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INVESTIGATION THE EXISTENCE OF A SOLUTION FOR A MULTI-SINGULAR FRACTIONAL DIFFERENTIAL EQUATION WITH MULTI-POINTS BOUNDARY CONDITIONS

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Abstract. We should try to increase our abilities in solving of complicate differential equations. One type of complicate equations are multi-singular pointwise defined fractional differential equations. We investigate the existence of solutions for a multi-singular pointwise defined fractional differential equation with multi-points boundary conditions. We provide an example to illustrate our main result.

1. Introduction

One possible way that the mathematics has effective role in the various fields the various fields of sciences is to become more powerful and flexible in modeling theory so that different types of phenomena with distinct parameters can be written in mathematical formulas. In this case, different softwares can be developed to allow for more cost-free testing and less material consumption. In this way, a method is working with complicate differential equations. Nowadays, many researchers are studying advanced fractional modelings and its related existence results and qualitative behaviors of solutions for distinct fractional differential equations and inclusions (see for example [1–24, 26–29, 31–34, 36–38]).

In 2013, the existence of solutions for the singular differential equation

$$D^\alpha u(t) + f(t, u(t)) = 0,$$

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with boundary conditions $u'(0) = u''(0) = \cdots = u^{n-1}(0) = 0$, $u(1) = \int_0^1 u(s)d\mu(s)$ studied by Vong, where $0 < t < 1$, $n \geq 2$, $\alpha \in (n-1, n)$, μ is a function of bounded variation with $\int_0^1 d\mu(s) < 1$, f may have singularity at $t = 1$ and D^α is the Caputo derivative [39]. In 2014, Jleli et al. proved the existence of positive solutions for the singular fractional problem $D^\alpha u(t) + f(t, u(t)) = 0$ with boundary value conditions $u(0) = u'(0) = 0$ and $u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i)$, where $0 < t < 1$, $2 < \alpha \leq 3$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $f : (0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t,x)$ is singular at $t = 0$ and D^α is the Caputo derivative [25].

In 2016, Shabibi et al. reviewed the multi-singular pointwise defined fractional integro-differential equation

$$D^\mu x(t) + f(t, x(t), x'(t), D^\beta x(t), I^p x(t)) = 0,$$

with boundary conditions $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$, where $\mu \in [2,3)$, $x'(0) = x(\xi)$, $x(1) = \int_0^\eta x(s)ds$ and $x^{(j)}(0) = 0$ for $j = 2, \ldots, [\mu] - 1$, $0 \leq t \leq 1$, $x \in C^1[0,1]$, $\beta, \xi, \eta \in (0,1)$, $p > 1$, D^μ is the Caputo fractional derivative of order μ and $f : [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ is a function such that $f(t, \cdot, \cdot, \cdot, \cdot)$ is singular at some points $t \in [0,1]$ [36]. In 2018, Baleanu et al. investigated the pointwise defined problem

$$D^\alpha x(t) + f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi)x(\xi)d\xi, \phi(x(t))\right) = 0,$$

with boundary conditions $x(1) = x(0) = x''(0) = x^n(0) = 0$, where $\alpha \geq 2$, $\lambda, \mu, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that

$$\|\phi(x) - \phi(y)\| \leq \theta_0 \|x - y\| + \theta_1 \|x' - y'\|,$$

for some non-negative real numbers θ_0 and $\theta_1 \in [0, \infty)$ and all $x, y \in X$, D^α is the Caputo fractional derivative of order α

$$f(t, x_1(t), \ldots, x_5(t)) = f_1(t, x_1(t), \ldots, x_5(t)),$$

for all $t \in [0, \lambda)$,

$$f(t, x_1(t), , x_5(t)) = f_2(t, x_1(t), \ldots, x_5(t)),$$

for all $t \in [\lambda, \mu]$ and

$$f(t, x_1(t), \ldots, x_5(t)) = f(t, x_1(t), \ldots, x_5(t)),$$

for all $t \in (\mu, 1]$, $f_1(t, \cdot, \cdot, \cdot, \cdot)$ and $f_3(t, \cdot, \cdot, \cdot, \cdot)$ are continuous on $[0, \lambda)$ and $(\mu, 1]$ and $f_2(t, \cdot, \cdot, \cdot, \cdot)$ is multi-singular [9].

By using idea of the works, we investigate the existence of solutions for the nonlinear fractional differential pointwise defined equation

$$D^\alpha x(t) = f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi)x(\xi)d\xi\right), \tag{1.1}$$

with boundary conditions $x(0) = 0$, $x^{(j)}(0) = 0$ for $j \geq 2$ while $j \neq k$ for one's $2 \leq k \leq n-1$ and $x(1) = \sum_{i=1}^m \lambda_i D^{\beta_i} x(\gamma_i)$, where $\alpha \geq 2$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 1$, $\beta_1, \ldots, \beta_m \in (0,1)$, $\lambda_1, \ldots, \lambda_m \in [0, \infty)$, $m \in \mathbb{N}$, D^α is the Caputo fractional derivative of order α, $n = [\alpha] + 1$, $h \in L^1$ and $f \in L^1$ is singular at some points $[0,1]$.

Recall that $D^\alpha x(t) + f(t) = 0$ is a pointwise defined equation on $[0,1]$ if there exists a set $E \subset [0,1]$ such that the measure of E^c is zero and the equation holds on E [36]. In this paper, we use $\|\cdot\|_1$ for the norm of $L^1[0,1]$, $\|\cdot\|$ for the sup norm of $Y = C[0,1]$ and $\|x\|_* = \max\{\|x\|, \|x'\|\}$ for the norm of $X = C^1[0,1]$.

The Riemann-Liouville integral of order p with the lower limit $a \geq 0$ for a function $f : (a,\infty) \to \mathbb{R}$ is defined by

$$I^p_{a^+} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s) ds,$$

provided that the right-hand side is pointwise define on (a,∞). We denote $I^p_{0^+} f(t)$ by $I^p f(t)$ [30]. The Caputo fractional derivative of order $\alpha > 0$ is defined by

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds,$$

where $n = [\alpha] + 1$ and $f : (a,\infty) \to \mathbb{R}$ is a function [30]. Let Ψ be the family of non-decreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < \infty$ for all $t > 0$. One can check that $\psi(t) < t$ for all $t > 0$ [35]. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two maps. Then T is called an α-admissible map whenever $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ [35]. Let (X,d) be a metric space, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ a map. A self-map $T : X \to X$ is called an α-ψ-contraction whenever

$$\alpha(x,y) d(Tx,Ty) \leq \psi(d(x,y)),$$

for all $x, y \in X$ [35]. We need next results.

Lemma 1.1 ([35]). *Let (X,d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0,\infty)$ a map and $T : X \to X$ an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.*

Lemma 1.2 ([30]). *Let $n-1 \leq \alpha < n$ and $x \in C(0,1)$. Then, we have*

$$I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i,$$

for some real constants $c_0, \dots, c_{n-1}$.

2. Main Results

Now, we are ready for preparing our main results.

Lemma 2.1. *Let $\alpha \geq 2$, $[\alpha] = n-1$, $m \in \mathbb{N}$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 1$, $\beta_1, \dots, \beta_m \in (0,1)$, $\lambda_1, \dots, \lambda_m \in [0,\infty)$ and $f \in L^1[0,1]$, then the solution of the problem $D^\alpha x(t) = f(t)$ with the boundary conditions $x(0) = 0$, $x^{(j)}(0) = 0$ for $j \geq 2$ while $j \neq k$ for one's $2 \leq k \leq n-1$ such that*

$$\sum_{i=1}^m \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} \neq \frac{1}{k!},$$

and $x(1) = \sum_{i=1}^{m} \lambda_i D^{\beta_i} x(\gamma_i)$ *is* $x(t) = \int_0^1 G(t,s) f(s) ds$*, where* $G(t,s)$ *is defined by*

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{t^k \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \leq s \leq t \leq 1$ *and* $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \leq s \leq \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ *for* $j = 1, 2, \ldots, m$*,*

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \leq s \leq t \leq 1$ *and* $\gamma_1 < \gamma_2 < \cdots < \gamma_m \leq s$*,*

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{t^k \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \leq t \leq s \leq 1$ *and* $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \leq s \leq \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ *for* $j = 1, 2, \ldots, m$*, and*

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \leq t \leq s \leq 1$ *and* $\gamma_1 < \gamma_2 < \cdots < \gamma_m \leq s$ *and*

$$\Delta := k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} - 1.$$

Proof. By using a similar method in [9], we can show that Lemma 1.1 holds on $L^1[0,1]$. Let $x(t)$ be a solution for the problem. Since $x^{(j)}(0) = 0$ for $j \geq 2$, by using Lemma 1.1, we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds + c_0 + c_1 t + \cdots + c_n t^n.$$

Since $x(0) = 0$, so $c_0 = 0$. Also since $x^{(j)}(0) = 0$ for $j \geq 2$ and $j \neq k$ so $c_2 = \cdots = c_j = \cdots = c_n = 0$ for $j \neq k$. Thus,

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds + c_k t^k. \tag{2.1}$$

Hence, we get

$$\begin{aligned} D^{\beta_i} x(t) &= \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} f(s) ds + c_k \frac{\Gamma(k+1)}{\Gamma(k+1-\beta_i)} t^{k-\beta_i} \\ &= \frac{1}{\Gamma(\alpha-\beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} f(s) ds + c_k \frac{k!}{\Gamma(k+1-\beta_i)} t^{k-\beta_i}, \end{aligned}$$

and so

$$\lambda_i D^{\beta_i} x(\gamma_i) = \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s) ds + c_k \lambda_i \frac{k!}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i},$$

for all $1 \le i \le m$. Therefore, we obtain

$$\sum_{i=1}^{m} \lambda_i D^{\beta_i} x(\gamma_i) = \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds + c_k k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i}.$$

On the other hand, by using (2.1) we have

$$x(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds + c_k.$$

Since $x(1) = \sum_{i=1}^{m} \lambda_i D^{\beta_i} x(\gamma_i)$, we get

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds + c_k = & \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds \\ & + c_k k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i}. \end{aligned}$$

Hence,

$$\begin{aligned} c_k \left[k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} - 1 \right] = & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds \\ & - \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds. \end{aligned}$$

Put $\Delta := k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} - 1$. Then, by using the assumption $\Delta \neq 0$, we have

$$c_k = \frac{1}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds - \frac{1}{\Delta} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds$$

and so

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{t^k}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds \\ & - \frac{t^k}{\Delta} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} f(s)ds. \end{aligned}$$

Thus,

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{t^k}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)ds \\ & - \frac{t^k}{\Delta} \frac{\lambda_1}{\Gamma(\alpha-\beta_1)} \int_0^{\gamma_1} (\gamma_1 - s)^{\alpha-\beta_1-1} f(s)ds \\ & - \cdots - \frac{t^k}{\Delta} \frac{\lambda_m}{\Gamma(\alpha-\beta_m)} \int_0^{\gamma_m} (\gamma_m - s)^{\alpha-\beta_m-1} f(s)ds. \end{aligned}$$

If $0 \le t \le \gamma_1 < \cdots < \gamma_m < 1$, then

$$\begin{aligned}
x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds \\
& + \frac{t^k}{\Delta\Gamma(\alpha)} \left(\int_0^t + \int_t^{\gamma_1} + \cdots + \int_{\gamma_m}^1 \right) (1-s)^{\alpha-1} f(s)ds \\
& - \frac{t^k \lambda_1}{\Delta\Gamma(\alpha-\beta_1)} \left(\int_0^t + \int_t^{\gamma_1} \right) (\gamma_1 - s)^{\alpha-\beta_1-1} f(s)ds \\
& - \cdots - \frac{t^k \lambda_m}{\Delta\Gamma(\alpha-\beta_m)} \\
& \times \left(\int_0^t + \int_t^{\gamma_1} + \cdots + \int_{\gamma_{m-1}}^{\gamma_m} \right) (\gamma_m - s)^{\alpha-\beta_m-1} f(s)ds.
\end{aligned}$$

If $0 < \gamma_1 \le t \le \gamma_2 < \cdots < \gamma_m < 1$, then

$$\begin{aligned}
x(t) = & \frac{1}{\Gamma(\alpha)} \left(\int_0^{\gamma_1} + \int_{\gamma_1}^t \right) (t-s)^{\alpha-1} f(s)ds \\
& + \frac{t^k}{\Delta\Gamma(\alpha)} \left(\int_0^{\gamma_1} + \int_{\gamma_1}^t + \int_t^{\gamma_2} + \cdots + \int_{\gamma_m}^1 \right) (1-s)^{\alpha-1} f(s)ds \\
& - \frac{t^k \lambda_1}{\Delta\Gamma(\alpha-\beta_1)} \int_0^{\gamma_1} (\gamma_1 - s)^{\alpha-\beta_1-1} f(s)ds \\
& - \cdots - \frac{t^k \lambda_m}{\Delta\Gamma(\alpha-\beta_m)} \\
& \times \left(\int_0^{\gamma_1} + \int_{\gamma_1}^t + \int_t^{\gamma_2} + \cdots + \int_{\gamma_{m-1}}^{\gamma_m} \right) (\gamma_m - s)^{\alpha-\beta_m-1} f(s)ds.
\end{aligned}$$

By continuing this, finally we get

$$\begin{aligned}
x(t) = & \frac{1}{\Gamma(\alpha)} \left(\int_0^{\gamma_1} + \int_{\gamma_1}^{\gamma_2} + \cdots + \int_{\gamma_m}^t \right) (t-s)^{\alpha-1} f(s)ds \\
& + \frac{t^k}{\Delta\Gamma(\alpha)} \left(\int_0^{\gamma_1} + \int_{\gamma_1}^{\gamma_2} + \cdots + \int_{\gamma_m}^t + \int_t^1 \right) (1-s)^{\alpha-1} f(s)ds \\
& - \frac{t^k \lambda_1}{\Delta\Gamma(\alpha-\beta_1)} \int_0^{\gamma_1} (\gamma_1 - s)^{\alpha-\beta_1-1} f(s)ds \\
& - \cdots - \frac{t^k}{\Delta} \frac{\lambda_m}{\Gamma(\alpha-\beta_m)} \int_0^{\gamma_m} (\gamma_m - s)^{\alpha-\beta_m-1} f(s)ds,
\end{aligned}$$

whenever $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m \le t \le 1$. Hence, $x(t) = \int_0^1 G(t,s) f(s)ds$, where

$$\begin{aligned}
G(t,s) = & \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k (1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k \lambda_1 (\gamma_1 - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_1)} \\
& - \frac{t^k \lambda_2 (\gamma_2 - s)^{\alpha-\beta_2-1}}{\Delta\Gamma(\alpha-\beta_2)} - \cdots - \frac{t^k \lambda_m (\gamma_m - s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},
\end{aligned}$$

when $0 \le s \le t \le 1$ and $s \le \gamma_1 < \gamma_2 < \cdots < \gamma_m$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_2(\gamma_2-s)^{\alpha-\beta_2-1}}{\Delta\Gamma(\alpha-\beta_2)} - \cdots - \frac{t^k\lambda_m(\gamma_m-s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 \le s \le \gamma_2 < \cdots < \gamma_m$, in the general case

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_j(\gamma_j-s)^{\alpha-\beta_j-1}}{\Delta\Gamma(\alpha-\beta_j)} - \cdots - \frac{t^k\lambda_m(\gamma_m-s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$, for $1 \le j \le m$, thus

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_m(\gamma_m-s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} \le s \le \gamma_m$, and

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \le s \le t \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \le s$,

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} + \frac{t^k\lambda_1(\gamma_1-s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_1)} - \frac{t^k\lambda_2(\gamma_2-s)^{\alpha-\beta_2-1}}{\Delta\Gamma(\alpha-\beta_2)} - \cdots - \frac{t^k\lambda_m(\gamma_m-s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le t \le s \le 1$ and $s \le \gamma_1 < \gamma_2 < \cdots < \gamma_m$,

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_2(\gamma_2-s)^{\alpha-\beta_2-1}}{\Delta\Gamma(\alpha-\beta_2)} - \cdots - \frac{t^k\lambda_m(\gamma_m-s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 \le s \le \gamma_2 < \cdots < \gamma_m$ and in the general case

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_j(\gamma_j-s)^{\alpha-\beta_j-1}}{\Delta\Gamma(\alpha-\beta_j)} - \cdots - \frac{t^k\lambda_m(\gamma_m-s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \le s \le \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $1 \le j \le m$, thus

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \frac{t^k\lambda_m(\gamma_m-s)^{\alpha-\beta_m-1}}{\Delta\Gamma(\alpha-\beta_m)},$$

when $0 \le t \le s \le 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{m-1} \le s \le \gamma_m$, and finally

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \leq t \leq s \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \leq s$. Thus,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{t^k \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \leq s \leq t \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \leq s \leq \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $j = 1, 2, \ldots, m$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \leq s \leq t \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \leq s$,

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{t^k \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \leq t \leq s \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \leq s \leq \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $j = 1, 2, \ldots, m$, and

$$G(t,s) = \frac{t^k(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \leq t \leq s \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \leq s$. □

One can check that

$$\frac{\partial G}{\partial t} = \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{kt^{k-1}(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{kt^{k-1} \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \leq s \leq t \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \leq s \leq \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $j = 1, 2, \ldots, m$,

$$\frac{\partial G}{\partial t} = \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{kt^{k-1}(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)},$$

when $0 \leq s \leq t \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \leq s$,

$$\frac{\partial G}{\partial t} = \frac{kt^{k-1}(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \sum_{i=j}^{m} \frac{kt^{k-1} \lambda_i (\gamma_i - s)^{\alpha-\beta_1-1}}{\Delta\Gamma(\alpha-\beta_i)},$$

when $0 \leq t \leq s \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_{j-1} \leq s \leq \gamma_j < \gamma_{j+1} < \cdots < \gamma_m$ for $j = 1, 2, \ldots, m$, and $\frac{\partial G}{\partial t} = \frac{kt^{k-1}(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}$, when $0 \leq t \leq s \leq 1$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_m \leq s$.

It is easy to see that G and $\frac{\partial}{\partial t}G$ are continuous with respect to t. Consider the space $X = C^1[0,1]$ with the norm $\|\cdot\|_*$, where $\|x\|_* = \max\{\|x\|, \|x'\|\}$ and $\|\cdot\|$ is the supremum norm on $C[0,1]$. Let f be a map on $[0,1] \times X^4$ such that is singular at some points of $[0,1]$. Define $F : X \to X$ as

$$\begin{aligned}
Fx(t) &= \int_0^1 \frac{\partial}{\partial t} G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\
&\quad + \frac{t^k}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\
&\quad - \frac{t^k}{\Delta} \sum_{i=1}^m \frac{\lambda_i(\gamma_i - s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} \\
&\quad \times \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds,
\end{aligned}$$

so

$$\begin{aligned}
F'x(t) &= \int_0^1 G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\
&= \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\
&\quad + \frac{kt^{k-1}}{\Delta\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \\
&\quad - \frac{kt^{k-1}}{\Delta} \sum_{i=1}^m \frac{\lambda_i(\gamma_i - s)^{\alpha-\beta_i-1}}{\Gamma(\alpha-\beta_i)} \\
&\quad \times \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-1} f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds.
\end{aligned}$$

It is notable that the singular pointwise defined (1.1) has a solution if and only if the map F has a fixed point.

Theorem 2.1. *Let $\alpha \geq 2$, $[\alpha] = n-1$, $m \in \mathbb{N}$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 1$, $\beta_1, \ldots, \beta_m \in (0,1)$, $\lambda_1, \ldots, \lambda_m \in [0,\infty)$, $h \in L^1[0,1]$ and $m_0 = \int_0^1 |h(s)|ds$. Assume that $f : [0,1] \times X^4 \to \mathbb{R}$ is a singular map on some points $[0,1]$ such that*

$$|f(t, x_1, \ldots, x_4) - f(t, y_1, \ldots, y_4)| \leq \Lambda(t, |x_1 - y_1|, \ldots, |x_4 - y_4|),$$

for all $x_1, \ldots, x_4, y_1, \ldots, y_1 \in X$ and almost all $t \in [0,1]$, where $\Lambda(t, x_1, \ldots, x_4)$ be a real mapping on $[0,1] \times X^4$ such that is non-decreasing with respect to $x_1, \ldots, x_4$,

$$\lim_{z \to 0^+} \frac{\Lambda(t, z, \ldots, z)}{H(z)} = \theta(t),$$

for almost all $t \in [0,1]$ in which $\theta : [0,1] \to \mathbb{R}^+$ is a mapping so that $\hat{\theta} \in L^1[0,1]$, with $\hat{\theta}(s) = (1-s)^{\alpha_i - 2}\theta(s)$, $H : [0,\infty) \to [0,\infty)$ is a linear mapping such that $\lim_{z \to 0^+} H(z) = 0$ and $\lim_{i \to \infty} H^i(t) < \infty$ for all $t \in [0,\infty)$. Here, H^i is the i-th *composition of H with itself. Let*

$$|f(t, x_1, \dots, x_4)| \leq \sum_{k=1}^{n_0} b_j(t) K_j(|x_1|, \dots, |x_4|),$$

almost everywhere on $[0,1]$ and all $x_1, \dots, x_4$, where $n_0 \in \mathbb{N}$, $b_j : [0,1] \to \mathbb{R}^+$, $\hat{b_j} \in L^1[0,1]$, $K_J : X^4 \to \mathbb{R}^+$ is a non-decreasing mapping with respect to all their components with

$$\lim_{z \to 0^+} \frac{K_j(z, \dots, z)}{z} = q_j,$$

for some $q_j \in \mathbb{R}^+$ and $1 \leq j \leq n_0$. If

$$\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \max\left\{\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} q_j, \|\hat{\theta}\|_{[0,1]}\right\} \in \left[0, \frac{1}{M}\right),$$

where $M = \max\left\{1, \frac{1}{\Gamma(2-\beta)}, m_0\right\}$, then the pointwise defined equation

$$D^\alpha x(t) = f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi) x(\xi) d\xi\right),$$

with boundary conditions $x(0) = 0$, $x^{(j)}(0) = 0$ for $j \geq 2$, while $j \neq k$, $2 \leq k \leq n-1$ and $x(1) = \sum_{i=1}^m \lambda_i D^{\beta_i} x(\gamma_i)$, has a solution.

Proof. First we show that F is continuous on X. Let $\epsilon > 0$ be given. Since $H(Mz) \to 0$ as $z \to 0^+$, there exists $\delta_1 > 0$ such that $z \in (0, \delta_1]$ implies that $H(Mz) < \epsilon$. Since

$$\lim_{z \to 0^+} \frac{\Lambda(t, Mz, \dots, Mz)}{H(Mz)} = \theta(t),$$

for almost all $t \in [0,1]$, there exists $\delta_2 > 0$ such that $z \in (0, \delta_2]$ implies that

$$\frac{\Lambda(t, Mz, \dots, Mz)}{H(z)} \leq \theta(t) + \epsilon.$$

Hence, $\Lambda(t, Mz, \dots, Mz) \leq (\theta(t) + \epsilon) H(Mz)$ almost everywhere on $[0,1]$. Let $\delta = \min\{\delta_1, \delta_2, \epsilon\}$ and $z := \|x - y\|_* < \delta$ for $x, y \in X$. Then, we have

$$\Lambda(t, M\|x-y\|_*, \dots, M\|x-y\|_*) \leq (\theta(t) + \epsilon) H(M\|x-y\|_*) < (\theta(t) + \epsilon)\epsilon.$$

So, for all $t \in [0,1]$ and $x, y \in X$ such that $\|x-y\|_* < \delta$ we have

$$\begin{aligned}
|Fx(t)-Fy(t)| &= \bigg|\int_0^1 G(t,s)\bigg[f\bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\bigg) \\
&\quad - f\bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\bigg)\bigg]ds\bigg| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg|f\bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\bigg) \\
&\quad - f\bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\bigg)\bigg|ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \bigg|f\bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\bigg) \\
&\quad - f\bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\bigg)\bigg|ds \\
&\quad + \frac{t^k}{|\Delta|}\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\quad \times \bigg|f\bigg(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\bigg) \\
&\quad - f\bigg(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\bigg)\bigg|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Lambda\bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|, \\
&\quad |D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\bigg)ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Lambda\bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|, \\
&\quad |D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\bigg)ds \\
&\quad + \frac{t^k}{|\Delta|}\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \\
&\quad \times \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1}\Lambda\bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|, \\
&\quad |D^\beta(x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)|d\xi\bigg)ds
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Lambda\Bigg(s, \|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\Bigg)ds$$
$$+ \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Lambda\Bigg(s, \|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\Bigg)ds$$
$$+ \frac{t^k}{|\Delta|}\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\Lambda(s, \|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|)ds.$$

Note that $|D^\beta(x-y)(s)| \leq \frac{\|x'-y'\|}{\Gamma(2-\beta)}$ and

$$\int_0^s h(\xi)|x(\xi)|d\xi \leq \|x\|\int_0^1 h(\xi)d\xi = m_0\|x\|.$$

Put $M = \max\left\{1, \frac{1}{\Gamma(2-\beta)}, m_0\right\}$. Now for each $t \in [0,1]$ and $x, y \in X$, with $\|x-y\|_* < \delta$, we obtain

$$\begin{aligned}
|Fx(t) - Fy(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Lambda\Bigg(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|_*\Bigg)ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \Lambda\Bigg(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|\Bigg)ds \\
&\quad + \frac{t^k}{|\Delta|}\sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1} \\
&\quad \times \Lambda\Bigg(s, \|x-y\|_*, \|x-y\|_*, \frac{\|x-y\|_*}{\Gamma(2-\beta)}, m_0\|x-y\|_*\Bigg)ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*)ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*)ds
\end{aligned}$$

$$
\begin{aligned}
& + \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1} \\
& \times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*)ds \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\theta(s)+\epsilon)\epsilon ds \\
& + \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\theta(s)+\epsilon)\epsilon ds \\
& + \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}(\theta(s)+\epsilon)\epsilon ds \\
\leq & \frac{1}{\Gamma(\alpha)}[\int_0^1 (1-s)^{\alpha_i-2}\theta(s)ds + \epsilon\int_0^t (t-s)^{\alpha_i-1}\theta(s)ds]\epsilon \\
& + \frac{t^k}{|\Delta|\Gamma(\alpha)}[\int_0^1 (1-s)^{\alpha-2}\theta(s)ds + \epsilon\int_0^1 (1-s)^{\alpha-1}\theta(s)ds]\epsilon \\
& + \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\int_0^1 (1-s)^{\alpha_i-2}\theta(s)ds\right. \\
& \left. +\epsilon\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\theta(s)ds\right]\epsilon \\
= & \frac{1}{\Gamma(\alpha)}\left[\|\hat{\theta}\|_{[0,1]} + \frac{\epsilon}{\alpha}t^{\alpha}\right]\epsilon + \frac{t^k}{|\Delta|\Gamma(\alpha)}\left[\|\hat{\theta}\|_{[0,1]} + \frac{\epsilon}{\alpha}\right]\epsilon \\
& + \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\|\hat{\theta}\|_{[0,1]} + \frac{\epsilon}{\alpha-\beta_i}\gamma_i^{\alpha-\beta_i}\right]\epsilon. \qquad (2.2)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|Fx-Fy\| \leq & \left[\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{|\Delta|\Gamma(\alpha)} + \frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\right. \\
& \left. + \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Delta|\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\right]\epsilon.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
|F'x(t)-F'y(t)| = & \left|\int_0^1 \frac{\partial G}{\partial t}(t,s)\left[f\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)d\xi\right)\right.\right. \\
& \left.\left. - f\left(s, y(s), y'(s), D^{\beta}y(s), \int_0^s h(\xi)y(\xi)d\xi\right)\right]ds\right| \\
\leq & \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} \\
& \times \left|f\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)d\xi\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&- f\left(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi\right)\Bigg| ds \\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\times \Bigg| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi\right) \\
&- f\left(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi\right)\Bigg| ds \\
&+ \frac{kt^{k-1}}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\times \Bigg| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi) x(\xi) d\xi\right) \\
&- f\left(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi\right)\Bigg| ds \\
\le& \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \Lambda\Bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|, \\
&|D^\beta (x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)| d\xi\Bigg) ds \\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Lambda\Bigg(s, |x(s)-y(s)|, |x'(s)-y'(s)|, \\
&|D^\beta (x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)| d\xi\Bigg) ds \\
&+ \frac{kt^{k-1}}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \Lambda\Bigg(s, |x(s)-y(s)|, \\
&|x'(s)-y'(s)|, |D^\beta (x-y)(s)|, \int_0^s h(\xi)|x(\xi)-y(\xi)| d\xi\Bigg) ds \\
\le& \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \\
&\times \Lambda\left(s, \|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\right) ds \\
&+ \frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\times \Lambda\left(s, \|x-y\|, \|x'-y'\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\\
&\times\Lambda\left(s,\|x-y\|,\|x'-y'\|,\frac{\|x'-y'\|}{\Gamma(2-\beta)},m_0\|x-y\|\right)ds\\
\le{}&\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\\
&\times\Lambda\left(s,\|x-y\|_*,\|x-y\|_*,\frac{\|x-y\|_*}{\Gamma(2-\beta)},m_0\|x-y\|_*\right)ds\\
&+\frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\\
&\times\Lambda\left(s,\|x-y\|_*,\|x-y\|_*,\frac{\|x-y\|_*}{\Gamma(2-\beta)},m_0\|x-y\|\right)ds\\
&+\frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\\
&\times\Lambda\left(s,\|x-y\|_*,\|x-y\|_*,\frac{\|x-y\|_*}{\Gamma(2-\beta)},m_0\|x-y\|_*\right)ds\\
\le{}&\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\\
&\times\Lambda(s,M\|x-y\|_*,M\|x-y\|_*,M\|x-y\|_*,M\|x-y\|_*)ds\\
&+\frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\\
&\times\Lambda(s,M\|x-y\|_*,M\|x-y\|_*,M\|x-y\|_*,M\|x-y\|_*)ds\\
&+\frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\\
&\times\Lambda(s,M\|x-y\|_*,M\|x-y\|_*,M\|x-y\|_*,M\|x-y\|_*)ds\\
\le{}&\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}(\theta(s)+\epsilon)\epsilon ds\\
&+\frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}(\theta(s)+\epsilon)\epsilon ds\\
&+\frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}(\theta(s)+\epsilon)\epsilon ds\\
\le{}&\frac{1}{\Gamma(\alpha-1)}\left[\int_0^1(1-s)^{\alpha_i-2}\theta(s)ds+\epsilon\int_0^t(t-s)^{\alpha_i-1}\theta(s)ds\right]\epsilon\\
&+\frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha_i-2}\theta(s)ds+\epsilon\int_0^1(1-s)^{\alpha-1}\theta(s)ds\right]\epsilon
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\int_0^1 (1-s)^{\alpha_i-2}\theta(s)ds\right.\\
&\left.+ \epsilon\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}\theta(s)ds\right]\epsilon\\
=&\frac{1}{\Gamma(\alpha-1)}\left[\|\hat{\theta}\|_{[0,1]}+\frac{\epsilon}{\alpha}t^{\alpha}\right]\epsilon+\frac{kt^{k-1}}{|\Delta|\Gamma(\alpha)}\left[\|\hat{\theta}\|_{[0,1]}+\frac{\epsilon}{\alpha}\right]\epsilon\\
&+\frac{kt^{k-1}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\|\hat{\theta}\|_{[0,1]}+\frac{\epsilon}{\alpha-\beta_i}\gamma_i^{\alpha-\beta_i}\right]\epsilon.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|F'x-F'y\|\leq&\left[\left(\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\right.\\
&\left.+\left(\frac{1}{\Gamma(\alpha)}+\frac{k}{|\Delta|\Gamma(\alpha+1)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\right]\epsilon,
\end{aligned}
$$

and so

$$
\begin{aligned}
\|Fx-Fy\|_*=&\max\{\|Fx-Fy\|,\|F'x-F'y\|\}\\
\leq&\max\left\{\left[\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{|\Delta|\Gamma(\alpha)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\right.\right.\\
&\left.+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Delta|\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\right]\epsilon,\\
&\left[\left(\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\right.\\
&\left.\left.+\left(\frac{1}{\Gamma(\alpha)}+\frac{k}{|\Delta|\Gamma(\alpha+1)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\right]\epsilon\right\}\\
=&\left[\left(\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\right.\\
&\left.+\left(\frac{1}{\Gamma(\alpha)}+\frac{k}{|\Delta|\Gamma(\alpha+1)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i\gamma_i^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i+1)}\right)\epsilon\right]\epsilon.
\end{aligned}
$$

This concludes that $\|Fx-Fy\|_*$ tends to zero as $\|x-y\|_*$ tends to zero and so F is continuous in X. Since for all $1\leq j\leq n_0$,

$$\lim_{z\to 0^+}\frac{K_j(Mz,\dots,Mz)}{Mz}=q_j,$$

for each $\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that

$$K_j(Mz,\dots,Mz)\leq(q_j+\epsilon)Mz,$$

for all $0 < z \leq \delta$ and $1 \leq j \leq n_0$. Since

$$M\left[\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\left[\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}q_j\right]<1,$$

there exists $\epsilon_0 > 0$ such that

$$M\left[\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\left(\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}(q_j+\epsilon_0)\right)<1.$$

Let $\delta_0 = \delta(\epsilon_0)$. On the other hand, for almost all $s \in [0,1]$ we have

$$\lim_{z\to 0^+}\frac{\Lambda(s,Mz,\dots,Mz)}{H(Mz)}=\theta(s).$$

For the given $\epsilon > 0$, there exists $\delta' = \delta'(\epsilon)$ such that for almost everywhere on $[0,1]$, $\Lambda(s, Mz, \dots, Mz) \leq (\theta(s)+\epsilon)H(Mz)$ for $0 < z \leq \delta'$. Since

$$M\left[\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\|\hat{\theta}\|_{[0,1]}<1,$$

there exists $\epsilon_1 > 0$ such that

$$\begin{aligned}&M\left[\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\|\hat{\theta}\|_{[0,1]}\\&+\frac{\epsilon_1 M}{\alpha-1}\left[\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]<1.\end{aligned}$$

Let $\delta_1 = \delta'(\epsilon_1)$ and $\delta_2 = \min\{\delta_0, \frac{\delta_1}{2}\}$. For each $z \in (0, \delta_2]$ and $1 \leq j \leq n_0$, we have $K_j(Mz, \dots, Mz) \leq (q_j + \epsilon_0)Mz$ and for each $z \in (0, \delta_1]$ we have

$$\Lambda(s,Mz,\dots,Mz) \leq (\theta(s)+\epsilon_1)H(Mz), \tag{2.3}$$

almost everywhere on $[0,1]$. Let $C = \{x \in X : \|x\|_* \leq \delta_2\}$. Define $\alpha : X^2 \to [0,\infty)$ by $\alpha(x,y) = 1$ whenever $x, y \in C$ and $\alpha(x,y) = 0$ otherwise. If $\alpha(x,y) \geq 1$, then

$x, y \in X$ and so $\|x\|_* \le \delta_2$ and $\|y\|_* \le \delta_2$. Thus, for each $t \in [0,1]$ we have

$$\begin{aligned}
|Fx(t)| &= \left| \int_0^1 G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) ds \right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \left| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) \right| ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \left| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) \right| ds \\
&\quad + \frac{t^k}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\quad \times \left| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right) \right| ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \sum_{j=1}^{n_0} b_j(s) K_j\left(|x(s)|, |x'(s)|, |D^\beta x(s)|, \left|\int_0^s h(\xi)x(\xi)d\xi\right|\right) ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times \sum_{j=1}^{n_0} b_j(s) K_j\left(|x(s)|, |x'(s)|, |D^\beta x(s)|, \left|\int_0^s h(\xi)x(\xi)d\xi\right|\right) ds \\
&\quad + \frac{t^k}{|\Delta|} \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha-\beta_i-1} \\
&\quad \times \sum_{j=1}^{n_0} b_j(s) K_j\left(|x(s)|, |x'(s)|, |D^\beta x(s)|, \left|\int_0^s h(\xi)x(\xi)d\xi\right|\right) ds \\
&\le \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n_0} \int_0^t (t-s)^{\alpha-1} b_j(s) \\
&\quad \times K_j\left(|x(s)|, |x'(s)|, \frac{\|x'\|}{\Gamma(2-\beta)}, \|x\| \int_0^s |h(\xi)x(\xi)| d\xi\right) ds \\
&\quad + \sum_{j=1}^{n_0} b_j(s) \frac{t^k}{|\Delta|\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\quad \times K_j\left(|x(s)|, |x'(s)|, \frac{\|x'\|}{\Gamma(2-\beta)}, \|x\| \int_0^s |h(\xi)x(\xi)| d\xi\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\sum_{j=1}^{n_0}\int_0^{\gamma_i}(\gamma_i-s)^{\alpha-\beta_i-1}b_j(s)\\
&\times K_j\left(|x(s)|,|x'(s)|,\frac{\|x'\|}{\Gamma(2-\beta)},\|x\|\int_0^s|h(\xi)x(\xi)|d\xi\right)ds\\
\le\ & \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n_0}\int_0^1(1-s)^{\alpha-1}b_j(s)\\
&\times K_j\left(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m_0\|x\|\right)ds\\
&+\sum_{j=1}^{n_0}\frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}b_j(s)\\
&\times K_j\left(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m_0\|x\|\right)ds\\
&+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\Bigg[\sum_{j=1}^{n_0}\int_0^{\gamma_i}(1-s)^{\alpha-2}\\
&\times b_j(s)K_j\left(\|x\|,\|x'\|,\frac{\|x'\|}{\Gamma(2-\beta)},m_0\|x\|\right)ds\Bigg]\\
\le\ & \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n_0}K_j(M\|x\|_*,\ldots,M\|x\|_*)\int_0^1(1-s)^{\alpha-2}b_j(s)ds\\
&+\frac{t^k}{|\Delta|\Gamma(\alpha)}\sum_{j=1}^{n_0}K_j(M\|x\|_*,\ldots,M\|x\|_*)\int_0^1(1-s)^{\alpha-2}b_j(s)ds\\
&+\frac{t^k}{|\Delta|}\sum_{j=1}^{n_0}[K_j(M\|x\|_*,\ldots,M\|x\|_*)\\
&\times\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^1(1-s)^{\alpha-2}b_j(s)ds]\\
\le\ & \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}K_j(M\delta_2,\ldots,M\delta_2)\\
&+\frac{t^k}{|\Delta|\Gamma(\alpha)}\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}K_j(M\delta_2,\ldots,M\delta_2)\\
&+\frac{t^k}{|\Delta|}\left(\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\sum_{j=1}^{n_0}\|\hat{b_j}\|_{[0,1]}K_j(M\delta_2,\ldots,M\delta_2)\\
=\ & \left[\frac{1}{\Gamma(\alpha)}+\frac{t^k}{|\Delta|\Gamma(\alpha)}+\frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]
\end{aligned}
$$

$$\times \left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} K_j(M\delta_2, \dots, M\delta_2)\right].$$

Hence,

$$\begin{aligned}\|Fx\| &\le \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{|\Delta|\Gamma(\alpha)} + \frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\cdot\left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} K_j(M\delta_2, \dots, M\delta_2)\right]\\
&\le \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{|\Delta|\Gamma(\alpha)} + \frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\cdot\left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} (q_j+\epsilon_0)\right] M\delta_2\\
&\le \left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\cdot\left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} (q_j+\epsilon_0)\right] M\delta_2 \le \delta_2.\end{aligned}$$

Similarly, one can concluded that

$$\begin{aligned}\|F'x\| &\le \left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\\
&\quad\times\left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} K_j(M\delta_2, \dots, M\delta_2)\right]\\
&\le \left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\\
&\quad\times\left[\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} (q_j+\epsilon_0)\right] M\delta_2 \le \delta_2,\end{aligned}$$

and so $\|Fx\|_* = \max\{\|Fx\|, \|F'x\|\} \le \delta_2$. Thus, $Fx \in C$. Similarly, we can show that $Fy \in C$. Hence, $\alpha(Fx, Fy) \ge 1$. It is obvious that $C \ne \phi$. For $x_0 \in C$, we have $Fx_0 \in C$ and so $\alpha(x_0, Fx_0) \ge 1$. Put

$$\lambda := M\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right]\|\hat{\theta}\|_{[0,1]}.$$

Let $x, y \in C$. Then, $\alpha(x, y) = 1$. On the other hand by using (2.2), for each $x, y \in X$ and $t \in [0,1]$ we have

$$\begin{aligned}|Fx(t) - Fy(t)| &\le \int_0^1 |G(t,s)| \left| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi\right)\right.\\
&\quad \left. - f\left(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi\right)\right| ds\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\\
&\quad \times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*)ds\\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\end{aligned}$$

$$\times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*)ds + \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\int_0^{\gamma_i}(\gamma_i - s)^{\alpha-\beta_i-1} \times \Lambda(s, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*, M\|x-y\|_*)ds.$$

If $x, y \in C$, then $\|x\|_* < \delta_1$ and $\|y\|_* < \delta_1$ and so

$$\|x-y\|_* < \|x\|_* + \|y\|_* < 2\delta_* \leq \delta_1.$$

Hence, by using (2.3) we have

$$\Lambda(s, M\|x-y\|_*, \ldots, M\|x-y\|_*) \leq (\theta(s)+\epsilon_1)H(M\|x-y\|_*).$$

Thus, for each $t \in [0,1]$ and $x, y \in C$ we have

$$\begin{aligned}
|Fx(t) - Fy(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\theta(s)+\epsilon_1)H(M\|x-y\|_*)ds \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(\theta(s)+\epsilon_1)H(M\|x-y\|_*)ds \\
&\quad + \frac{t^k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)} \\
&\quad \times \int_0^{\gamma_i}(\gamma_i - s)^{\alpha-\beta_i-1}(\theta(s)+\epsilon_1)H(M\|x-y\|_*)ds \\
&\leq \frac{1}{\Gamma(\alpha)}H(M\|x-y\|_*) \\
&\quad \times \left[\int_0^1 (1-s)^{\alpha-2}\theta(s)ds + \epsilon_1\int_0^1 (1-s)^{\alpha-2}\theta(s)ds\right] \\
&\quad + \frac{t^k}{|\Delta|\Gamma(\alpha)}H(M\|x-y\|_*) \\
&\quad \times \left[\int_0^1 (1-s)^{\alpha-2}\theta(s)ds + \epsilon_1\int_0^1 (1-s)^{\alpha-2}\theta(s)ds\right] \\
&\quad + \frac{t^k}{|\Delta|}H(M\|x-y\|_*) \\
&\quad \times \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\left[\int_0^1 (1-s)^{\alpha-2}\theta(s)ds + \epsilon_1\int_0^1 (1-s)^{\alpha-2}\theta(s)ds\right] \\
&= H(M\|x-y\|_*)\Bigg[\left(\frac{\|\hat{\theta}\|_{[0,1]}}{\Gamma(\alpha)} + \frac{t^k\|\hat{\theta}\|_{[0,1]}}{|\Delta|\Gamma(\alpha)} + \frac{t^k\|\hat{\theta}\|_{[0,1]}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right) \\
&\quad + \frac{\epsilon_1}{\alpha-1}\left(\frac{\|\hat{\theta}\|_{[0,1]}}{\Gamma(\alpha)} + \frac{t^k\|\hat{\theta}\|_{[0,1]}}{|\Delta|\Gamma(\alpha)} + \frac{t^k\|\hat{\theta}\|_{[0,1]}}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\Bigg].
\end{aligned}$$

Hence,

$$\begin{aligned}\|Fx-Fy\| &\leq \Bigg[\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{|\Delta|\Gamma(\alpha)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\\ &\quad+\frac{\epsilon_1}{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{|\Delta|\Gamma(\alpha)}+\frac{1}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\Bigg]H(M\|x-y\|_*)\\ &\leq M\Bigg[\left(\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\|\hat{\theta}\|_{[0,1]}\\ &\quad+\frac{\epsilon_1 M}{\alpha-1}\left(\frac{1}{\Gamma(\alpha-1)}+\frac{k}{|\Delta|\Gamma(\alpha)}+\frac{k}{|\Delta|}\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right)\Bigg]H(\|x-y\|_*)\\ &=\lambda H(\|x-y\|_*).\end{aligned}$$

Similarly, we conclude that $\|F'x-F'y\|\leq\lambda H(\|x-y\|_*)$. Hence,

$$\begin{aligned}\|Fx-Fy\|_* &= \max\{\|Fx-Fy\|,\|F'x-F'y\|\}\\ &\leq \lambda H(\|x-y\|_*)=\psi(\|x-y\|_*),\end{aligned}$$

where $\psi:[0,\infty)\to[0,\infty)$ is defined as $\psi(t)=\lambda H(t)$. Since H is non-decreasing and λ is positive, ψ is non-decreasing. Also,

$$\sum_{i=1}^{\infty}\psi^i(t)=H^{\infty}(t)\frac{\lambda}{1-\lambda},$$

where $H^{\infty}(t)=\lim_{i\to\infty}H^i(t)$. If $x\neq C$ or $y\neq C$, then $\alpha(x,y)=0$ and so $\alpha(x,y)d(Fx,Fy)\leq\psi(d(x,y))$. Thus, $\alpha(x,y)d(Fx,Fy)\leq\psi(d(x,y))$ for all $x,y\in C$. Now by using Lemma 1.1, F has a fixed point which is the solution of the problem. □

Now, we provide an example to illustrate our main result.

Example 2.1. Consider the pointwise defined problem

$$D^{\frac{7}{2}}x(t)=f\left(t,x(t),x'(t),D^{\frac{1}{2}}x(t),\int_0^t\xi x(\xi)d\xi\right), \tag{2.4}$$

with boundary conditions $x(0)=0$, $x^{(j)}(0)=0$ for $j\geq 2$ and $j\neq 3$ and

$$x(1)=\frac{1}{4}D^{\frac{1}{3}}x\left(\frac{1}{10}\right)+\frac{1}{3}D^{\frac{1}{2}}x\left(\frac{1}{5}\right),$$

where

$$f(t,x_1,\dots,x_4)=\frac{t}{4p(t)}(|x_1|+\cdots+|x_4|),$$

$p(t)=0$ whenever $t\in[0,1]\cap\mathbb{Q}$ and $p(t)=1$ whenever $t\in[0,1]\cap\mathbb{Q}^c$. Put $h(t)=t$, $\Lambda(t,x_1,\dots,x_4)=f(t,x_1,\dots,x_4)$, $H(z)=z$, $\theta(t)=\frac{t}{p(t)}$, $n_0=1$, $b_1(t)=\frac{t}{4p(t)}$, $K_1(x_1,\dots,x_4)=|x_1|+\cdots+|x_4|$ and $q_1=4$. Then

$$m_0=\int_0^1 h(\xi)d(\xi)=\int_0^1\xi d(\xi)=\frac{1}{2},$$

$\Lambda(t, x_1, \dots, x_4)$ is a positive and non-decreasing mapping with respect to $x_1, \dots, x_4$ and

$$\lim_{z\to 0^+} \frac{\Lambda(t, z, \dots, z)}{H(z)} = \theta(t),$$

for almost all $t \in [0,1]$, $H : [0,\infty) \to [0,\infty)$ is a linear mapping, $\lim_{z\to 0^+} H(z) = 0$ and $\lim_{i\to\infty} H^i(t) = t < \infty$ for all $t \in [0,\infty)$, $\|\hat{\theta}\|_{[0,1]} \le \frac{2}{5}$,

$$|f(t, x_1, \dots, x_4)| \le \sum_{k=1}^{n_0} b_j(t) K_j(|x_1|, \dots, |x_4|) = b_1(t) K_1(|x_1|, \dots, |x_4|),$$

almost everywhere on $[0,1]$, $K_1(|x_1|, \dots, |x_4|)$ is a positive and non-decreasing mapping with respect to $x_1, \dots, x_4$, $\lim_{z\to 0^+} \frac{K_1(z,\dots,z)}{z} = 4 = q_1$ and $\|\hat{b_1}\|_{[0,1]} \le \frac{2}{20}$. Also we have

$$M = \max\left\{1, \frac{1}{\Gamma(2-\beta)}, m_0\right\} = \max\left\{1, \frac{1}{\Gamma(\frac{3}{2})}, \frac{1}{2}\right\} = \frac{2}{\sqrt{\pi}}$$

and

$$\begin{aligned}|\Delta| &:= \left| k! \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(k+1-\beta_i)} \gamma_i^{k-\beta_i} - 1 \right| \\ &= \left| 3! \left[\frac{\frac{1}{4}}{\Gamma(4-\frac{1}{3})} \left(\frac{1}{10}\right)^{4-\frac{1}{3}} + \frac{\frac{1}{3}}{\Gamma\left(4-\frac{1}{2}\right)} \left(\frac{1}{5}\right)^{4-\frac{1}{2}} \right] - 1 \right| \ge 0.997.\end{aligned}$$

Since

$$\begin{aligned}&\left[\frac{1}{\Gamma(\alpha-1)} + \frac{k}{|\Delta|\Gamma(\alpha)} + \frac{k}{|\Delta|} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(\alpha-\beta_i)}\right] \max\left\{\sum_{j=1}^{n_0} \|\hat{b_j}\|_{[0,1]} q_j, \|\hat{\theta}\|_{[0,1]}\right\} \\ \le&\left[\frac{1}{\Gamma(\frac{7}{2})} + \frac{3}{0.997\Gamma(\frac{7}{2})} + \frac{3}{0.997}\left(\frac{\frac{1}{4}}{\Gamma(\frac{7}{2}-\frac{1}{3})} + \frac{\frac{1}{3}}{\Gamma\left(\frac{7}{2}-\frac{1}{2}\right)}\right)\right] \max\left\{\frac{2}{20} \times 4, \frac{2}{5}\right\} \\ <&\left[\frac{8}{15\sqrt{\pi}} + \frac{8}{0.997 \times 5\sqrt{\pi}} + \frac{3}{0.997}\left(\frac{\frac{1}{4}+\frac{1}{3}}{6}\right)\right] \times \frac{2}{5} \\ <&0.604 < \frac{1}{M}.\end{aligned}$$

By using Theorem 2.1, we conclude that the problem (2.4) has a solution.

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