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SOME INTERPOLATORY POLYNOMIALS ON TCHEBYCHEFF ABSCISSAS — I

R. B. Saxena

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1. In this paper we shall deal with the following interpolation problem: Let x_i be a system of $(2n+1)$ distinct points in $[-1, 1]$ such that

$$(1.1)\qquad 1=x_1>x_2>\cdots>x_{2n}>x_{2n+1}=-1$$

and arbitrary numbers:

$$(1.2)\qquad y_i\,(i=1, 2, \ldots, 2n+1), \qquad y_i^*\,(i=0, 1, 2, \ldots, n+1),$$
$$y_i^{**}\,(i=1, 2, \ldots, n),$$

we seek to find a polynomial $g(x)$ of degree $4n+2$ atmost such that

$$g(x_i)=y_i \quad (i=1, 2, \ldots, 2n+1)$$
$$(1.3)\qquad g'(x_{2i})=y_i^* \quad (i=1, 2, \ldots, n), \qquad g'(x_1)=y_0^*, \quad g'(x_{2n+1})=y_{n+1}^*$$
$$g'''(x_{2i})=y_i^{**} \quad (i=1, 2, \ldots, n).$$

We call this process of interpolation the (0; 0, 1, 3) interpolation.

We shall however solve the above interpolation problem when the abscissas (1. 1) are the points.

$$(1.4)\qquad x_i=\cos\frac{i\pi}{2n} \qquad i=0, 1, 2, \ldots, 2n$$

which are the zeros of

$$(1.5)\qquad \pi_{2n+1}(x)\overset{\text{def}}{=}(1-x^2)\,U_{2n-1}(x)$$

where

$$(1.6)\qquad U_{2n-1}(x)=\frac{\sin 2n\theta}{\sin\theta}, \quad x=\cos\theta, \quad -1\leqslant x\leqslant 1$$

stands for the $(2n-1)$th Tchebycheff polynomial of second kind.

A similar problem of interpolation for the abscissas which are the zeros of

$$(1-x^2)\,P_{n-1}(x)$$

has been solved by the author in his works [1, 2, 3]. The results of A. K. Varma [5, 6] in this direction on Tchebycheff abscissas deserve a reference here.

2. With the representation (1. 1) of $(2n+1)$ points in $[-1, 1]$, the point $x=0$ either falls to x_{2i}'s or it falls to x_{2i+1}'s according as n is odd or even.

The following theorem 1. shows that in the case of odd number of distinct symmetrical points of the form $4m+3$ — both the problems of exsistence and uniqueness have a negative solution.

Theorem I. *If n is odd and the $(2n+1)$ points in $[-1, 1]$ have the representation* (1. 1) *with*

$$x_j = x_{2n+2-j} \qquad (j=1, 2, \ldots, n) \tag{2.1}$$

then to given numbers (1. 2) *there is in general no polynomial of degree $\leqq 4n+2$ such that* (1. 3) *is satisfied. If there exists such a polynomial then there are infinity of them.*

The proof of this theorem is obvious from theorem 1. in [4]. Naturally we omit the details.

3. Let n be even and the points $x_i\,(i=1, 2, \ldots, 2n+1)$ in (1. 1) be given by (1. 4). We characterise the points

$$x_{2i} = \cos\left(i-\frac{1}{2}\right)\frac{\pi}{n}, \quad i=1, 2, \ldots, n \tag{3.1}$$

in (1. 4) as the zeros of $T_n(x)$ where

$$T_n(x) = \cos n\theta, \qquad x=\cos\theta \qquad -1 \leqq x \leqq 1 \tag{3.2}$$

is the nth Tchebycheff polynomial of first kind and

$$x_{2i+1} = \cos\frac{i\pi}{n}, \quad i=0, 1, 2, \ldots, n \tag{3.3}$$

as the zeros of $(1-x^2)\,U_{n-1}(x)$ where

$$U_{n-1}(x) = \frac{\sin n\theta}{\sin\theta}, \qquad x=\cos\theta, \qquad -1 \leqq x \leqq 1 \tag{3.4}$$

stands for the Tchebycheff polynomial of second kind. So that

$$T_n(x_{2i}) = 0, \qquad i=1, 2, \ldots, n \tag{3.5}$$

and

$$U_{n-1}(x_{2i+1}) = 0, \qquad i=1, 2, \ldots, n-1. \tag{3.6}$$

We shall prove the

Theorem II. *If n is even and the $(2n+1)$ points in $[-1, 1]$ are given* by (1. 4) *then to prescribed numbers* (1. 2) *there is a uniquely determined polynomial $g(x)$ of degree $\leqq 4n+2$ such that* (1. 3) *holds.*

4. Before proving theorem II we collect some results on Tchebycheff polynomials of first and second kind which we shall have occassion to use in the sequel.

The differential equation satisfied by $T_n(x)$ is:

$$(1-x^2)\,T_n''(x) - x\,T_n'(x) + n^2\,T_n(x) = 0 \tag{4.1}$$

and that by $U_{n-1}(x)$ is:

$$(1-x^2)\,U_{n-1}''(x) - 3x\,U_{n-1}'(x) + (n^2-1)\,U_{n-1}(x) = 0. \tag{4.2}$$

From (3. 2) and (3. 4) we have

(4. 3) $$T_n(1)=1=(-1)^n T_n(-1), \quad T_n'(1)=n^2=(-1)^{n-1} T_n'(-1)$$

(4. 4) $$U_{n-1}=n=(-1)^{n-1} U_{n-1}(-1); \quad U'_{n-1}(1)=\frac{n(n^2-1)}{3}=(-1)^n U'_{n-1}(-1).$$

Let us denote

(4. 5) $$\omega_{3n+1}(x)=\omega(x)\overset{\text{def}}{=}(1-x^2)\,T_n^2(x)\,U_{n-1}(x)$$

so that owing to (3. 5) and (3. 6)

(4. 6) $$\omega(x_i)=0, \qquad i=1, 2, \ldots, 2n+1$$
$$\omega'(x_{2i})=0, \qquad i=1, 2, \ldots, n.$$

Now

$$\omega'''(x_{2i})=6\,T_n'(x_{2i})\,T_n''(x_{2i})\left(1-x_{2i}^2\right)U_{n-1}(x_{2i})+$$
$$+6\,T_n'^2(x_{2i})\left[\left(1-x_{2i}^2\right)U'_{n-1}(x_{2i})-2x_{2i}\,U_{n-1}(x_{2i})\right]$$
$$=6T_n'(x_{2i})\left[U_{n-1}(x_{2i})\left\{\left(1-x_{2i}^2\right)T_n''(x_{2i})-2x_{2i}\,T_n'(x_{2i})\right\}+\right.$$
$$\left.+\left(1-x_{2i}^2\right)T_n'(x_{2i})\,U'_{n-1}(x_{2i})\right]$$

(4. 7) $$=6\,T_n'^2(x_{2i})\left[\left(1-x_{2i}^2\right)U'_{n-1}(x_{2i})-x_{2i}\,U_{n-1}(x_{2i})\right].$$

Now from the identity

$$\sin\theta\, U_{n-1}(\cos\theta)=\sin n\theta,$$

we hawe on differentiation

$$-\sin^2\theta\, U'_{n-1}(\cos\theta)+\cos\theta\, U_{n-1}(\cos\theta)=n\cos n\theta,$$

ie.

$$(1-x^2)\,U'_{n-1}(x)-x\,U_{n-1}(x)=-n\,T_n(x)$$

or

(4. 8) $$\left(1-x_{2i}^2\right)U'_{n-1}(x_{2i})-x_{2i}\,U_{n-1}(x_{2i})=0, \qquad i=1, 2, \ldots, n.$$

Also

(4. 9) $$T_n'(x)=n\,U_{n-1}(x).$$

Hence from (4. 7) and (4. 8) we have

(4. 10) $$\omega'''(x_{2i})=0, \qquad i=1, 2, \ldots, n.$$

We can also verify

(4. 11) $$\omega''(x_{2j})=2\,T_n'^2(x_{2j})\,U_{n-1}(x_{2j})\left(1-x_{2j}^2\right), \qquad j=1, 2, \ldots, n$$

and

(4. 12) $$\omega'(1)=-2n=(-1)^n\,\omega'(-1).$$

We shall also need the

Lemma 4. 1. *We have*

(4. 13) $$\int_1^x T_n(x)\,dx=\frac{1}{2}\left[\frac{T_{n-1}(x)}{n-1}-\frac{T_{n+1}(x)}{n+1}+\frac{2}{n^2-1}\right].$$

Proof.
$$\int T_n(x)\,dx=\int -T_n(\cos\theta)\sin\theta\,d\theta$$
$$=\int -\cos n\theta\sin\theta\,d\theta$$
$$=\frac{1}{2}\left[\frac{\cos(n+1)\theta}{n+1}-\frac{\cos(n-1)\theta}{n-1}\right]+c$$
$$=\frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1}-\frac{T_{n-1}(x)}{n-1}\right]+c.$$

Hence owing to (4. 3)
$$\int_1^x T_n(x)\,dx=\frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1}-\frac{T_{n-1}(x)}{n-1}+\frac{2}{n^2-1}\right].$$

As a corollary to this lemma we have

(4. 14)
$$\int_{-1}^{1} T_n(x)\,dx=0 \quad\text{or}\quad -\frac{2}{n^2-1}$$

according as n is odd or even.

5. *Proof of theorem* II. In order to prove theorem II we show that in case

(5. 1)
$$\begin{aligned} y_i&=0, & i&=1,2,\dots,2n+1,\\ y_i^*&=0, & i&=0,1,2,\dots,n+1,\\ y_i^{**}&=0, & i&=1,2,\dots,n, \end{aligned}$$

the only polynomial of degree $\leqq 4n+2$ which satisfy (1. 3) is $g(x)\equiv 0$[1].

Consider the polynomial

(5. 2)
$$g(x)=\omega(x)\,q_{n+1}(x)$$

where $\omega(x)$ is defined by (4. 5) and $q_{n+1}(x)$ is a fixed polynomial of degree $n+1$. Owing to (4. 6) we see that first two conditions of (5. 1) are satisfied except $g'(\pm 1)=0$. Now $g'''(x_{2i})=0$ gives

(5. 3)
$$\omega'''(x_{2i})\,q_{n+1}(x_{2i})+3\,\omega''(x_{2i})\,q'_{n+1}(x_{2i})=0$$

which owing to (4. 9) and (4. 10) gives
$$q'_{n+1}(x_{2i})=0,\quad i=1,2,\dots,n$$
ie.
$$q'_{n+1}(x)=c_1T_n(x)$$
with a numerical c_1. Hence if $c_1\neq 0$[2]
$$q_{n+1}(x)=\left[c_1\int_1^x T_n(x)\,dx+c_2\right]$$
ie. we have

(5. 4)
$$g(x)=\omega(x)\left[c_1\int_1^x T_n(x)\,dx+c_2\right].$$

We shall determine the constants c_1 and c_2 by the conditions $g'(\pm 1)=0$.

[1] The results (4. 6) and (4. 9) show that there is a non-trivial polynomial $\omega(x)$ of degree $3n+1$ which satisfies almost all the conditions (5. 1) except the two $\omega'(\pm 1)=0$.

[2] For if $c_1=0$, $g(x)=$ constant $\omega(x)$ which does not fulfil all requirments in (1. 3).

Thus

$$g'(x) = \omega'(x)\left[c_1 \int_1^x T_n(x)\,dx + c_2\right] + c_1\,\omega(x)\,T_n(x) \tag{5.5}$$

so that

$$g'(1) = 0 \text{ gives } c_2 = 0 \text{ and } g'(-1) = 0 \text{ gives}$$

$$-c_1\,\omega'(-1) \int_{-1}^{1} T_n(x)\,dx = 0. \tag{5.6}$$

For n even, on account of (4. 14) we have $c_1 = 0$. Hence $g(x) \equiv 0$ and our theorem is proved.

We incidently see owing to (4. 13) that for n odd the conditions $g'''(\pm 1) = 0$ determine only $c_2 = 0$ and then there is left a constant c_1 undetermined.

Thus in the case of infinitely many solutions in theorem I for n odd, the general form of the solution is

$$g(x) = c_1\,\omega(x) \int_1^x T_n(x)\,dx$$

or alternately the form

$$g(x) = c_1\,\omega(x)\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1} - \frac{2}{n^2-1}\right].$$

6. *The interpolatory polynomials.* In the following articles we proceed to obtain the unique polynomials defined for even values of n in theorem II i. e. to obtain the polynomials $X_n(x)$ of degree $\leqslant 4n+2$ (n even)[3] such that for prescribed numbers α_i, β_i, γ_i's

$$\begin{aligned} &X_n(x_i) = \alpha_i, \quad i = 1, 2, \ldots, 2n+1; \\ &X_n'(+1) = \beta_0, \quad X_n'(-1) = \beta_{2n+2}, \quad X_n'(x_{2i}) = \beta_i \\ &X_n'''(x_{2i}) = \gamma_i, \quad i = 1, 2, \ldots, n \end{aligned} \tag{6.1}$$

where x_i's are defined by (1. 4)[4]

For this we shall collect in § 7 some more results on Tchebycheff polynomials. In §§ 8—10 we introduce the fundamental polynomials $k(x)$, $\Lambda(x)$, $M(x)$ and establish some of their properties. Finally in § 11 we shall give the explicit form of the polynomials $X_n(x)$.

7. (a) Let

$$\lambda_{2i} = \frac{(1-x^2)\,T_n(x)}{\left(1-x_{2i}^2\right) T_n'(x_{2i})\,(x-x_{2i})}, \qquad i = 1, 2, \ldots, n \tag{7.1}$$

represent the fundamental polynomial (degree $n+1$) of Lagrange interpolation based on our x_{2j} — points in (3. 1) satisfying for $j = 1, 2, \ldots, n$

$$\lambda_{2i}(x_{2j}) = \begin{matrix} 0 \\ 1 \end{matrix} \text{ for } \begin{matrix} j \neq i \\ j = i \end{matrix}, \quad \lambda_{2i}(\pm 1) = 0. \tag{7.2}$$

[3] From now onward we shall take n to be even.

[4] Later distinguished by (3. 1) and (3. 3).

One can also see that

$$\lambda'_{2i}(x_{2j}) = \frac{\left(1-x_{2j}^2\right) T_n'(x_{2j})}{\left(1-x_{2i}^2\right) T_n'(x_{2i})(x_{2j}-x_{2i})}, \qquad j \neq i$$

(7. 3)
$$\lambda'_{2i}(x_{2i}) = -\frac{3x_{2i}}{\left(1-x_{2i}^2\right)}$$

$$\lambda''_{2i}(x_{2i}) = -\frac{x_{2i}^2}{\left(1-x_{2i}^2\right)^2} - \frac{n^2+5}{3\left(1-x_{2i}^2\right)}.$$

Also let

(7. 4)
$$\lambda_{2i+1}(x) = \frac{U_{n-1}(x)}{(x-x_{2i+1})\, U'_{n-1}(x_{2i+1})}, \qquad i = 1, 2, \ldots, n-1$$

represent the fundamental polynomial of degree $n-2$ of Lagrange interpolation based on the points x_{2j+1} $(1 \leqslant j \leqslant n-1)$ given by (3. 3). So that

(7. 5)
$$\lambda_{2i+1}(x_{2j+1}) = \begin{matrix} 0 \\ 1 \end{matrix} \text{ for } \begin{matrix} j \neq i \\ j = i \end{matrix}, \quad j = 1, 2, \ldots, n-1.$$

(b) We denote by

(7. 6)
$$p(x) \overset{\text{def}}{=} (1-x^2)\, T_n'(x) - \left[\left(1-x_{2i}^2\right) + \frac{1}{2} x_{2i}(x-x_{2i})\right] T_n'(x_{2i})\, \lambda_{2i}(x),$$

a polynomial of degree $n+2$ satisfying the conditions:

$$p(x_{2j}) = \left(1-x_{2j}^2\right) T_n'(x_{2j}), \qquad j \neq i$$

(7. 7)
$$p(x_{2i}) = p'(x_{2i}) = 0, \qquad i = 1, 2, \ldots, n.$$

So that

(7. 8)
$$\frac{p(x)}{(x-x_{2i})^2}$$

is a polynomial of degree n.

(c) Further if

(7. 9)
$$h(x) \overset{\text{def}}{=} (1-x^2)^2\, T_n'(x) -$$

$$-\left[c_3 (x-x_{2i})^2 - \frac{3}{2} x_{2i}\left(1-x_{2i}^2\right)(x-x_{2i}) + \left(1-x_{2i}^2\right)\right] T_n'(x_{2i})\, \lambda_{2i}(x)$$

where

(7. 10)
$$c_3 = -\frac{1}{12}\left[(4n^2+5)\left(1-x_{2i}^2\right) + 3\right]$$

denote a polynomial of degree $n+3$ satisfying

(7. 11)
$$h(x_{2j}) = \left(1-x_{2j}^2\right)^2 T_n'(x_{2j}), \qquad j \neq i$$

$$h(x_{2i}) = h'(x_{2i}) = h''(x_{2i}) = 0, \qquad i = 1, 2, \ldots, n.$$

Then

(7. 12)
$$\frac{h(x)}{(x-x_{2i})^3}$$

is a polynomial of degree n.

8. *The polynomials* $K_i(x)$, $1 \leqslant i \leqslant 2n+1$

(8. 1) $$K_1(x) = \frac{U_{n-1}(x)}{U_{n-1}(1)}[(2-x)\,T_n(x) - (1-x^2)\,T_n'(x)]\left[\frac{1+x}{2}T_n(x)\right]^2 + \frac{(16n^2-1)(n^2-1)}{12}\int_{-1}^{x} T_n(x)\,dx$$

(8. 2) $$K_{2n+1}(x) = \frac{U_{n-1}(x)}{U_{n-1}(-1)}[(2+x)\,T_n(x) + (1-x^2)\,T_n'(x)]\left[\frac{1+x}{2}T_n(x)\right]^2 + \frac{(16n^2-1)(n^2-1)}{12}\int_{x}^{1} T_n(x)\,dx,$$

for $1 \leqslant i \leqslant n-1$

(8. 3) $$K_{2i+1}(x) = \frac{(1-x^2)^2\,T_n^3(x)\,\lambda_{2i+1}(x)}{\left(1-x_{2i+1}^2\right)^2 T_n^3(x_{2i+1})} - \frac{n\,\omega(x)}{\left(1-x_{2i+1}^2\right)^3 T_n^3(x_{2i+1})}\left[\int_{-1}^{x}(1-x^2)\,\lambda_{2i+1}(x)\,dx + C_4\int_{-1}^{x} T_n(x)\,dx\right]$$

and for $1 \leqslant i \leqslant n$

(8. 4) $$K_{2i}(x) = \frac{U_{n-1}(x)\,\lambda_{2i}^3(x)}{U_{n-1}(x_{2i})} - \frac{\omega(x)}{U_{n-1}(x_{2i})\left(1-x_{2i}^2\right)^3 T_n'^3(x_{2i})}\times$$
$$\times\left[\int_{-1}^{x}\frac{h(x)}{(x-x_{2i})^3}\,dx + C_5\int_{-1}^{x}\frac{T_n(x)}{x-x_{2i}}\,dx + C_6\int_{-1}^{x} T_n(x)\,dx\right],$$

where

(8. 5) $$C_4 = \frac{n^2-1}{2}\int_{-1}^{1}(1-x^2)\,\lambda_{2i+1}(x)\,dx$$

(8. 6) $$C_5 = \frac{1}{24}\left(\frac{42}{1-x_{2i}^2} + 22n^2 - 15\right)x_{2i}$$

(8. 7) $$C_6 = \frac{n^2-1}{2}\left\{\int_{-1}^{1}\frac{h(x)}{(x-x_{2i})^3}\,dx + c_5\int_{-1}^{1}\frac{T_n(x)}{x-x_{2i}}\,dx\right\}$$

and $\lambda_{2i}(x)$, $\lambda_{2i+1}(x)$, $\omega(x)$ and $h(x)$ are given by (7. 1), (7. 4), (4. 5) and (7. 8) respectively.

The remark (7. 11) shows that the expressions for $K_{2i}(x)$ in (8. 4) is a polynomial of degree $4n+2$ while the expressions for $K_1(x)$, $K_{2n+1}(x)$ and $K_{2i+1}(x)$ are also polynomials each of degree $4n+2$, is obvious. We shall verify that the polynomials $K_{2i+1}(x)$, $0 \leqslant i \leqslant n$ in (8. 1), (8. 2) and (8. 3) satisfy the following conditions:

(8. 8) $$K_{2i+1}(x_{2j+1}) = {0 \atop 1} \text{ for } {j \neq i \atop j = i}; \quad K'_{2i+1}(\pm 1) = 0;$$
$$K_{2i+1}(x_{2j}) = K'_{2i+1}(x_{2j}) = K'''_{2i+1}(x_{2j}) = 0$$
$$(x_{2j+1} : 0 \leqslant j \leqslant n) \qquad (x_{2j} : 1 \leqslant j \leqslant n)$$

and the polynomials $K_{2i}(x)$ $1 \leqslant i \leqslant n$ in (8. 4) satisfy

(8. 9) $$K_{2i}(x_{2j+1}) = 0; \quad K_{2i}(x_{2j}) = {0 \atop 1} \text{ for } {j \neq i \atop j = i}; \, K'_{2i}(\pm 1) = 0,$$
$$K'_{2i}(x_{2j}) = K'''_{2i}(x_{2j}) = 0$$
$$(x_{2j+1} : 0 \leqslant j \leqslant n) \qquad (x_{2j} : 1 \leqslant j \leqslant n)$$

For the (8. 8) we start with the form (8. 3) with constant C_4 to be chosen suitabely. Due to (7. 5), (4. 6), the first two conditions of (8. 8) and $K'_{2i+1}(-1) = 0$ is satisfied at any choice of C_4 which is fixed in (8. 7) by the requirement that $K'_{2i+1}(1) = 0$. To see that $K'''_{2i+1}(x_{2j}) = 0$, we have on differentiation

$$K'''_{2i+1}(x_{2j}) = \frac{6\left(1-x_{2j}^2\right)^2 T_n'^3(x_{2j})}{\left(1-x_{2j+1}^2\right)^2 T_n'^3(x_{2j+1})} \lambda_{2i+1}(x_{2j}) - \frac{3n\,\omega''(x_{2j})\left(1-x_{2j}^2\right)}{\left(1-x_{2j+1}^2\right)^2 T_n'^3(x_{2j+1})} \lambda_{2i+1}(x_{2j})$$
$$= \frac{6\left(1-x_{2j}^2\right)^2 T_n'^2(x_{2j})}{\left(1-x_{2j+1}^2\right)^2 T_n'^3(x_{2j+1})} \{T_n'(x_{2j}) - n\,U_{n-1}(x_{2j})^2\} \lambda_{2i+1}(x_{2j}) = 0$$

by the use of (4. 9) and (4. 11).

The verification of (8. 8) for $K_1(x)$ and $K_{2n+1}(x)$ is not very difficult.

For (8. 9) we start with the form (8. 4) for $K_{2i}(x)$ with constants C_5 and C_6 to be adjusted later. As before we see that owing to (7. 2), (4. 6) the first three conditions and $K_{2i}(-1) = 0$ are satisfied at any choice of C_5 and C_6. Now for $j \neq i$ we have

$$K'''_{2i}(x_{2j}) = \frac{1}{U_{n-1}(x_{2i})} \left[U_{n-1}(x)\lambda_{2i}^3(x)\right]'''_{x=x_{2j}} - \frac{3\omega(x_{2j})}{U_{n-1}(x_{2i})\left(1-x_{2i}^2\right) T_n'^3(x_{2i})} \cdot \frac{h(x_{2j})}{(x_{2j}-x_{2i})^3}$$
$$= \frac{6\,U_{n-1}(x_{2j})\lambda_{2i}'^3(x_{2j})}{U_{n-1}(x_{2i})} - \frac{6\left(1-x_{2j}^2\right)^3 T_n'^3(x_{2j}) U_{n-1}(x_{2j})}{U_{n-1}(x_{2i})\left(1-x_{2i}^2\right)^3 T_n'^3(x_{2i})(x_{2j}-x_{2i})^3}$$
$$= 6\frac{U_{n-1}(x_{2j})}{U_{n-1}(x_{2i})} \left[\lambda_{2i}'^3(x_{2j}) - \left\{\frac{\left(1-x_{2j}^2\right) T_n'(x_{2j})}{\left(1-x_{2i}^2\right) T_n'(x_{2i})(x_{2j}-x_{2i})}\right\}^3\right] = 0.$$

The constant C_5 is determined by the condition $K'''_{2i}(x_{2i}) = 0$, $i = 1, 2, \ldots, n$. Thus we have

(8. 10) $$\left[U_{n-1}(x)\lambda_{2i}^3(x)\right]'''_{x=x_{2i}} - \frac{3\omega''(x_{2i})}{\left(1-x_{2i}^2\right)^3 T_n'^3(x_{2i})} \left[\lim_{x=x_{2i}} \frac{h(x)}{(x-x_{2i})^3} + C_5 T_n'(x_{2i})\right] = 0.$$

Now we can verify that

(8. 11) $$\left[U_{n-1}(x)\lambda_{2i}^3(x)\right]'''_{x=x_{2i}} = \frac{3}{4} x_{2i} \left[\frac{17}{\left(1-x_{2i}^2\right)^3} + \frac{20n^2+6}{\left(1-x_{2i}^2\right)^2}\right] U_{n-1}(x_{2i})$$

and

(8. 12) $$\lim_{x=x_{2i}} \frac{h(x)}{(x-x_{2i})^3} = \frac{1}{12} x_{2i} \left[\frac{4n^2+9}{2} - \frac{3}{1-x_{2i}^2}\right] T_n'(x_{2i}).$$

Therefore from (8. 10), (8. 11) and (8. 12) we at once get the value of C_5 in (8. 6). Lastly with the known value of C_5 the condition $K_{2i}'(1)=0$ gives C_6 as given in (8. 7).

9. *The polynomials* $\Lambda_{2i}(x)$, $0\leqslant i\leqslant n+1$

(9. 1) $$\Lambda_0(x)=-\frac{n^2-1}{4n}\,\omega(x)\int_{-1}^{x} T_n(x)\,dx$$

(9. 2) $$\Lambda_{2n+2}(x)=-\frac{n^2-1}{4n}\,\omega(x)\int_{x}^{1} T_n(x)\,dx$$

and for $1\leqslant i\leqslant n$,

(9. 3) $$\Lambda_{2i}(x)=\frac{T_n(x)\,U_{n-1}(x)}{T_n'(x_{2i})\,U_{n-1}(x_{2i})}\,\lambda_{2i}^2(x)-$$

$$-\frac{\omega(x)}{\left(1-x_{2i}^2\right)^2 T_n'^3(x_{2i})\,U_{n-1}(x_{2i})}\left[\int_{-1}^{x}\frac{p(x)}{(x-x_{2i})^2}dx+C_7\int_{-1}^{x}\frac{T_n(x)}{x-x_{2i}}\,dx+C_8\int_{-1}^{x}T_n(x)\,dx\right],$$

where

(9. 4) $$C_7=\frac{1}{2}\left[\frac{1}{1-x_{2i}^2}+\frac{(2n-7)(n+2)}{3}\right],$$

(9. 5) $$C_8=\frac{n^2-1}{2}\left[\int_{-1}^{1}\frac{p(x)}{(x-x_{2i})^2}\,dx+C_7\int_{-1}^{1}\frac{T_n(x)}{x-x_{2i}}\,dx\right]$$

and $\lambda_{2i}(x)$, $p(x)$ and $\omega(x)$ are given by (7. 1), (7. 6) and (4. 5) respectively.

On account of the remark made in (7. 8), the expressions in (9. 3) and also in (9. 1) and (9. 2) are polynomials each of degree $4n+2$.

We can see after an easy calculation and using (4. 14) that the polynomials $\Lambda_0(x)$ and $\Lambda_{2n+2}(x)$ verify the conditions:

(9. 6) $$\Lambda_0(x_j)=0;\quad \Lambda_0'(1)=1,\quad \Lambda_0'(-1)=0;\quad \Lambda_0'(x_{2j})=\Lambda_0'''(x_{2j})=0;$$

$$\Lambda_{2n+2}(x_j)=0;\quad \Lambda_{2n+2}'(-1)=1,\quad \Lambda_{2n+2}'(1)=0;$$

$$\Lambda_{2n+2}'(x_{2j})=\Lambda_{2n+2}'''(x_{2j})=0$$

$$(x_j:1\leqslant j\leqslant 2n+1)\qquad (x_{2j}:1\leqslant j\leqslant n).$$

In the following we shall show that the polynomials $\Lambda_{2i}(x)$, $1\leqslant i\leqslant n$ in (8. 3) fulfil the requirements:

(9. 7) $$\Lambda_{2i}(x_j)=0,\quad \Lambda_{2i}'(\pm 1)=0;\quad \Lambda_{2i}'(x_{2j})=\begin{matrix}0\\1\end{matrix}\ \text{for}\ \begin{matrix}j\neq i\\ j=i\end{matrix};$$

$$\Lambda_{2i}'''(x_{2j})=0$$

$$(x_j:1\leqslant i\leqslant 2n+1)\qquad (x_{2j}:1\leqslant j\leqslant n).$$

The third is seen to be true on account of (7. 2) and (4. 6). Now for $j\neq i$

$$\Lambda_{2i}'''(x_{2j})=\frac{\left[T_n(x)\,U_{n-1}(x)\,\lambda_{2i}^2(x)\right]'''_{x=x_{2j}}}{T_n'(x_{2i})\,U_{n-1}(x_{2i})}-\frac{3\,\omega''(x_{2j})}{\left(1-x_{2i}^2\right)^2 T_n'^3(x_{2i})\,U_{n-1}(x_{2i})}\cdot\frac{p(x_{2j})}{(x_{2j}-x_{2i})^2}$$

$$=\frac{6\,U_{n-1}(x_{2j})\,T_n'(x_{2j})\,\lambda_{2i}'^2(x_{2j})}{T_n'(x_{2i})\,U_{n-1}(x_{2i})}-\frac{6\left(1-x_{2j}^2\right)^2 T_n'^3(x_{2j})\,U_{n-1}(x_{2j})}{\left(1-x_{2i}^2\right)^2 T_n'^3(x_{2i})\,U_{n-1}(x_{2i})\,(x_{2j}-x_{2i})^2}$$

$$=\frac{6\,U_{n-1}(x_{2j})\,T_n'(x_{2j})}{T_n'(x_{2i})\,U_{n-1}(x_{2i})}\left[\lambda_{2i}'^2(x_{2j})-\left\{\frac{\left(1-x_{2j}^2\right)T_n'(x_{2j})}{\left(1-x_{2i}^2\right)T_n'(x_{2i})\,(x_{2j}-x_{2i})}\right\}^2\right]$$

$$=0,\quad j\neq i.$$

at any choice of C_7 and C_8 , on using (4. 11), (7. 7) and (7. 3).

$\Lambda_{2i}'''(x_{2i})=0$ gives on account of (4. 11),

$$(9.8)\qquad \left(1-x_{2i}^2\right)\left[T_n(x)\,U_{n-1}(x)\,\lambda_{2i}^2(x)\right]'''_{x=x_{2i}}-\left[\lim_{x=x_{2i}}\frac{p(x)}{(x-x_{2i})^2}+C_7\,T_n'(x_{2i})\right]=0.$$

The value of C_7 is at once determined when one calculates and simplifies:

$$(9.9)\qquad \left[T_n(x)\,U_{n-1}(x)\,\lambda_{2i}^2(x)\right]'''_{x=x_{2i}}=-\left[\frac{3x_{2i}^2}{2\left(1-x_{2i}^2\right)^2}+\frac{2n^2+3}{1-x_{2i}^2}\right]T_n'(x_{2i})\,U_{n-1}(x_{2i})$$

and

$$\lim_{x=x_{2i}}\frac{p(x)}{(x-x_{2i})^2}=\frac{1}{4}\left[\frac{3}{1-x_{2i}^2}-\frac{4n^2+5}{3}\right]T_n'(x_{2i}).$$

The condition $\Lambda_{2i}'(-1)=0$ holds at any choice of C_8 while $\Lambda_{2i}'(1)=0$ gives C_8 as in (9. 5).

10. *The polynomials* $M_{2i}(x)$, $1\leqslant i\leqslant n$

$$(10.1)\qquad M_{2i}(x)=\frac{\omega(x)}{\omega''(x_{2i})\,T_n''(x_{2i})}\left[\int_{-1}^{x}\frac{T_n(x)}{x-x_{2i}}\,dx+C_9\int_{-1}^{x}T_n(x)\,dx\right]$$

where

$$(10.2)\qquad C_9=\frac{n^2-1}{2}\int_{-1}^{1}\frac{T_n(x)}{x-x_{2i}}\,dx$$

and $\omega(x)$ is given by (4. 5).

For these polynomials of degree $4n+2$, it is easily verified that

$$M_{2i}(x_j)=0;\quad M_{2i}'(\pm 1)=0;\quad M_{2i}'(x_{2j})=0,$$

$$(10.3)\qquad M_{2i}'''(x_{2j})=\begin{matrix}0\\1\end{matrix}\ \text{for}\ \begin{matrix}j\neq i\\ j=i\end{matrix}.$$

$$(x_j:1\leqslant j\leqslant 2n+1)\qquad (x_{2j}:1\leqslant j\leqslant n.)$$

We shall omit the details.

11. *The polynomials* $X_n(x)$. The fundamental polynomials determined in §§ 8—10 very easily lead to the explicit expression of the required interpolatory polynomials $X_n(x)$ in (6. 1). Thus we have:

$$X_n(x) = \sum_{i=1}^{2n+1} \alpha_i K_i(x) + \sum_{i=0}^{n+1} \beta_i \Lambda_{2i}(x) + \sum_{i=1}^{n} \gamma_i M_{2i}(x). \tag{11. 1}$$

Where $K_i(x)$, $\Lambda_{2i}(x)$ and $M_{2i}(x)$ are the fundamental polynomials each of degree $4n+2$ given by (8. 1) — (8. 4), (9. 1) — (9. 3) and (10. 1) respectively. Owing to the properties of fundamental polynomials established in (8. 8), (8. 9); (9. 6), (9. 7), (10. 3) and the uniqueness theorem, the only polynomials of degree $\leq 4n+2$ satisfying (6. 1) are $X_n(x)$ given by (11. 1).

The convergence behavior of the sequence of polynomials $X_n(x)$ will be dealt in the next comunication.

REFERENCES

[1] R. B. Saxena: *On mixed type lacunary interpolation* — I. Ganita, Vol. 11 (1960) pp. 65—81.

[2] R. B. Saxena: *On mixed type lacunary interpolation* — II. Acta Math. Vol. 14 (1963), pp. . .

[3] R. B. Saxena: *On some interpolatory properties of Legendre polynomials.*

[4] J. Surnya and P. Turan: *Notes on interpolation* — I. Acta Math. Acad. Sci. Hung., 6 (1955), pp. 66—79.

[5] A K. Varma: *Some interpolatory properties of Tchebycheff polynomials,* (0, 1, 3) *case.* Duke Math. Jour. Vol. 28 (1961), pp. 449—462.

[6] A. K. Varma: *Some interpolatory properties of Tchebycheff polynomials.* Pub. Math. Vol. 8 (1961), pp. 336—349.