SOME APPLICATIONS OF A LEMMA ON FOURIER SERIES

by

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My interest in the problems I am going to consider has arisen from the study of some papers by Erdös and Turán [1—4]. It is my intention to show that some of their theorems can be derived from a simple lemma on the Fourier coefficients of bounded integrable functions. By formulating the results in terms of potential theory we obtain the theorems in a more general setting. Some potential-theoretic aspects of the Erdös-Turán theorem are also given in an interesting paper [7] by Rosenbloom.

The class of functions to be considered consists of the functions of period 2π which are real, bounded and integrable. If V is a function of this kind, we shall use the notations

$$V_m=(2\pi)^{-1}\int_0^{2\pi} e^{-im\tau}\, V(\tau)\, d\tau$$

and

$$\omega(\delta)=\omega(\delta; V)=\sup_{t\leqq\tau\leqq t+\delta}\{V(\tau)-V(t)\}.$$

If a function h is integrable we shall also use the notation

$$\|h\| = (2\pi)^{-1}\int_0^{2\pi} |h(\tau)|\, d\tau.$$

The letter C will denote a constant not necessarily the same each time but every theorem of this paper is true if we put $C=100$.

LEMMA. *If V is a real integrable function of period 2π and if $n>0$ is an integer, then*

$$\sup_t |V(t)| \leqq C\left\{\sum_{m=0}^{n-1}(1-m/n)\,|V_m| + \omega(n^{-1}; V)\right\}.$$

The most interesting special case that will be applied below is obtained by assuming that $V(t)-Kt$ is non-increasing for some K. Then $\omega(n^{-1}; V) \leqq K n^{-1}$ and the result follows from theorem IV in my paper [6]. Here I give another proof which is based on the same idea as the proof of theorem III by Erdös and Turán [2].

Let $\{\sigma_n\}$ denote the arithmetic means of the partial sums of the Fourier series for V so that

$$\sigma_n(t) = V_0 + \sum_{m=0}^{n-1} (1-m/n)\ (V_m e^{imt} + V_{-m} e^{-imt}).$$

Then

$$\sigma_n(t) = \int_0^{2\pi} \psi_n(\tau)\, V(t-\tau)\, d\tau,$$

where ψ_n is the Fejér kernel,

$$\psi_n(t) = (2\pi n)^{-1} \left(\sin \frac{1}{2} t\right)^{-2} \sin^2 \frac{1}{2} nt.$$

We observe that

$$\int_\delta^{2\pi-\delta} \psi_n(\tau)\, d\tau \leqq (\pi n)^{-1} \int_\delta^{\pi} \left(\sin \frac{1}{2}\tau\right)^{-2} d\tau \leqq 4\,(\pi n \delta)^{-1},$$

if $0 < \delta < \pi$. Since $\int_0^{2\pi} \psi_n(\tau)\, d\tau = 1$, we get

$$\int_{-\delta}^{\delta} \psi_n(\tau)\, d\tau \geqq 1 - 4\,(\pi n \delta)^{-1}.$$

Put $W = \sup_t |V(t)|$ and suppose that $W = \sup_t V(t)$. The other possibility $W = -\inf_t V(t)$ can be treated in a similar way. To every $\varepsilon > 0$ we can find a T such that $V(T) > W - \varepsilon$. Then

$$V(t) > W - \varepsilon - \omega(2\delta) \text{ if } T - 2\delta \leqq t \leqq T.$$

Since $\psi_n \geqq 0$ it follows that

$$\sigma_n(T-\delta) =$$
$$= \int_0^{2\pi} \psi_n(\tau)\, V(T-\delta-\tau)\, d\tau > (W - \varepsilon - \omega(2\delta)) \int_{-\delta}^{\delta} \psi_n(\tau)\, d\tau - W \int_\delta^{2\pi-\delta} \psi_n(\tau)\, d\tau.$$

We now choose $\delta = 16(\pi n)^{-1}$. It may be supposed that $n > 1$ since the conclusion is trivially valid in the case $n=1$ as the inequality $W \leqq |V_0| + \omega(2\pi) \leqq |V_0| + 7\omega(1)$ shows. If $n > 1$, then $\delta < \pi$ and we can use the estimates for the integrals of ψ_n given above. We must assume that $W - \varepsilon - \omega(2\delta) \geqq 0$, but that can be done since the contrary assumption immediately implies the lemma. We find that

$$\sigma_n(T-\delta) > \frac{3}{4}(W - \varepsilon - \omega(2\delta)) - \frac{1}{4}W,$$

and hence

$$W < 2\sigma_n(T-\delta) + 3\omega(\delta) + 2\varepsilon.$$

Now ε is arbitrary and by aid of simple inequalities we find that

$$W \leqq 4\sum_{m=0}^{n-1}(1-m/n)|V_m| + 18\omega(n^{-1}),$$

and the lemma is proved.

THEOREM 1. *Let μ be a positive measure on the unit circle, with total mass* 1, *and put*

$$c_m = (2\pi)^{-1}\int_0^{2\pi} e^{-im\tau}\, d\mu(\tau).$$

Let ν be the ordinary arc measure divided by 2π and let γ be an arc of the unit circle. For every integer $n > 1$ it holds that

$$|\mu(\gamma) - \nu(\gamma)| \leqq C\left(n^{-1} + \sum_{m=1}^{n-1}(1-m/n)\, m^{-1}|c_m|\right).$$

To prove this theorem we introduce the function

$$V(t) = \nu(t) - \mu(t) - A,$$

where $\nu(t)$ and $\mu(t)$ denote the masses on the arc $0 < \tau \leqq t$, and A is a constant such that

$$2\pi V_0 = \int_0^{2\pi} V(\tau)\, d\tau = 0.$$

Evidently

$$\omega(\delta;\, V) \leqq \nu(t+\delta) - \nu(t) = (2\pi)^{-1}\delta,$$

and for $m > 0$ it holds that

$$V_m = (2\pi)^{-1} \int_0^{2\pi} e^{-im\tau} V(\tau)\, d\tau = (2\pi i m)^{-1} \int_0^{2\pi} e^{-im\tau} dV(\tau) = -(im)^{-1} c_m .$$

Application of the lemma gives

$$\sup_t |V(t)| \leqq C \left\{ \sum_{m=0}^{n-1} (1 - m/n)\, m^{-1} |c_m| + (2\pi n)^{-1} \right\}.$$

If the end-points of γ are e^{it_1} and e^{it_2}, then

$$|\mu(\gamma) - \nu(\gamma)| = |V(t_1) - V(t_2)| \leqq 2 \sup_t |V(t)|,$$

and theorem 1 is proved.

If μ consists of N equal point-masses we obtain a special case which has been considered by Erdös and Turán [2, theorem III]. Their theorem implies Weyl's well-known criterion for uniform distribution and is a finite counterpart of that theorem.

Our further results can be obtained as special cases of the following variant of the lemma. In this formulation our theorems may be considered as a kind of tauberian remainder theorems for periodic functions.

THEOREM 2. *Let V be a real periodic function of bounded variation, and let k be a real periodic integrable function with the Fourier coefficients* $k_m = (2\pi)^{-1} \int_0^{2\pi} e^{-im\tau} k(\tau)\, d\tau$. *Suppose $k_m \neq 0$ if $m \neq 0$. Let us consider the convolution*

$$h(t) = \int_0^{2\pi} k(t-\tau)\, dV(\tau),$$

which evidently belongs to L. If n is a positive integer, then

$$\sup_t |V(t) - V_0| \leqq C \left\{ \|h\| \sum_{m=1}^{n-1} |mk_m|^{-1} + \omega(n^{-1}; V) \right\}.$$

The proof is immediate by aid of our lemma. We can change the order of integration by absolute convergence and we obtain

$$h_m = (2\pi)^{-1} \int_0^{2\pi} e^{-im\tau} h(\tau)\, d\tau = 2\pi i m k_m V_m .$$

Since $|h_m| \leqq \|h\|$ we get

$$|V_m| \leqq (2\pi)^{-1} |mk_m|^{-1} \|h\|.$$

Application of the lemma to $V(t)-V_0$ gives the conclusion of theorem 2.

The following result comes out if we choose $k(t)=\log|2\sin\frac{1}{2}t|$ in theorem 2, but we prefer to prove it more directly.

THEOREM 3. *Let V be a real periodic function with mean value zero and suppose that $\omega(\delta; V) \leqq K\delta$ for all δ. If U is a conjugate function of V and if we put $H=\|U\|$, then*

$$\sup_t |V(t)| \leqq C(HK)^{\frac{1}{2}}.$$

Our assumptions imply that V has bounded variation over a period. It follows that the conjugate functions belong to L so that $H<\infty$. The simple relations between the Fourier coefficients of conjugate functions show that $|V_m|=|U_m| \leqq H$. Hence our lemma gives

$$|V(t)| \leqq C\{(n-1)H+Kn^{-1}\}.$$

We take $n=1+[(K/H)^{1/2}]$ and obtain

$$|V(t)| \leqq C(HK)^{1/2}.$$

COROLLARY. (cf. Ganelius [5, p. 18]). *Suppose that $f(z)=u+iv$ is an analytic function regular for $|z|<1$ and that $f(0)=0$. If $(z=\rho e^{it})$*

$$u<H,\ \partial v/\partial t<K,\ \textit{for}\ \rho<1,$$

then

$$|v|<C(HK)^{1/2}\ \textit{for}\ \rho<1.$$

The corollary follows from theorem 3 if we consider $u(\rho e^{it})$ and $v(\rho e^{it})$ as functions of t. We put $u^+=\max(u,0)$ and $u^-=\min(u,0)$ Let the norm sign denote integration with respect to t. Since $u<H$ and $\int_0^{2\pi} u(\rho e^{it})\,dt=0$, we find that $-\int_0^{2\pi} u^-(\rho e^{it})\,dt=\int_0^{2\pi} u^+(\rho e^{it})\,dt<2\pi H$, and hence $\|u(\rho e^{it})\|<2H$. That $\omega(\delta; v) \leqq K\delta$ follows from $\partial v/\partial t<K$.

THEOREM 4. *Let ρ, t be polar coordinates and let $V(\rho, t)$ be a harmonic function regular for $\rho < 1$. Let r be a fixed number, $0 < r < 1$. If $\partial V/\partial t < 1$ for $\rho < 1$, and if*

$$(2\pi)^{-1} \int_0^{2\pi} |V(r, t)|\, dt \leqq H,$$

then

$$|V(\rho, t)| \leqq C(\log^+ (1/H))^{-1} \log (2/r) \text{ for } \rho < 1.$$

If $H \geqq 1$, then we interprete the right side of the conclusion as $+\infty$. and the result is trivial.

For the proof we use the following variant of Poisson's formula which is easily proved by calculation of the Fourier coefficients of the expression on the right:

$$V(r, t) - V(0) = \pi^{-1} \int_0^{2\pi} \arg (1 - e^{i(t-\tau)} r \rho^{-1})\, dV(\rho, \tau).$$

Putting $k(t) = \pi^{-1} \arg (1 - e^{it} r \rho^{-1})$ we find that the Fourier coefficients satisfy $2\pi |mk_m| = (r/\rho)^m$.

Since our assumptions on V also imply that $|V_0| = |V(0)| \leqq H$, it follows from theorem 2 that

$$|V(\rho, t)| \leqq C\left(H \sum_{m=0}^{n-1} (\rho/r)^m + n^{-1}\right) \leqq C n^{-1} (1 + Hn^2 r^{-(n-1)}),$$

and since $n^2 \leqq 2^{n+1}$ we have

$$|V(\rho, t)| \leqq C n^{-1} (1 + H(2/r)^{n-1}).$$

If we take $n = [(\log (2/r))^{-1} \log^+ (1/H)] + 1$, then $H(2/r)^{n-1} \leqq 1$, and the conclusion of theorem 4 follows.

If we substitute the conjugate kernel $\pi^{-1} \log |1 - e^{it} r/\rho|$ for k in the proof we obtain the following

COROLLARY. *Let U be a conjugate harmonic function to V. If the assumption $\| V(r, \cdot) \| \leqq H$ in theorem 4 is replaced by $\| U(r, \cdot) \| \leqq H$, $V(0) = 0$, then the conclusion of that theorem is still valid.*

That the estimate for $|V(\rho, t)|$ given in theorem 4 cannot be essentially improved is seen from the following example. Consider the functions $\{V_m\}_{m=1}^{\infty}$, where $V_m(\rho, t) = m^{-1}\rho^m \cos mt$. Then $H_m = \| V_m(r, \cdot)\| = (2/\pi) m^{-1} r^m$, further $\sup_t V_m(1, t) = m^{-1}$ and $\partial V/\partial t < 1$. Now

$$\log(1/H_m) \sup_t |V_m(1, t)| \to \log(1/r)$$

if m tends to infinity.

The next two theorems are formulations in terms of potential theory of theorem 3 and the corollary of theorem 4.

THEOREM 5. *Let μ be a positive mass-distribution of total mass 1 on the unit circle E. Denote the potential of μ by u and let ν be the equilibrium distribution of the unit mass on E, i. e. the ordinary arc measure divided by 2π. Let γ be an arc of E. Then*

$$|\mu(\gamma) - \nu(\gamma)| \leqq C |\inf_E u|^{\frac{1}{2}}.$$

We introduce polar coordinates ρ, t with vertex in the center of the circle. If $\rho < 1$, then

$$u(\rho, t) = -\int_0^{2\pi} \log|\rho e^{it} - e^{i\tau}| \, d\mu(\tau) = \int_0^{2\pi} \log|1 - \rho e^{i(t-\tau)}| \, d\xi(\tau),$$

where $\xi = \nu - \mu$.

We now apply theorem 2 to this convolution. The Fourier coefficients k_m of $\log|1 - \rho e^{it}|$ satisfy $|2mk_m| = \rho^{+m}$ and $\omega(n^{-1}; \xi) \leqq \omega(n^{-1}; \nu) = (2\pi n)^{-1}$. Hence

$$\sup_t |\xi(t) - \xi_0| \leqq C\left\{\|u(\rho, \cdot)\| \sum_{m=1}^{n-1} \rho^{-m} + n^{-1}\right\}.$$

If the end-points of γ are e^{it_1} and e^{it_2}, then

$$|\mu(\gamma) - \nu(\gamma)| = |\xi(t_1) - \xi(t_2)| \leqq 2 \sup |\xi(t) - \xi_0|$$
$$\leqq 2C\left\{\|u(\rho, \cdot)\| \sum_{m=1}^{n-1} \rho^{-m} + n^{-1}\right\}.$$

Since $\int_0^{2\pi} u(\rho, t)\, dt = 0$, it follows as in the corollary of theorem 3 that

$$\|u(\rho, \cdot)\| \leqq 2 |\inf_t u(\rho, t)| \leqq 2H,$$

if $H = -\inf_E u$. Letting ρ tend to 1, we find that

$$|\mu(\gamma) - \nu(\gamma)| \leqq C((n-1)H + n^{-1})$$

for every integer $n > 0$. The cho ce $n = 1 + [H^{-1/2}]$ yields the bound given in the theorem.

The following result is proved by obvious modifications in the last lines of the proof above.

THEOREM 6. *Under the assumptions of theorem* 5 *and if* $0 < r < 1$ *it holds that*

$$|\mu(\gamma) - \nu(\gamma)| \leqq C\,|\log^-(-\inf_t u(r,t))|^{-1} \log(2/r).$$

If theorems 5 and 6 are specialized by the assumption that μ consists of N point-masses of weight N^{-1}, we obtain two theorems on the zeros of polynomials given by Erdös and Turán [2, 3, 4]. More precisely we obtain the corresponding theorems in the crucial case when all the zeros are on the unit circle.

This is easily seen if we consider the potential u_0 of a mass-distribution μ_0 with point-masses N^{-1} in the points $\{e^{i\tau_k}\}_{n=1}^{N}$ on the unit circle. Then

$$u_0(z) = -\int_0^{2\pi} \log|z - e^{i\tau}|\, d\mu_0(\tau) = -N^{-1} \log|P_N(z)|,$$

where

$$P_N(z) = \prod_{k=1}^{N} (z - e^{i\tau_k}).$$

It follows that

$$-\inf_{|z|=1} u_0(z) = N^{-1} \log \operatorname{Max}_{|z|=1} |P_N(z)|,$$

and if this expression is introduced in our two theorems we get the theorems of Erdös and Turán.

Our last theorem deals with mass-distributions on the interval $I = [-1, 1]$. (The corresponding theorem by Erdös and Turán is given in [1].)

In this case the equilibrium distribution ν satisfies $\nu(\gamma) = \pi^{-1}(\arccos x_1 - \arccos x_2)$ if $\gamma = [x_1, x_2]$ and $0 \leqq \arccos x \leqq \pi$.

THEOREM 7. *Let* μ *be a positive distribution of the mass* 1 *on* I *and let* u *denote the potential of* μ. *Then*

$$|\mu(\gamma)-\nu(\gamma)|\leqq C|\log c+\inf_{I} u|^{\frac{1}{2}},$$

where $c=1/2$ *is the capacity of* I.

This result will be deduced from theorem 5. We define a mass-distribution μ^* on the unit circle in the following way. If $0<\varphi\leqq\psi<\pi$, then we prescribe that if the arc γ^* has the end-points $e^{i\varphi}$ and $e^{i\psi}$, then $\mu^*(\gamma)=1/2\,\mu(\gamma)$, where $\gamma=(\cos\psi,\ \cos\varphi)\subset I$. The definition is completed on the lower half of the unit circle so that μ will be symmetric with respect to the real axis. Corresponding to ν we get $\nu^*(\gamma^*)=(2\pi)^{-1}(\psi-\varphi)$. Then

$$u^*(\rho, t)=-\int_0^{2\pi}\log|\rho\, e^{it}-e^{i\tau}|\,d\mu^*(\tau)=$$

$$=-\int_0^{\pi}\log|(\rho\, e^{it}-e^{i\tau})\ (\rho\, e^{it}-e^{-i\tau})|\,d\mu^*(\tau),$$

and

$$2\,u^*(1,t)=-\int_0^{\pi}\log(2|\cos t-\cos\tau|)\,d\mu^*(\tau)=\log\frac{1}{2}+u(\cos t).$$

Hence

$$2\inf_{t} u^*(1,t)=\log\frac{1}{2}+\inf_{I} u,$$

and theorem 7 follows from theorem 5.

Theorems 5 and 6 deal with distributions on the unit circle. In fact similar theorems are true for every suitably regular curve in a formulation indicated in theorem 7. The proof of this result is postponed, since the simple tools of the present paper are not quite sufficient for that purpose.

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