{x, Tx, T^2x, \ldots\}) \leqslant \frac{1}{1-q} \operatorname{diam}(\{x, Tx, T^2x, \ldots, T^k x\}), \tag{1}$$

where k is such that (B) is satisfied. Let n be an arbitrary positive integer and i and j $(i<j)$ are such integrers that

$$d(T^i x, T^j x) = \operatorname{diam}(\{x, Tx, T^2x, \ldots, T^n x\}). \tag{2}$$

Putt $r(x) = \operatorname{diam}(\{x, Tx, T^2x, \ldots, T^k x\})$ and $O_n(x) = \{x, Tx, \ldots T^n x\}$. If $j \leqslant k$, then $\operatorname{diam}(O_n(x)) \leqslant r(x) < \frac{1}{1-q} r(x)$. Let now be $j > k$ and $i < k$. Then by (B)

$$\begin{aligned} d(T^i x, T^j x) &\leqslant d(T^i x, T^k x) + d(T^k x, T^j x)) \leqslant r(x) + \\ &\quad q \operatorname{diam}(\{T^k x, T^{k-1} x, \ldots, x, T^{j-k} x, \ldots, T^j x\}) \\ &\leqslant r(x) + q \operatorname{diam}(O_n(x)) \end{aligned}$$

and hence

$$\operatorname{diam}(O_n(x)) \leqslant \frac{1}{1-q} r(x).$$

The case $i \geqslant k$ is imposible, as then

$$\begin{aligned} d(T^i x, T^j x) &= d(T^k T^{i-k} x, T^k T^{j-k} x) \\ &\leqslant q \operatorname{diam}(\{T^i x, T^{i-1} x, \ldots, T^{i-k} x, T^{j-k} x, T^{j-k+1} x, \ldots, T^j x\}) \\ &\leqslant q \operatorname{diam}(\{x, Tx, \ldots, T^n x\}) = q \operatorname{diam}(O_n(x)) \end{aligned}$$

and being $q < 1$, this will contradict (2).

Therefore, we proved that

$$\operatorname{diam}(O_n(x)) \leqslant \frac{1}{1-q} r(x)$$

and by letting n tend to infinity we obtain (1).

Now we shall show that $\operatorname{diam}(\{T^n x, T^{n+1} x, \ldots, \})$ tends to O when $n \to \infty$. Denote $\{T^n x, T^{n+1} x, \ldots, \}$ by $O(T^n x)$ and let $T^i x, T^j x \in O(T^{nk} x)$. Then by (B)

$$\begin{aligned} d(T^i x, T^j x) &= d(T^k T^{i-k} x, T^k T^{j-k} x) \\ &\leqslant q \operatorname{diam}(\{T^i x, T^{i-1} x, \ldots, T^{i-k} x, T^{j-k} x, \ldots, T^j x\}) \leqslant q \operatorname{diam}(O(T^{i-k} x)) \\ &\leqslant q \operatorname{diam}(O(T^{(n-1)k} x)), \end{aligned}$$

as $O(T^{i-k}x) \subseteq O(T^{(n-1)k}x)$. Hence

$$\text{diam}(O(T^{nk}x)) \leqslant q \ \text{diam}\ (O(T^{(n-1)k}x) \leqslant \cdots \leqslant q^n \ \text{diam}\ (O(x)).$$

Since $q<1$ and diam $(O(x))$ is bounded, it follows that $\lim\limits_{n\to\infty} \text{diam}\ (O(T^{nk}x))=0$. As sequence

$$\text{diam}\ (O(x)) \geqslant \text{diam}\ (O(Tx)) \geqslant \cdots \geqslant \text{diam}\ (O(T^n x) \geqslant \cdots$$

is monotone and contains a convergent subsequence $\left\{\text{diam}\ O(T^{nk}x))\right\}_{n=1}^{\infty}$, it follows that it is convergent and

$$\lim_{n\to\infty} \text{diam}\ (O(T^n x))=0,$$

i.e, that $\left\{T^n x\right\}_{n=0}^{\infty}$ is a Cauchy sequence. Since M is complete, there exists a point u such that $u=\lim\limits_{n\to\infty} T^n x$.

It now remains to be shown that u is a fixed point of T. Since $\lim\limits_{n\to\infty} T^n x=u$ and $\lim\limits_{n\to\infty} \sup D(T^n x)=\lim\limits_{n\to\infty} \sup d(T^n x, T^{n+1}x)=0$, it follows that $O \geqslant D(u)= =d(u, Tu)$ and hence $Tu=u$. The uniquenees of the fixed point may be established by use of (B). This completes the proof.

The example, following the definition of the contractive condition (B), shows that one can not delete the stated condition for a functional $D(x)= =d(x, Tx)$.

Theorem 1 holds if some of the conditions are relaxed. So we hawe

Theorem 2. *Let T be a selfmapping of a T-orbitally complete metric space (M, d) into itself. If T is orbitally continuous and satisfies* (B), *then T has a unique fixed point, say u, and $u=\lim\limits_{n\to\infty} T^n x$ for every $x \in M$.*

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