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THE LAWS OF CONSERVATION OF ONE DIMENSIONAL MOTION OF NONLINEAR CONTINUUM

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Introduction. The classical laws of conservation of the linear momentum, the moment of momentum of motion and of energy reflect the homogeneous and isotropic structure of space and time. They are the sequence allowance of the Galilei group. Namely, the invariance of laws of motion in relation to this group involves the existence of the mentioned laws of coservation. They are generaly valid, and also for motions of any continuous media. But the concrete continuous media possesses specific properties. These properties and pecularities reflect to the concrete laws of behaviour, for instance, to the equations of motions, to the rheologic equations etc. On the other side, the equations of motion of concrete continuous media can allow, i. e. can be invariant in relation to broader group, which is an embracing group for the Galilei group. Using the Noether theorem [1] it is clear that such an enlargement of group will bring to new laws of conservation. These supplementary laws, we can say, are on internal way connected to the concrete media. Therefore, they represent an important characteristic. Their knowledge gives the fundament for a more deeply understanding of the process of motion. The purpose of this paper is to study and form such an enlarged set of laws of conservation for the case of one dimensional motion of nonlinear continuum.

1 The laws of conservation with geometrical nonlinearity.

The formulation of the laws of conservation is in direct connection with the first Noether's theorem [1] The conditions of applicability of this theorem, as is known, require the variational formulation of the corresponding problem [2]. In our case, that means, to give the variational formulation to the problem, whose equations are:

$$\frac{\partial \rho}{\partial t}+\frac{\partial (\rho v)}{\partial x}=0, \tag{1}$$

(2) $$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}-v=0,$$

(3) $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{\lambda+2\mu}{\rho}\frac{\partial^2 u}{\partial x^2},$$

where t-time, x-coordinates, ρ -thed ensity, u-the displacement and v-the velocity of particle.

The formulation of integral of action, respectively, Lagrangian density for the problem, whose equations are given in advance, is not easy, and sometimes even it is impossible. But there exist problems in which the Lagrangian's densities are easy to find, under the condition, of course, that this problem permit the variational formulation. Especially the problem of finding of Lagrangian's density is simplified, if we have to deal only with one equation. Therefore, let us reduce the equations (1), (2) and (3) to only one equation, introducing the substitutions

(4) $$\rho=1-\frac{\partial u}{\partial x}=1-u_x, \qquad \rho v=\frac{\partial u}{\partial t}=u_t$$

in the above equations. It is easy to see that equations (1) and (2) proceed to identities and the equation (3) into equation of the form

(5) $$\frac{\partial}{\partial t}\left(\frac{u_t}{1-u_x}\right)+\frac{u_t}{1-u_x}\frac{\partial}{\partial x}\left(\frac{u_t}{1-u_x}\right)=\frac{\lambda+2\mu}{1-u_x}\frac{\partial^2 u}{\partial x^2},$$

or into the equivalent equation

(6) $$\frac{\partial}{\partial t}\left(\frac{u_t}{1-u_x}\right)+\frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{u_t}{1-u_x}\right)^2+\alpha\ln(1-u_x)\right]=0,$$

where $\alpha=\lambda+2\mu$, is the characteristic of the material. Thus we have reduced the problem of variational formulation of one dimensional motion to the problem of variational formulation of the equation (6). Now, it is easy to see that the equation (6) is the Euler-Lagrange's equation for the integral of action in the form:

(7) $$L=\int_{t_1}^{t_2}\int_{x_1}^{x_2}\mathscr{L}\,dx\,dt=\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\frac{1}{2}\frac{u_t^2}{1-u_x}-\alpha(1-u_x)\ln(1-u_x)-\alpha u_x\right]dx\,dt,$$

where $\mathscr{L}$ is the Lagrange's density, t_1, t_2, x_1, and x_2 are the boundary of dominion of integration.

From the theory of Lie groups [3, 4], it is known in what way are constructed the groups of invariant of functional (7). By application of such a proceeding, which we will not cite here, we find that the group of invariant of functional (7) is given by the "general" generator of the group in the form:

(8) $$X=(at+b_1)\frac{\partial}{\partial t}+(ax+b_2)\frac{\partial}{\partial x}+(au+b_3)\frac{\partial}{\partial u},$$

where a, b_1, b_2, and b_3 are parameters of the group, by whose correct choice, we obtain four mutually independent generators of the Lie group:

$$X_1=\frac{\partial}{\partial t}, \quad X_2=\frac{\partial}{\partial x}, \quad X_3=\frac{\partial}{\partial u}, \quad X_4=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}. \tag{9}$$

By applying the first Noether's theorem [2] we obtain the "general" (parametric) law of conservation in the form

$$\frac{\partial A_\alpha^0}{\partial t}+\frac{\partial A_\alpha}{\partial x}=0, \tag{10}$$

or in the equivalent integral formulation, in the form

$$\frac{d}{dt}\int_{x_1}^{x_2}\left(S_\alpha\frac{\partial \mathscr{L}}{\partial u_t}+\mathscr{L}\xi_\alpha^0\right)dx+\left[S_\alpha\frac{\partial \mathscr{L}}{\partial u_x}+\mathscr{L}\xi_\alpha\right]_{x_1}^{x_2}=0, \tag{11}$$

where

$$A_\alpha^0=S_\alpha\frac{\partial \mathscr{L}}{\partial u_t}+\mathscr{L}\xi_\alpha^0=(\eta_\alpha-u_t\xi_\alpha^0-u_x\xi_\alpha)\frac{\partial \mathscr{L}}{\partial u_t}+\mathscr{L}\xi_\alpha^0, \tag{12}$$

$$A_\alpha=S_\alpha\frac{\partial \mathscr{L}}{\partial u_x}+\mathscr{L}\xi_\alpha=(\eta_\alpha-u_t\xi_\alpha^0-u_x\xi_\alpha)\frac{\partial \mathscr{L}}{\partial u_t}+\mathscr{L}\xi_\alpha, \tag{13}$$

$$\xi_\alpha^0=at+b_1, \quad \xi_\alpha=ax+b_2, \quad \eta_\alpha=au+b_3. \tag{14}$$

while $\mathscr{L}$ is the Lagrange's density in (6). If we substitute the corresponding expressions from (12), (13) and (14) in (11), then for the integral form of the "general" law of conservation, we have

$$\begin{aligned}&\frac{d}{dt}\int_{x_1}^{x_2}\Big\{[a(u-t\rho v+\rho x-x)-b_1 v\rho-b_2+b_3+\rho b_2]v+\\ &\qquad+\left[\frac{1}{2}\rho v^2-\alpha\rho\ln\rho-(1-\rho)\alpha\right](at+b_1)\Big\}dx+\\ &+\Big\{[a(u-t\rho v+\rho x-x)-b_1 v\rho-b_2+b_3+\rho b_2]\left(\frac{v^2}{2}+\alpha\ln\rho\right)+\\ &\qquad+\left[\frac{1}{2}\rho v^2-\alpha\rho\ln\rho-\alpha(1-\rho)\right](ax+b_2)\Big\}\Big|_{x_1}^{x_2}=0\end{aligned} \tag{15}$$

in which we have used and substituted the magnitude from (4) and the value $\mathscr{L}$ of Lagrange's density from (7).

From the fact that that the physical sense have only the laws of conservation of distinct generators of the group, substituting the parameters group in expression (15), we finally obtain the following laws of conservation:

(a) $a=1,\ b_1,=b_2,=b_3=0$

(16)
$$\frac{d}{dt}\int_{x_1}^{x_2}\left\{[u-\rho v t-xu_x]v+\left[\rho\frac{v^2}{2}-\alpha\rho\ln\rho-\alpha u_x t\right]\right\}dx+$$
$$+\left\{[u-\rho v t-xu_x]\left(\frac{1}{2}v^2+\alpha\ln\rho\right)+\left(\frac{\rho v^2}{2}-\alpha\rho\ln\rho-\alpha u_x\right)x\right\}\Bigg|_{x_1}^{x_2}=0,$$

(b) $b_1=1,\ a=b_2=b_3=0$

(17)
$$\frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}\rho v^2+\alpha\rho\ln\rho+\alpha u_x\right)dx+(\rho v^3-\alpha\rho v\ln\rho)\Bigg|_{x^1}^{x_2}=0,$$

(c) $b_2=1,\ a=b_1=b_3=0$

(18)
$$\frac{d}{dt}\int_{x_1}^{x_2}u_x v\,dx+\left[u_x\left(\frac{1}{2}v^2+\alpha\ln\rho\right)+\frac{\rho v^2}{2}-\alpha\rho\ln\rho-\alpha u_x\right]\Bigg|_{x_1}^{x_2}=0,$$

d) $b_3=1,\ a=b_1=b_2=0$

(19)
$$\frac{d}{dt}\int_{x_1}^{x_2}v\,dx+\left(\frac{1}{2}v^2+\alpha\ln\rho\right)\Bigg|_{x_1}^{x_2}=0.$$

2. The laws of conservation with physical nonlinearity.

The physical nonlinearity, as it is known, derives from the nonlinear constitutive equations.

The case of onedimensional motion was studied by G. Kirchhoff and G. Kauderer [4, 5]. Kirhhoff has found that equations of onedimensional motion with nonlinear physical relation are of the form:

(20)
$$\frac{\partial^2 u}{\partial t^2}-\alpha^2\left[1+\lambda\left(\frac{\partial u}{\partial x}\right)^2\right]\frac{\partial^2 u}{\partial x^2}=0,$$

where α and λ, are the material characteristic of the media.

Kirchhoff also has constated that the equation (20) can be obtained as the Euler-Lagrange's equation from the functional of the form

(21)
$$L=\int_{t_1}^{t_2}\int_{x_1}^{x_2}\mathcal{L}\,dx\,dt=\int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{1}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2-\alpha^2\left[1+\frac{\lambda}{6}\left(\frac{\partial u}{\partial x}\right)^2\right]\left(\frac{\partial u}{\partial x}\right)^2\right\}dx\,dt,$$

where

(22) $$\mathscr{L}=\frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2-\alpha^2\left[1+\frac{\lambda}{6}\left(\frac{\partial u}{\partial x}\right)^2\right]\left(\frac{\partial u}{\partial x}\right)^2\right\},$$

is the Lagrange's density of the the integral of action (21).

By applying the known procedure [1, 3, 4], the forming of the Lie group of the invariant of equation (20) and the functional (21) is shown that the group is determined by the expression (8).

If we use the relations (10) and (11) and substitute it in (12) and (13) and if we substitute the corresponding value from (22) then we obtain the following laws of conservations:

(23) $$\frac{d}{dt}\int_{x_1}^{x_2}\{(u-tu_t-xu_x)\,u_t+\mathscr{L}\cdot t\}\,dx+ +\left\{(u-tu_t-xu_x)\,\alpha^2\left(1+\frac{\lambda}{6}\,u_x^2\right)u_x+\mathscr{L}x\right\}\Bigg|_{x_1}^{x_2}=0.$$

(24) $$\frac{d}{dt}\int_{x_1}^{x_2}\{-u_t+\mathscr{L}\}\,dx+\left[\alpha^2 u_t u_x\left(1+\frac{\lambda}{3}\,u_x^2\right)\right]\Bigg|_{x_1}^{x_2}=0,$$

(25) $$\frac{d}{dt}\int_{x_1}^{x_2}u_t u_x\,dx+\mathscr{L}\big|_{x_1}^{x_2}=0,$$

(26) $$\frac{d}{dt}\int_{x_1}^{x_2}u_t\,dx-\alpha^2\left[(1+\lambda u_x^2)\frac{\partial^2 u}{\partial x^2}\right]\Bigg|_{x_1}^{x_2}=0.$$

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