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ON ALMOST-TRACTABLE SPACES I

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The objective of the present paper is to examine a class of topological spaces characterised by the property that none of the nonvoid proper regularly closed sets in such a space is a fixed set of the group of autohomeomorphisms of the space. Such spaces are said to be almost-tractable and the class of almost--tractable spaces contains the class of tractable spaces introduced by the present author in [2]. In the first theorem of the present paper we have proved a necessary and sufficient condition for a topological space to be almost-tractable. Together with other results it has been observed that almost-tractable spaces are productive. In a paper [1] we have introduced the notion of schistic space and here it has been shown that a schistic space with infinite dispersion character is an almoss-tractable space. However, a schistic space with finite dispersion character need not be almost--tractable. Further, we have also proved a necessary and sufficient condition for a regular almost-tractable space to be tractable and it has been observed that every nonprincipal ultraspace which are maximal elements of the lattice of topologies are maximal almost-tractable T_1 topologies.

A topological space $(X, \mathcal{J})$ is said to be a tractable space if for every nonvoid proper closed set K there exists an autohomeomorphism g of $(X, \mathcal{J})$ such that $g(K)\neq K$. If $\mathcal{F}$ be the collection of all the images of a nonvoid open set O of the space $(X, \mathcal{J})$, then $\mathcal{F}$ is said to be an of-collection of $(X, \mathcal{J})$ generated by O or simply an of-collection of the space, when there is no chance of confusion. The union of the members of $\mathcal{F}$ is said to be an of-set of the space generated by O or simply an of-set of the space. The least cardinal number of a nonvoid open set of a topological space is said to be the dispersion character of the space. The set of all the images of a point of a topological space $(X, \mathcal{J})$ is said to be an orbit of the space. An infinite subset of a topological space is said to be an *i*-set of the space if its complement is also infinite. An infinite topological space $(X, \mathcal{J})$ is said to be a schistic space if for every *i*-set A of the space there exists an autohomeomorphism g of the space $(X, \mathcal{J})$ such that $g(A)\subset A$, where $\subset$ denotes proper inclusion. For other terms and definitions readers are referred to [3], [4], [5], [6] and [7].

D e f i n i t i o n. A topological space $(X, \mathcal{J})$ is said to be an almost-tractable space if for every proper nonvoid regularly closed set K there exists an autohomeomorphism g of the space $(X, \mathcal{J})$ such thas $g(K)\neq K$.

In the following theorem we prove a necessary and sufficient condition for a topological space to be almost-tractable.

Theorem 1. *A topological space $(X, \mathcal{J})$ is almost-tractable if and only if every of-set of the space is dense.*

Proof. Let the space $(X, \mathcal{J})$ be almost-tractable. If it is assumed that there exists an of-set U of the space which is nondense, then CU is a nonvoid proper regularly closed subset of $(X, \mathcal{J})$. As the space $(X, \mathcal{J})$ is almost-tractable, there exists an autohomeomorphism g of the space $(X, \mathcal{J})$ such that $g(CU) \neq CU$. Consequently, we have $g(U) \neq U$. But, as U is an of-set of the space, it is a fixed set of the group of autohomeomorphisms of the space $(X, \mathcal{J})$. Hence we have $g(U) = U$, a contradiction. It follows that every of-set of the space is dense.

To establish the converse result, let us assume that every of-set of the space $(X, \mathcal{J})$ is dense. If it is assumed that the space $(X, \mathcal{J})$ is not an almost--tractable space, then there exists a nonvoid proper regularly closed set K such that for any autohomeomorphism g^* of the space $(X, \mathcal{J})\, g^*(K) = K$. Now it is obvious that Int K is an of-set of the space. But it is nondense, a contradiction. Hence it follows that the space $(X, \mathcal{J})$ is almost-tractable.

Theorem 2. *If $(X, \mathcal{J})$ be an almost-tractable space containing a set of second category, then the space $(X, \mathcal{J})$ is a Baire space.*

Proof. If it is assumed that the space $(X, \mathcal{J})$ is not a Baire space, then there exists an open set O' of $(X, \mathcal{J})$ which is a set of first category. Let $\mathcal{F}$ be the of-collection of the space $(X, \mathcal{J})$ generated by O'. Then clearly every member of the of-collection $\mathcal{F}$ is a set of first category. Also it is well known [see [7]] that the union of any family of open sets of first category is a set of first category. Hence it follows that the union U of the members of $\mathcal{F}$ is a set of first category. Also U is an of-set of the space and consequently CU is nowhere dense. Hence it follows that X is a set of first category. It contradicts the hypothesis that the space $(X, \mathcal{J})$ contains a set of second category. Hence it follows that the space $(X, \mathcal{J})$ is a Baire space.

In the following the class of all the of-sets of a topological space $(X, \mathcal{J})$ will be denoted by $\mathcal{S}$ and the intersection of all the members of $\mathcal{S}$ is said to be the *s*-set of the space $(X, \mathcal{J})$.

Lemma 1. *If $(X, \mathcal{J})$ be an almost-tractable space, then every orbit of the space is either dense or nowhere dense.*

Proof. Let A be an orbit of the space $(X, \mathcal{J})$. If it is assumed that A is neither dense nor nowhere dense, then Int $Cl\,A \neq \emptyset$ and $Cl\,A$ is a proper nonvoid regularly closed subset of $(X, \mathcal{J})$. Hence, as the space $(X, \mathcal{J})$ is almost--tractable, there exists an autohomeomorphism g of the space $(X, \mathcal{J})$ such that $g(ClA) \neq ClA$. As A is an orbit of the space, it is obvious that ClA is the union of a collection of orbits of the space. It follows that $g(ClA) = ClA$, a contradiction. Hence it follows that every orbit of the space $(X, \mathcal{J})$ is either dense or nowhere dense.

Lemma 2. *If $(X, \mathcal{J})$ be an almost-tractable space and if there exists a nonvoid open set which, as a subspace, satisfies the second axiom of countability, then the s-set of the space is the intersection of a countable subcollection of the collection of of-sets of the space.*

P r o o f. Let O be an open set which as a subspace satisfies the second axiom of countability. Let $\mathcal{B}=\{O_i : i\in N\}$ be a countable base of the subspace O. Let O^* be a nonvoid open set of $(X, \mathcal{J})$. Since the space $(X, \mathcal{J})$ is almost-tractable there exists an autohomoeomorphism g of the space $(X, \mathcal{J})$ such that $g(O^*)\cap O\neq\emptyset$. It follows that there exists a member O_j of $\mathcal{B}$ such that $O_j\subseteq g(O^*)$ and consequently the of-set of the space $(X, \mathcal{J})$ generated by O_j is contained in the of-set of the space $(X, \mathcal{J})$ generated by O^*. Now, if $\mathcal{S}'$ be the collection of all the of-sets of the space $(X, \mathcal{J})$ generated by the members of $\mathcal{B}$, then clearly every member of the collection of of-sets $\mathcal{S}$ of the space $(X, \mathcal{J})$ contains a member of $\mathcal{S}'$. Also $\mathcal{S}'$ is countable. It follows that the s-set of the space $(X, \mathcal{J})$ is the intersection of a countable family $\mathcal{S}'$ of of-sets of the space $(X, \mathcal{J})$.

T h e o r e m 3. *Let $(X, \mathcal{J})$ be an almost-tractable space and O be a nonvoid open subset which, as a subspace, satisfies the second axiom of countability. Then the space $(X, \mathcal{J})$ is a Baire space if and only if the s-set of the space, as a subspace, is a Baire space.*

P r o o f. Let S be the s-set of the space $(X, \mathcal{J})$ and let S be, as a subspace, a Baire space. Then obviously $(X, \mathcal{J})$ contains a set of second category and consequently it follows from the Theorem 2 that the space, $(X, \mathcal{J})$ is a Baire space.

To establish the converse result, let us assume that the space $(X, \mathcal{J})$ is a Baire space. Since S is the intersection of a countable family of of-sets of the space $(X, \mathcal{J})$ and as the complement of every of-sets of the space $(X, \mathcal{J})$ is nowhere dense subset of $(X, \mathcal{J})$, CS is a set of first category. Hence, as the space $(X, \mathcal{J})$ is a Baire space, S is obviously nonvoid and it is a set of second category. Further, it is obvious that S is the union of the class of all the dense orbits of the space $(X, \mathcal{J})$. Again, it is obvious that the restriction of every autohomeomorphism of the space $(X, \mathcal{J})$ to S is an autohomeomorphism of the subspace S. It follows that every of-collection of the subspace S forms a cover of the subspace S and consequently it follows from the Theorem 1 of [2] that the space S is a tractable space. Since every tractable space is almost-tractable, it follows from the Theorem 2 that the subspace S is a Baire space.

C o r o l l a r y. *If $(X, \mathcal{J})$ be an almost-tractable complete metric space satisfying the second axiom of countability, then the s-set of the space is, as a subspace, a topologically complete tractable space.*

P r o o f. As the space $(X, \mathcal{J})$ is a complete metric space, it is a Baire space. Again, as the space satisfies the second axiom of countability, it is obvious from the Lemma 2 that the s-set S of the space $(X, \mathcal{J})$ is a G_δ-set of $(X, \mathcal{J})$. Also CS is a set of first category and consequently S is nonvoid. Hence it follows from the Theorem 1.2 of [7] that S is topologically complete. Also it follows from the above theorem that the subspace S is tractable. Hence we have the required result.

T h e o r e m 4. *The one-point compactification, of a Hausdorff tractable space $(X, \mathcal{J})$ is an almost-tractable space.*

P r o o f. Let $(X^*, \mathcal{J}^*)$ be the one-point compactification [see [5]] of the Hausdorff tractable space $(X, \mathcal{J})$ and let $X^*-X=\{\omega\}$. Now, if O^* be a nonvoid open set of the space $(X^*, \mathcal{J}^*)$, then $O=O^*\cap X$ is a nonvoid open set of the space $(X, \mathcal{J})$. Further, if g is an autohomeomorphism of the space $(X, \mathcal{J})$ and g^* be a one-one correspondence of the space $(X^*, \mathcal{J}^*)$ onto itself such that for any point x of X, $g^*(x)=g(x)$ and $g^*(\omega)=\omega$, then g^* is obviously an autohomeomorphism of the space $(X^*, \mathcal{J}^*)$. Further, as the space $(X, \mathcal{J})$ is tractable, it is obvious from the above arguments that every of-set of the space $(X^*, \mathcal{J}^*)$ contains X. Consequently every of-set of the space $(X^*, \mathcal{J}^*)$ is dense in $(X^*, \mathcal{J}^*)$ and the space $(X^*, \mathcal{J}^*)$ is almost-tractable.

In this connection it is to be remarked that every infinite discrete space being a Hausdorff tractable space its one-point compactification is an almost--tractable space. Further, it can be easily shown that the subspace $J=(0, 1)$ of the real line is a Hausdorff tractable space and consequently its one-point compactification is an almost-tractable space.

The following theorem shows that almost-tractable spaces are productive.

Theorem 5. *The product space of a family of almost-tractable spaces is an almost-tractable space.*

Proof. Let $(X, \mathcal{J})$ be the product space of a family of almost-tractable spaces $\{X_\beta, \mathcal{J}_\beta): \beta\in\Gamma\}$. If it is assumed that the product space $(X, \mathcal{J})$ is not an almost-tractable space, then there exists an of-set U of the space $(X, \mathcal{J})$ which is nondense in $(X, \mathcal{J})$. Let us write $U=\prod_{\beta\in\Gamma} 0_\beta$, where $0_\beta\in\mathcal{J}_\beta$ and $0_\beta=X_\beta$ for all but finite number of indices. Now, as U is a nonvoid nondense open subest of the product space $(X, \mathcal{J})$, there exists a member O_{β^*} of the family of sets $\{O_\beta:\beta\in\Gamma\}$ such that O_{β^*} is a nonvoid nondense open subset of the corresponding factor space $(X_{\beta^*}, \mathcal{J}_{\beta^*})$. As, by hypothesis, the space $(X_{\beta^*}, \mathcal{J}_{\beta^*})$ is almost-tractable, there exists an autohomeomorphism f_{β^*} of the space $(X_{\beta^*}, \mathcal{J}_{\beta^*})$ such that $f_{\beta^*}(Cl_{\mathcal{J}\beta^*} O_{\beta^*})\neq Cl_{\mathcal{J}\beta^*} O_{\beta^*}$ and consequently we have $f_{\beta^*}(O_{\beta^*})\neq O_{\beta^*}$. Let $\{g_\beta:\beta\in\Gamma\}$ be a class of autohomeomorphisms of the factor spaces such that $g_{\beta^*}=f_{\beta^*}$ and $g_\beta=\sigma_\beta$, for every $\beta\neq\beta^*$, where $\{\sigma_\beta:\beta\in\Gamma\}$ is the class of identity homeomorphisms of the factor spaces. Let (x_β) be a point of the product space $(X, \mathcal{J})$. It is well known that the function $g^*:(X, \mathcal{J})\to(X, \mathcal{J})$ defined by the $g^*((x_\beta))=(g_\beta(x_\beta))$ maps the product space homeomorphically onto itself. Also it is obvious from the construction of the function g^* that $g^*(U)\neq U$. But, as U is an of-set of the product space, we have $g^*(U)=U$, a contradiction. Hence it follows that U is a dense subset of the product space and consequently it follows from the Theorem 1 that $(X, \mathcal{J})$ is almost-tractable.

Let us consider the two point space $X=\{0, 1\}$ having its topology $\mathcal{J}=\{\emptyset, \{0, 1\}, \{1\}\}$. Indeed the space X is an almost-tractable space and consequently it follows from the above theorem that an Alexandroff cube of order n which is the product space of n copies of two point space X is an almost--tractable space. Further, a two point discrete space $D=\{0, 1\}$ is obviously a tractable space. A Cantor cube of order $m\geq\aleph_0$ is the product space of m copies of two point space D. As tractable spaces are productive [see [2]], a Cantor cube of order m is a tractable space. Further, as every tractable space is almost-tractable, a topological space which is the product space of an Alexandroff cube of order n and a Cantor cube of order m is an almost-tractable space.

Theorem 6. *If $(X, \mathcal{J})$ be a schistic space with infinite dispersion character, then the space $(X, \mathcal{J})$ is an almost-tractable space.*

P r o o f. It it assumed that the space $(X, \mathcal{J})$ is not an almos-tractable space, then there exists an of-set of the space U such that $ClU \neq X$. Consequently CU has nonvoid interior. Further, as the dispersion character of the space is infinite CU is an i-set of the space. Hence there exist an autohomeomorphism g of the space $(X, \mathcal{J})$ such that $g(CU) \subset CU$. It follows that $g(U) \neq U$ and consequently U is not an of-set of the space, a contradiction. Hence it follows that the space $(X, \mathcal{J})$ is an almost-tractable space.

In this connection it is to be remarked that if the dispersion of a schistic space is finite, then the space need not be an almost-tractable space. Thus if $(X, \mathcal{J})$ be an infinite topological space such that $\mathcal{J} = \mathcal{C}(x, \mathcal{U}(y))$, where $x \neq y$ and $\mathcal{C}(x, \mathcal{U}(y))$ is a principal ultraspace [see [4, 8]], then it can be easily shown that the space $(X, \mathcal{J})$ is a schistic space with finite dispersion character. However, the space $(X, \mathcal{J})$ is not an almost-tractable space. For, although $\{x, y\}$ is a proper nonvoid regularly closed subset of the space $(X, \mathcal{J})$, $g\{x, y\} = \{x, y\}$ for every autohomeomorphism g of the space $(X, \mathcal{J})$.

D e f i n i t i o n. A decomposition of a topological space $(X, \mathcal{J})$ is said to be an almost upper semicontinuous decomposion if for every nonvoid regularly open set O, the union of all the members of the decomposition contained in O is open.

It is well known that the class of orbits of a topological space $(X, \mathcal{J})$ forms a decomposition of the space. Here the decomposition of a topological space consisting of the orbits of a topological space $(X, \mathcal{J})$ will be denoted by $\mathcal{D}$.

In the following theorem we prove a necessary and sufficient condition for a regular almost-tractable space to be a tractable space.

T h e o r e m 7. *A regular almost-tractable space $(X, \mathcal{J})$ is tractable if and only if the decomposition $\mathcal{D}$ is almost upper semi-continuous.*

P r o o f. To prove the "if" part of the theorem, let us assume that the decomposition $\mathcal{D}$ is almost upper semi-continuous. If it is assumed that the space $(X, \mathcal{J})$ is not a tractable space, then there exists an orbit P of the space $(X, \mathcal{J})$ which is nondense. Since the space is regular there exists a proper regularly open set O such that $ClP \subseteq O$. Let V be the union of all the members of the decomposition $\mathcal{D}$ which are contained in O. As, by hypothesis, the decomposition $\mathcal{D}$ is almost upper semi-continuous, V is a proper regularly open subset of $(X, \mathcal{J})$. Further, as V is the union of a nonvoid collection of members of $\mathcal{D}$, ClV is obviously the union of a nonvoid collection of membes of the decomposition $\mathcal{D}$. Also ClV is a proper subset of $(X, \mathcal{J})$. Hence there exists an autohomeomorphism g of the space $(X, \mathcal{J})$ such that $g(ClV) \neq ClV$. But, as ClV is the union of a collection of orbits of the space, we have $g(ClV) = ClV$, a contradiction. Hence it follows that every orbit of the space is dense. Consequently every of-collection of the space is a cover of the space and it follows from the Theorem 1 of [2] that the space $(X, \mathcal{J})$ is tractable.

The "only if" part of the theorem can be easily established.

C o r o l l a r y. *A regular almost-tractable space $(X, \mathcal{J})$ is homogeneous if and only if the decomposition $\mathcal{D}$ is upper semi-continuous.*

P r o o f. If the space $(X, \mathcal{J})$ is homogeneous, then the decomposition $\mathcal{D}$ is obviously upper semi-continuous.

To establish the converse result, let us consider that the decomposition $\mathcal{D}$ is upper semi-continuous [see [6]]. Indeed every upper semi-continuous decompostition is almost upper semi-continuous. Therefore, it follows from the above theorem that the space $(X, \mathcal{J})$ is tractable. Now, if the space $(X, \mathcal{J})$ is assumed to be non-homogeneous, then $\mathcal{D}$ is non-degenerate. Let P be a member of $\mathcal{D}$ and x be a point in P. As the space $(X, \mathcal{J})$ is regular, it satisfies R_1-axiom [see [3]]. Consequently, $Cl\{x\}$ is an e-set of the space [see [2]]. Since P is an orbit of the space $Cl\{x\}$ is clearly contained in P Now, $C(Cl\{x\})$ contains all the members of the decomposition $\mathcal{D}$ excepting P. As $\mathcal{D}$ is nondegenerate and upper semi-continuous, the union of the collection of the members of $\mathcal{D}$ contained in $C(Cl\{x\})$ is a nonvoid open set. It follows that P is a proper closed subset of $(X, \mathcal{J})$. But, as the space $(X, \mathcal{J})$ is tractable P is dense, a contradiction. Hence it follows that $\mathcal{D}$ is degenerate and the space $(X, \mathcal{J})$ is homogeneous.

Let $\mathcal{C}(p, \mathcal{U})$ be a nonprincipal ultraspace [see [4, 8]] on a set X. We call the point p of X to be the centre of the ultraspace $\mathcal{C}(p, \mathcal{U})$.

Theorem 8. *If $(X, \mathcal{J})$ be a topological space such that the topology $\mathcal{J}$ is the infimum of finite number of nonprincipal ultraspaces, then the space $(X, \mathcal{J})$ is an almost-tractable space.*

Proof. Let $\mathcal{J} = \bigwedge_{i=1}^{n} \mathcal{C}(p_i, \mathcal{U}_i)$ and let A_p be the set of centres of the members of the collection of nonprincipal ultraspaces $\{\mathcal{C}(p_i, \mathcal{U})\}_{1 \leq i \leq n}$. Let K be a nonvoid proper regularly closed subest of the space $(X, \mathcal{J})$. Obviously K is a clopen subset of the space. Let a point $x \in K$ such that $x \notin A_p$ and a point $y \in CK$ such that $y \notin A_p$. Let g be a one-one correspondence of $(X, \mathcal{J})$ onto itself such that $g(x) = y$, $g(y) = x$ and $g(z) = z$, for every $z \in X - \{x, y\}$. Let O be an open set. We write $O = O_\alpha \cup O_\beta$, where $O_\alpha = (X - \{x, y\}) \cap O$ and $O_\beta = O - (X - \{x, y\})$. As $(X - \{x, y\})$ is a clopen subset of the space, O_α and O_β are obviously open sets of the space $(X, \mathcal{J})$. Further, from the construction of the mapping g we have $g(O) = g(O_\alpha) \cup g(O_\beta) = O_\alpha \cup g(O_\beta)$. If $g(O_\beta)$ is nonvoid, then obviously it consists of isolated points of the space $(X, \mathcal{J})$. It follows that $g(O)$ and $g^{-1}(O)$ are open sets of the space $(X, \mathcal{J})$. Hence it follows that g is an autohomeomorphism of the space $(X, \mathcal{J})$. Also we have $g(K) \neq K$ and consequently the space $(X, \mathcal{J})$ is an almost-tractable space.

It has been shown by A. K. Steiner in [8] that every T_1-topology is the infimum of all the nonprincipal ultraspaces finer than it. Now, from the above theorem it follows that a topological space having its topology as a nonprincipal ultraspace is an almost-tractable space. From the above facts it follows that every maximal [see [9]] almost-tractable T_1-topology is a noprincipal ultraspace.

Further, it is to be mentioned that the space $(X, \mathcal{J})$ of the Theorem 8 can be easily shown to be a regular space. However, the decomposition of the space $\mathcal{D}$ is not upper semi-continuous. For, every singleton contained in $X - A_p$ is a clopen subset of $(X, \mathcal{J})$ and it can be easily shown that $X - A_p$ is a member of the decompoisition $\mathcal{D}$ and consequently, if $\mathcal{D}$ is to be an almost upper semi-continuous decomposition, A_p should be an open set. However, it is not true and, therefore, $\mathcal{D}$ is not an almost upper semi-continuous decomposition.

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