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ON THE ALPHA-MODEL OF TURBULENCE

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Summary. One considers the structure of stellar discs with an α-model of turbulence. One obtains the equation of the energetic balance for turbulent pulsations and shows that $\alpha(r)$ is, in the central and peripheral parts of the disc, a monotonous, slowly increasing and bounded function. Quick changes occur in the narrow band of the disc near the star's surface.

T. Angelov, O ALFA-MODELU TURBULENCIJE — Razmatra se struktura zvezdanih diskova sa α-modelom turbulencije. Izvodi se jednačina energetske ravnoteže za turbulentne pulsacije i pokazuje se da je $\alpha(r)$ — u centralnom i perifernom delu diska — monotona, sporo rastuća i ograničena funkcija; brze promene dešavaju se u uskoj oblasti diska blizu površine zvezde.

1. INTRODUCTION

Since 1973. — with the theory of disc accretion assuming a turbulent mechanism (Shakura, Sunyaev, 1973) — theoretical models of discs were built up under the assuption of a constant ratio between the velocity of turbulent pulsations and the velocity of sound: in every point of the disc $v_t = \alpha v_s$, $\alpha = \text{const.}$ (Bath and Pringle, 1981, with endpapers). It is natural, of course, to take $\alpha = \alpha(r)$, but the theory and the observations did not allow, as yet, to determine that function.

In this paper the problem of the determination of $\alpha(r)$ is formulated, and the equation of energetic balance for turbulent pulsations obtained. From it the behaviour of $\alpha(r)$ is studied, in a rough approximation, within the α-model.

2. BASIC EQUATIONS

One considers the disc-approximation of the stellar envelope, so that we use a cylindrical system with the origin in the star's centre: the *r*-axis is in the orbital plane of the envelope (the equatorial plane of the star), the z-axis being taken in the sense of rotation of the star. The equation of the conservation of mass and the equation of motion for the φ-coordinate (Angelov, 1981b), in the case of axial symmetry, are:

$$\frac{\partial \sigma}{\partial t}+\frac{1}{r}\frac{\partial r \sigma u}{\partial r}=0, \tag{2.1}$$

$$\sigma\frac{\partial p}{\partial t}+\sigma u\frac{\partial p}{\partial r}=\frac{1}{r}\frac{\partial r^2 N_{r\varphi}}{\partial r}, \tag{2.2}$$

where σ is the surface density of the gas; $p = r\, v$ — the specific angular momentum; u, v — the velocity components (v_r, $_\varphi$); $N_{r\varphi}=\eta\, r\frac{\partial \omega}{\partial}$ — the component of the viscous stress tensor; $\eta=\sigma\nu$; ν — the kinematical viscosity coefficient; ω — the angular velocity.

Every unknown function in (2.1) and (2.2) will be represented as the sum of its average value and the pulsating component: $f=<f>+f'$. After a simple transformation in the equations obtained and the subtraction of corresponding averaged equations, one obtains:

$$\frac{\partial \sigma'}{\partial t}+\frac{\partial(\sigma u)'}{\partial r}+\frac{(\sigma u)'}{r}=0, \tag{2.3}$$

$$\frac{\partial(\sigma p)'}{\partial t}+\frac{\partial(\sigma u p)'}{\partial r}=-\frac{1}{r}(\sigma u p)'+\frac{1}{r}\frac{\partial r^2 N'_{r\varphi}}{\partial r}. \tag{2.4}$$

For arbitrary functions: f, g, h, we make use of developments:

$$(fg)'=<f>g'+<g>f'+f'g'-<f'g'>,$$
$$(fgh)'=<fg>h'+<h>(fg)'+(fg)'h'-<(fg)'h'>. \tag{2.5}$$

Equation (2.4) can be written, with the help of (2.3) and (2.5), as

$$<\sigma>\frac{\partial p'}{\partial t}+<\sigma u>\frac{\partial p'}{\partial r}=-(\sigma u)'\frac{\partial <p>}{\partial r}+\frac{1}{r}\frac{\partial r^2 N'_{r\varphi}}{\partial r}-$$
$$-(\sigma u)'\frac{\partial p'}{\partial r}-\sigma'\frac{\partial <p>}{\partial t}-\sigma'\frac{\partial p'}{\partial t}+\frac{\partial <\sigma' p'>}{\partial t}+\frac{1}{r}\frac{\partial r<(\sigma u)' p'>}{\partial r}. \tag{2.6}$$

Let us multiply (2.6) with 2p', then average and transform it by (2.3). One obtains:

$$<\sigma>\frac{\partial <p'p'>}{\partial t}+<\sigma u>\frac{\partial <p'p'>}{\partial r}=-2<(\sigma u)'p'>\frac{\partial <p>}{\partial r}-$$
$$-2r<N'_{r\varphi}\frac{\partial p'}{\partial r}>+\frac{1}{r}\frac{\partial}{\partial r}\left\{r^3\left[2<v'N'_{r\varphi}>-<(\sigma u)'v'v'>\right]\right\}-$$
$$-2<\sigma'p'>\frac{\partial <p>}{\partial t}-\frac{\partial <\sigma'p'p'>}{\partial t}. \tag{2.7}$$

The estimate of some expressions in (2.7) is:

$$<(\sigma u)'p'> \approx <\sigma><u'p'>, \quad <N_r{}'_\varphi \frac{\partial p'}{\partial r}> \approx r<\eta>< \frac{\partial \omega'}{\partial r} \cdot \frac{\delta p'}{\delta r}>,$$

$$<\upsilon' N_r{}'_\varphi> \approx <\eta><p' \frac{\partial \omega'}{\partial r}>, \quad <(\sigma u)'\upsilon'\upsilon'> \approx <\sigma><u'\upsilon'\upsilon'>.$$

If we write $<\sigma u>/<\sigma> \equiv <u>$ and put f instead of $<f>$, equation (2.7) for steady motion becomes, having in mind $<\eta> = <\sigma>\nu$:

$$\sigma u \frac{\partial <p'p'>}{\partial r} = -2r^2\sigma\left\{<u'\omega'>\frac{\partial p}{\partial r} + \nu < \frac{\partial \omega'}{\partial r} \cdot \frac{\partial p'}{\partial r}>\right\} +$$

$$+ \frac{1}{r} \cdot \frac{\partial}{\partial r}\left\{r^5\sigma\left[\nu \frac{\partial <\omega'\omega'>}{\partial r} - <u'\omega'\omega'>\right]\right\}. \tag{2.8}$$

The equation obtained is a form of the equation of energy balance for turbulent pulsations. The first and second terms on the right hand side give the generation and dissipation of turbulent energy and the last two terms — the diffusion of turbulence.

3. THE DETERMINATION OF $\alpha(r)$

We shall make use of (2.8) for the determination of $\alpha(r)$ in the α-model of turbulence:

$$\upsilon_t = \alpha \upsilon_s (\alpha < 1), \quad L = \gamma r (\gamma = \upsilon_s | \upsilon_\varphi < 1), \quad \nu_t = L \upsilon_t. \tag{3.1}$$

We assume, in a rough approximation, the equilibrium between the generation of turbulence and the turbulent dissipation:

$$<u'\omega'> \frac{\partial p}{\partial r} + \nu < \frac{\partial \omega'}{\partial r} \cdot \frac{\partial p'}{\partial r}> = 0. \tag{3.2}$$

Since $\partial \upsilon / \partial r \sim \sqrt{<\upsilon'^2>}/L$, we have:

$$\frac{\partial \omega}{\partial r} \sim \frac{1}{r} \frac{\partial \upsilon}{\partial r}, \quad \frac{\partial p}{\partial r} \sim r^2 \frac{\partial \omega}{\partial r}, \quad <u'\omega'> = \frac{1}{r} <u'\upsilon'> \sim \frac{1}{r} \upsilon_t^2. \tag{3.3}$$

Moreover:

$$< \frac{\partial \omega'}{\partial r} \cdot \frac{\partial p'}{\partial r}> = <\left(\frac{\partial \upsilon'}{\partial r}\right)^2 - \left(\frac{\upsilon'}{r}\right)^2> \sim \left(\frac{\upsilon_t}{L}\right)^2\left[1 - \left(\frac{L}{r}\right)^2\right],$$

so that for $L \ll r$: $\nu < \frac{\partial \omega'}{\partial r} \cdot \frac{\partial p'}{\partial r}> \sim \upsilon_{ef}\left(\frac{\upsilon_t}{L}\right)^2$. If for small R_e (the Reynolds nomber): $\nu_{ef} \approx \nu$, for great R_e: $\nu_{ef} \sim \nu_t = \beta \nu_t$, $\beta = \text{const}$, one obtain for arbitrary R_e:

$$\nu < \frac{\partial \omega'}{\partial r} \cdot \frac{\partial p'}{\partial r} > \sim (\nu + \beta \nu_t) \left(\frac{v_t}{L} \right)^2. \tag{3.4}$$

Condition (3.2) becomes, with the help of (3.3) and (3.4):

$$\beta \nu_t = \frac{A_1}{A_2} \cdot \frac{L^2}{r} \cdot \frac{\partial p}{\partial r} - \nu, \qquad A_i = \text{const}, \tag{3.5}$$

and, with the model (3.1) and for $A_1 \approx A_2$,

$$\beta \alpha = \frac{d \ln p}{d \ln r} - \frac{1}{\gamma^2} \cdot \frac{\nu}{p}; \ \beta, \gamma < 1. \tag{3.6}$$

The last equation offer the possibility of determining $\alpha(r)$.

When expressing ν in units $\nu_0 = 2.5 \times 10^{-16} T_0^{5/2}/\rho_0 = 7.9 \times 10^6$ cm²/s ($T_0 = 10^5$K, $\rho_0 = 10^{-10}$ gr/cm³) and p in units $p_0 = \sqrt{GM_* r_*} = 1.42 \times 10^{16}$ cm²/s $\left(M_* = 1.5\, M_\odot, r_* = 10^6 \text{ cm} \text{ — proton star} \right)$, equation (3.6) can be written as

$$\beta \alpha = \frac{d \ln p}{d \ln r} - CM^2 \frac{\nu}{r^{1/2}}. \tag{3.7}$$

Where r is in units r_*, $M = 1/\Upsilon$ is Mach number, $C = 5.58 \times 10^{-10}$.

The numerical simulation of the time-dependent structure of stellar discs with $\alpha =$ const. (Angelov, 1981a) has given $L/r = (2-4)\%$, whick determines the Mach nomber in the interval (25 — 50). For the Kepler motion of gas in the disc, $d\ln p/d\ln r = 0.5$, so that $\alpha < 0.5/\beta$ $(0.5 < \beta < 1)$.

One can see, from (3.7), that the dependence $\alpha(r)$ exists in fact for very high values of the coefficient of molecular viscosity. In the central and the peripheral parts of the disc ($r \gg 1$) $\alpha(r)$ is monotonous, slowly increasing and bounded. Quick changes occur in the narrow band of the disc near the star's surface.

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