APPLICATION OF JACOBI POLYNOMIAL AND MULTIVARIABLE ALEPH-FUNCTION IN HEAT CONDUCTION IN NON-HOMOGENEOUS MOVING RECTANGULAR PARALLELEPIPED

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ABSTRACT. The present paper deals with an application of Jacobi polynomial and multivariable Aleph-function to solve the differential equation of heat conduction in non-homogeneous moving rectangular parallelepiped. The temperature distribution in the parallelepiped, moving in a direction of the length (x-axis) between the limits $x = -1$ and $x = 1$ has been considered. The conductivity and the velocity have been assumed to be variables. We shall see two particular cases and the cases concerning Aleph-function of two variables and the I-function of two variables.

1. INTRODUCTION AND PRELIMINARIES

We suppose the parallelepiped has heat conductivity K, density ρ, diffusivity k and specific heat σ. The partial differential equation satisfied by the temperature $v(x, y, z, t)$ at any time t in a homogeneous parallelepiped bounded by the planes $y = 0$ and $y = b$, $z = 0$ and $z = c$, moves with a constant velocity U in the direction of its length (x-axis) between the limits $x = -1$ and $x = 1$, on the lines of Carslaw and Jaeger [4, page 155, (1)] is

$$(1.1) \qquad k \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] - U \frac{\partial v}{\partial x} - \frac{\partial v}{\partial t} = 0,$$

where $k = \frac{K}{\rho \sigma}$. If we consider a non-homogeneous parallelepiped of variable conductivity $k'(1 - x^2)$ and the velocity $k_0 \left[(\alpha - \beta) + (\alpha + \beta) x \right]$, where k', k_0, α and β are

constants, the partial differential equation (1.1) reduces to

$$(1.2) \quad \frac{\partial v}{\partial t} = k_0 \left[\left(1 - x^2\right) \frac{\partial^2 v}{\partial x^2} + \left((\beta - \alpha) - (\alpha + \beta + 2)x\right) \frac{\partial v}{\partial x} \right] + k \left[\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] = 0,$$

where $k_0 = \frac{k'}{\rho\sigma}$, $\operatorname{Re}(\alpha) > -1$, $\operatorname{Re}(\beta) > -1$.

As physical example we can consider the temperature distribution of the moving mercury parallelepiped between the planes $x = -1$, $x = 1$, $y = 0$ and $y = b$, $z = 0$ and $z = c$ connected by two reservoirs of the mercury at the two ends. The variable flow in the mercury at the end $x = -1$ with a certain speed. The initial temperature distribution in the parallelepiped of mercury can be taken to be $f(x, y, z)$. The surfaces $y = 0$ and $y = b$, $z = 0$ and $z = c$ of the parallelepiped are supposed to be insulated. The ends $x = -1$ and $x = 1$ of the mercury parallelepiped should also be insulated as the conductivity vanishes there.

2. Solution of the Problem

By assuming the solution of the partial differential equation (1.2) as $v(x, y, z, t) = X(x) Y(y) Z(z) T(t)$, the solution of the partial differential equation (1.2) reduces to

$$\frac{1}{T} \frac{dT}{dt} = \frac{k_0}{X} \left[\left(1 - x^2\right) \frac{d^2 X}{dx^2} + (-\alpha + \beta - (\alpha + \beta + 2)x) \frac{dX}{dx} \right] + \frac{k}{Y} \frac{d^2 Y}{dy^2} + \frac{k}{Z} \frac{d^2 Z}{dz^2}.$$

Now, taking

$$\frac{k_0}{X} \left[\left(1 - x^2\right) \frac{d^2 X}{dx^2} + (-\alpha + \beta - (\alpha + \beta + 2)x) \frac{dX}{dx} \right] = - k_0 n (n + \alpha + \beta + 1),$$

$$\frac{k}{Y} \frac{d^2 Y}{dy^2} = - k\lambda^2,$$

and $\frac{k}{Z} \frac{d^2 Z}{dz^2} = -k\upsilon^2$, λ, υ being constants, n being positive integer, we obtain the following equations

$$(2.1) \quad \left(1 - x^2\right) \frac{d^2 X}{dx^2} + (-\alpha + \beta - (\alpha + \beta + 2)x) \frac{dX}{dx} + n(n + \alpha + \beta + 1) X = 0,$$

$$(2.2) \quad \frac{d^2 Y}{dy^2} + \lambda^2 y = 0,$$

$$(2.3) \quad \frac{d^2 Z}{dz^2} + \upsilon^2 z = 0,$$

and

$$(2.4) \quad \frac{dT}{dt} = \left[-n(n + \alpha + \beta + 1) - k\left(\lambda^2 + \upsilon^2\right) \right] T.$$

The (2.1) is the differential equation of Jacobi polynomials and its solution is

$$X = P_n^{(\alpha,\beta)}(x).$$

The solution of (2.2), (2.3) and (2.4) are

$$Y = A \cos \lambda y + B \sin \lambda y,$$
$$Z = C \cos \upsilon z + D \sin \upsilon z$$

and

$$T = E \exp \left[- \left(k_0 \, n \, (n + \alpha + \beta + 1) + k \left(\lambda^2 + \upsilon^2 \right) \right) t \right],$$

where A, B, C, D, E are constants.

Hence, the general solution of (1.2), the temperature distribution at any point $M(x, y, z)$ of the parallelepiped at time t is given by

$$v(x, y, z, t) = \exp \left[- \left(k_0 \, n \, (n + \alpha + \beta + 1) + k \left(\lambda^2 + \upsilon^2 \right) \right) t \right] P_n^{(\alpha, \beta)}(x)$$

(2.5)
$$\times \, [A \cos \lambda y + B \sin \lambda y] \, [C \cos \upsilon z + D \sin \upsilon z],$$

if no heats flows from the surfaces $y = 0$ and $y = b$, $z = 0$ and $z = c$, $\left(\frac{\partial v}{\partial y} \right)_{y=b=0} = 0$, $\left(\frac{\partial v}{\partial z} \right)_{z=c=0} = 0$ for all x and t. These demand $B = 0$, $D = 0$, $\lambda = \frac{m\pi}{b}$, $\upsilon = \frac{l\pi}{c}$, where $m, l = 0, 1, 2, \ldots$.

Therefore, the solution (2.5) reduces to

$$v(x, y, z, t) = \sum_{n,m,l=0}^{\infty} A_{nml} \exp \left[- \left(k_0 \, n \, (n + \alpha + \beta + 1) + k \left(\lambda^2 + \upsilon^2 \right) \right) t \right]$$

$$\times \, P_n^{(\alpha, \beta)}(x) \cos \frac{m\pi}{b} y \cos \frac{l\pi}{c} z.$$

Here, $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$ and A_{nml} are constants. If the initial temperature distribution in parallelepiped is given by

(2.6)
$$\sum_{n,m,l=0}^{\infty} A_{nml} P_n^{(\alpha, \beta)}(x) \cos \frac{m\pi}{b} y \cos \frac{l\pi}{c} z.$$

Now, multiplying both sides of (2.6) by $(1-x)^\alpha (1+x)^\beta \, P_n^{(\alpha, \beta)}(x) \cos \frac{m\pi}{b} y \cos \frac{l\pi}{c} z$, integrating both sides between $x = -1$ and $x = 1$, $y = 0$ and $y = b$, $z = 0$ and $z = c$, and applying the result [5, Vol. II. Page 285, (5)]

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \left[P_n^{(\alpha, \beta)}(x) \right]^2 \mathrm{d}x = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! \, (\alpha+\beta+2n+1) \Gamma(\alpha+\beta+n+1)},$$

with $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\beta) > -1$, we obtain

$$A_{mnl} = \frac{n! \, (\alpha+\beta+2n+1) \Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}$$

$$\times \int_0^c \int_0^b \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \, P_n^{(\alpha, \beta)}(x) \cos \frac{m\pi}{b} y \cos \frac{l\pi}{c} z \, f(x, y, z) \, dx \, dy \, dz.$$

Hence, the temperature distribution in the non-homogeneous moving rectangular parallelepiped is given by

(2.7)

$$
v(x, y, z, t)
$$

$$
= \frac{1}{2^{\alpha+\beta-1}bc} \sum_{n,m,l=0}^{\infty} \frac{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}
$$

$$
\times \exp\left[-\left(k_0\, n\,(n+\alpha+\beta+1)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right)t\right] P_n^{(\alpha,\beta)}(x)\cos\frac{m\pi}{b}y\cos\frac{l\pi}{c}z
$$

$$
\times \int_0^c \int_0^b \int_{-1}^1 (1-x)^\alpha (1+x)^\beta\, P_n^{(\alpha,\beta)}(x)\cos\frac{m\pi}{b}y\cos\frac{l\pi}{c}z\, f(x,y,z)\,dx\,dy\,dz,
$$

with $\operatorname{Re}(\alpha) > -1$, $\operatorname{Re}(\beta) > -1$.

Remark 2.1. Prasad and Maurya [10] have given application of Jacobi polynomial and multivariable H-function in heat conduction in non-homogeneous moving rectangular parallelepiped; Simões et al. [9] have studied Green's functions for heat conduction for unbounded and bounded rectangular spaces.

3. Multivariable Aleph-Function

For an illustration, if we take $f(x,y,z) = f_1(x)\,f_2(y)\,f_3(z)$, $f_2(y) = e^{-\mu y}$, $f_3(z) = e^{-\delta z}$ and $f_1(x)$ to be the most general special function in the form of multivariable Aleph-function.

The multivariable Aleph-function is a generalization of the multivariable H-function defined by Srivastava and Panda [14, 15]. The multivariable Aleph-function is defined by means of the multiple contour integral [3, 7]:

$$
\aleph(z_1, \ldots, z_r)
$$

$$
= \aleph_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\ldots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}^{0,\,\mathfrak{n}:m_1,n_1,\ldots,m_r,n_r}
\left(
\begin{array}{c}
z_1 \\ \vdots \\ z_r
\end{array}
\left|
\begin{array}{c}
\left[\left(a_j; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)_{1,\mathfrak{n}}\right], \\
\cdots,
\end{array}
\right.
\right.
$$

$$
\left.
\begin{array}{c}
\left[\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \ldots, \alpha_{ji}^{(r)}\right)_{\mathfrak{n}+1,p_i}\right] : \left[\left(c_j^{(1)}\right), \left(\gamma_j^{(1)}\right)_{1,n_1}\right], \left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right]; \\
\left[\tau_i\left(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)}\right)_{m+1,q_i}\right] : \left[\left(d_j^{(1)}\right), \left(\delta_j^{(1)}\right)_{1,m_1}\right], \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right]; \\
\cdots; \left[\left(c_j^{(r)}\right), \left(\gamma_j^{(r)}\right)_{1,n_r}\right], \left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right] \\
\cdots; \left[\left(d_j^{(r)}\right), \left(\delta_j^{(r)}\right)_{1,m_r}\right], \left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]
\end{array}
\right)
$$

(3.1)

$$
= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \ldots, s_r) \prod_{k=1}^r \theta_k(s_k)\, z_k^{s_k}\, ds_1 \cdots ds_r,
$$

with $\omega = \sqrt{-1}$,

$$\psi\left(s_1, \ldots, s_r\right) = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma\left(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k\right)}{\sum_{i=1}^{R}\left[\tau_i \prod_{j=\mathfrak{n}+1}^{p_i} \Gamma\left(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k\right) \prod_{j=1}^{q_i} \Gamma\left(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k\right)\right]}$$

and

$$\theta_k\left(s_k\right) = \frac{\prod_{j=1}^{m_k} \Gamma\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) \prod_{j=1}^{n_k} \Gamma\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma\left(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k\right) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma\left(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k\right)\right]}.$$

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by

$$|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi,$$

where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$(3.2) \qquad + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0,$$

with $k = 1, \ldots, r$, $i = 1, \ldots, R$, $i^{(k)} = 1, \ldots, R^{(k)}$.

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. For convenience, we shall use the following notations in this paper.

$$V = m_1, n_1; \ldots; m_r, n_r$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \ldots; p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)},$$

$$A = \left\{\left(a_j; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)_{1,n}\right\}, \left\{\tau_i\left(a_{ji}; \alpha_{ji}^{(1)}, \ldots, \alpha_{ji}^{(r)}\right)_{n+1,p_i}\right\} : \left\{\left(c_j^{(1)}; \gamma_j^{(1)}\right)_{1,n_1}\right\},$$

$$\left\{\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}}\right\}; \ldots; \left\{\left(c_j^{(r)}; \gamma_j^{(r)}\right)_{1,n_r}\right\}, \left\{\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}}\right\},$$

$$B = \left\{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \ldots, \beta_{ji}^{(r)})_{m+1,q_i}\right\} : \left\{\left(d_j^{(1)}; \delta_j^{(1)}\right)_{1,m_1}\right\}, \left\{\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}}\right\};$$

$$\ldots; \left\{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\right\}, \left\{\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}}\right\}.$$

Let

$$f_1(x) = \aleph_{p_i, q_i, \tau_i; R:W}^{0,\mathbf{n}:V}\left(\begin{array}{c} p_1\left(1-x\right)^{m_1'}\left(1+x\right)^{m_1''} \\ \vdots \\ p_r\left(1-x\right)^{m_r'}\left(1+x\right)^{m_r''} \end{array} \middle| \begin{array}{c} A \\ \vdots \\ B \end{array}\right).$$

Substituting for $f_1(x)$, $f_2(y)$ and $f_3(z)$ in equation (2.7), which is justifiable under the given conditions, we evaluate the y and z-integrals, first and the write the multivariable Aleph-function into the Mellin-Barnes contour integral with the help of (3.1), apply the result [5, Vol. II, page 284, (3)],

$$(3.3) \quad \int_{-1}^{1} (1-x)^l (1+x)^\sigma \, P_n^{(\alpha,\beta)}(x) dx$$

$$= \frac{2^{l+\sigma+1}\Gamma(l+1)\Gamma(\sigma+1)}{\Gamma(l+\sigma+2)} \times {}_3F_2\left(-n, \alpha+\beta+n+1, l+1; \alpha+1, l+\sigma+2; 1\right),$$

with $\operatorname{Re}(\alpha) > -1$, $\operatorname{Re}(\beta) > -1$, and finally interpret the resulting Γ-functions with the definition of multivariable Aleph-function. The temperature distribution in a non-homogeneous moving rectangular parallelepiped is then

$$v(x,y,z,t) = \frac{\mu\delta e^{-(\mu b + \delta c)}\Gamma(\alpha+1)}{bc2^{\alpha+\beta-1}} \sum_{n,m,l,N=0}^{\infty} \frac{\left(1-(-)^m\right)\left(1-(-)^l\right)}{(\mu^2+m^2\pi^2/b^2)(\delta^2+l^2\pi^2/c^2)}$$

$$\times \frac{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+N+1)\,\Gamma(-n+N)}{N!\Gamma(\alpha+N+1)\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\Gamma(-n)}$$

$$\times \exp\left[-\left\{k_0\,n\,(n+\alpha+\beta+1) + k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right\}t\right]$$

$$\times P_n^{(\alpha,\beta)}(x)\cos\frac{m\pi}{b}y\cos\frac{l\pi}{c}z\,\aleph_{p_i+2,q_i+1,\tau_i;R:W}^{0,\,\mathbf{n}+2:V}\left(\begin{array}{c} p_1 2^{m_1'+m_1''} \\ \vdots \\ p_r 2^{m_r'+m_r''} \end{array}\right.$$

$$(3.4) \qquad \left.\begin{array}{c} (-\alpha-N:m_1',\ldots,m_r'),(-\beta:m_1'',\ldots,m_r''),A \\ \vdots \\ (-\alpha-\beta-N-1:m_1'+m_1'',\ldots,m_r'+m_r''),B \end{array}\right).$$

Provide that $\operatorname{Re}(\alpha) > -1$, $\operatorname{Re}(\beta) > -1$, m_i', $m_i'' > 0$ for $i = 1, \ldots, r$, and

$$\operatorname{Re}(\alpha+1) + \sum_{i=1}^{r} m_i' \min_{1 \leqslant j \leqslant m_i} \operatorname{Re}\left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0,$$

$$\operatorname{Re}(\beta+1) + \sum_{i=1}^{r} m_i'' \min_{1 \leqslant j \leqslant m_i} \operatorname{Re}\left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0,$$

$|\arg p_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (3.2).

Remark 3.1. For detail and applications of Aleph-function, the reader can refer recent work [2, 8, 16].

4. Particular Cases

(a) When the rectangular parallelepiped moves with uniform velocity, the partial differential equation (1.2) reduces to the unsteady case of the partial differential equation (1) of Carslaw and Jaeger [4] with no radiation but with variable conductibility. We have $\alpha + \beta = 0$ and

$$\frac{\partial v}{\partial t} = k_0 \left(\left(1 - x^2\right) \frac{\partial^2 v}{\partial x^2} + (\beta - \alpha - 2x) \frac{\partial v}{\partial x} \right) + k \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = 0,$$

and the temperature distribution in the parallelepiped between $x = -1$ and $x = 1$, $y = 0$ and $y = b$, $z = 0$ and $z = c$ is given by

$$v\left(x, y, z, t\right)$$

$$= \frac{2\mu\delta e^{-(\mu b + \delta c)} \Gamma\left(\alpha + 1\right)}{bc} \sum_{n,m,l,N=0}^{\infty} \frac{\left(1 - (-)^m\right)\left(1 - (-)^l\right)}{\left(\mu^2 + m^2\pi^2/b^2\right)\left(\delta^2 + l^2\pi^2/c^2\right)}$$

$$\times \frac{n!\,(2n+1)\,\Gamma\left(n+N+1\right)\Gamma\left(-n+N\right)}{N!\,\Gamma\left(\alpha+N+1\right)\Gamma\left(\alpha+n+1\right)\Gamma\left(\beta+n+1\right)\Gamma\left(-n\right)}$$

$$\times \exp\left[-\left(k_0\,n\,(n+1) + k\pi^2\left(\frac{m^2}{b^2} + \frac{l^2}{c^2}\right)\right)t\right] P_n^{(\alpha,\beta)}(x) \cos\frac{m\pi}{b}y \cos\frac{l\pi}{c}z$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i;R:W}^{0,\mathbf{n}+2:V} \left(\begin{array}{c} p_1 2^{m_1'+m_1''} \\ \vdots \\ p_r 2^{m_r'+m_r''} \end{array} \middle| \begin{array}{c} \left(-\alpha-N : m_1', \ldots, m_r'\right), \left(-\beta : m_1'', \ldots, m_r''\right), A \\ \vdots \\ \left(-N-1 : m_1'+m_1'', \ldots, m_r'+m_r''\right), B \end{array} \right),$$

under the same condition that (3.4) with $\alpha + \beta = 0$.

(b) When the parallelepiped is stationary between $x = -1$ and $x = 1$, we have $\alpha = \beta = 0$ and the partial differential equation (1.2) reduces to

$$\frac{\partial v}{\partial t} = k_0 \left(\left(1 - x^2\right) \frac{\partial^2 v}{\partial x^2} - 2x\frac{\partial v}{\partial x} \right) + k \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = 0,$$

and the temperature distribution in the parallelepiped between $x = -1$ and $x = 1$, $y = 0$ and $y = b$, $z = 0$ and $z = c$ is given by

$$v\left(x, y, z, t\right)$$

$$= \frac{2\mu\delta e^{-(\mu b + \delta c)}}{bc} \sum_{n,m,l,N=0}^{\infty} \frac{\left(1 - (-)^m\right)(1 - (-)^l)}{\left(\mu^2 + m^2\pi^2/b^2\right)\left(\delta^2 + l^2\pi^2/c^2\right)}$$

$$\times \frac{n!\,(2n+1)\,\Gamma\left(n+N+1\right)\Gamma\left(-n+N\right)}{N!\,\Gamma\left(N+1\right)\left(\Gamma(n+1)\right)^2\Gamma\left(-n\right)} \exp\left[-\left(k_0\,n\,(n+1) + k\pi^2\left(\frac{m^2}{b^2} + \frac{l^2}{c^2}\right)\right)t\right]$$

$$\times P_n(x) \cos\frac{m\pi}{b}y \cos\frac{l\pi}{c}z$$

$$\times \ \aleph^{0,\,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W} \left(\begin{array}{c} p_1 2^{m_1'+m_1''} \\ \vdots \\ p_r 2^{m_r'+m_r''} \end{array} \middle| \begin{array}{c} (-N:m_1',\cdots,m_r'),\,(0:m_1'',\ldots,m_r''),\,A \\ \vdots \\ (-N-1:m_1'+m_1'',\ldots,m_r'+m_r''),\,B \end{array} \right),$$

where $P_n(x)$ is a Legendre's polynomial, under the same condition that (3.4) with $\alpha = \beta = 0$.

5. Aleph-Function of Two Variables

If $r = 2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [13](see also, [6]) and the general solution is

$v(x,y,z,t)$

$$= \frac{\mu\delta e^{-(\mu b+\delta c)}\Gamma(\alpha+1)}{bc2^{\alpha+\beta-1}} \sum_{n,m,l,N=0}^{\infty} \frac{(1-(-)^m)\left(1-(-)^l\right)}{(\mu^2+m^2\pi^2/b^2)(\delta^2+l^2\pi^2/c^2)}$$

$$\times \ \frac{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+N+1)\,\Gamma(-n+N)}{N!\,\Gamma(\alpha+N+1)\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\Gamma(-n)}$$

$$\times \ \exp\left[-\left(k_0\,n\,(n+\alpha+\beta+1)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right)t\right] P_n^{(\alpha,\beta)}(x)\,\cos\frac{m\pi}{b}y\,\cos\frac{l\pi}{c}z$$

$$\times \ \aleph^{0,\,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W} \left(\begin{array}{c} p_1 2^{m_1'+m_1''} \\ \vdots \\ p_2 2^{m_2'+m_2''} \end{array} \middle| \begin{array}{c} (-\alpha-N:m_1',m_2'),\,(-\beta:m_1'',m_2''),\,A \\ \vdots \\ (-\alpha-\beta-N-1:m_1'+m_1'',m_2'+m_2''),\,B \end{array} \right),$$

under the same condition that (3.4) with $r = 2$.

6. I-Function of Two Variables

If $r = 2$ and $\tau_i, \tau_{i'}, \tau_{i''} \to 1$ the multivariable Aleph-function reduces to I-function of two variables defined by Sharma and Mishra [12] (see also, [11]) and the general solution is

$v(x,y,z,t)$

$$= \frac{\mu\delta e^{-(\mu b+\delta c)}\Gamma(\alpha+1)}{bc2^{\alpha+\beta-1}} \sum_{n,m,l,N=0}^{\infty} \frac{(1-(-)^m)\left(1-(-)^l\right)}{(\mu^2+m^2\pi^2/b^2)(\delta^2+l^2\pi^2/c^2)}$$

$$\times \ \frac{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+N+1)\,\Gamma(-n+N)}{N!\,\Gamma(\alpha+N+1)\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\Gamma(-n)}$$

$$\times \ \exp\left[-\left(k_0\,n\,(n+\alpha+\beta+1)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right)t\right] P_n^{(\alpha,\beta)}(x)\,\cos\frac{m\pi}{b}y\,\cos\frac{l\pi}{c}z$$

$$\times \ I^{0,\,\mathbf{n}+2:V}_{p_i+2,q_i+1,R:W} \left(\begin{array}{c} p_1 2^{m_1'+m_1''} \\ \vdots \\ p_2 2^{m_2'+m_2''} \end{array} \middle| \begin{array}{c} (-\alpha-N:m_1',m_2'),\,(-\beta:m_1'',m_2''),\,A \\ \vdots \\ (-\alpha-\beta-N-1:m_1'+m_1'',m_2'+m_2''),\,B \end{array} \right),$$

under the same condition that (3.4) with $r = 2$ and $\tau_i, \tau_{i'}, \tau_{i''} \to 1$.

7. Concluding Remarks

Specializing the parameters of the multivariable Aleph-function, we can obtain a large number of results involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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