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DENSITY PROBLEMS IN SOBOLEV'S SPACES ON TIME SCALES

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Abstract. In this paper, we present a generalization of the density some of the functional spaces on the time scale, for example, spaces of rd-continuous function, spaces of Lebesgue Δ-integral and first-order Sobolev's spaces.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was originally introduced by Stefan Hilger in his Ph.D. Thesis in 1988 in order to unify, extend and generalize continuous and discrete analysis (see Hilger [4]).

Recently, the Lebesgue Δ-integral has been introduced by Bohner and Guseinov in [2, Chapter 5]. For the fundamental relationship between Riemann and Lebesgue Δ-integrals see A. Cabada, D. Vivero [3]. The first study Sobolev's spaces on time scales R. Agarwal et al. (see [7]).

In this paper, we study the density relationship between some of the functional spaces on the time scale, for example, spaces of rd-continuous function, spaces of Lebesgue Δ-integral and first-order Sobolev's spaces.

2. Preliminaries

We will briefly recall some basic definitions and facts from time scale calculus that we will use in the sequel.

Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

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(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$, otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$, otherwise, set $\mathbb{T}^k = \mathbb{T}$.

Definition 2.1 ([1])**.** The function $\varphi : \mathbb{T} \to \mathbb{R}$ will be called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write $\varphi \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.2 ([1])**.** Assume $\varphi : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define φ^Δ to be the number (provided it exists), with the property that given any $\varepsilon > 0$, there is a neighbourhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$) for some $\delta > 0$ such that

$$|\varphi(\sigma(t)) - \varphi(s) - \varphi^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call φ^Δ the delta (or Hilger) derivative of φ at t.

Lemma 2.1 ([3])**.** *The set of all right-scattered points of $\mathbb{T}$ is at most countable, that is, there are $J \subset N$ and $\{t_j\}_{j\in J} \subset \mathbb{T}$ such that*

$$\mathcal{R} := \{t \in \mathbb{T}, \sigma(t) > t\} = \{t_j\}_{j\in J}.$$

In order to do this, given a function $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}}$, we need an auxiliary function which extends φ to the interval $[a, b]$ defined as

$$\widetilde{\varphi}(t) := \begin{cases} \varphi(t), & \text{if } t \in \mathbb{T}, \\ \varphi(t_j), & \text{if } t \in (t_j, \sigma(t_j)) \text{ for all } j \in J. \end{cases} \tag{2.1}$$

Let $E \subset \mathbb{T}$, we define

$$J_E = \{j \in J : t_j \in E \cap \mathcal{R}\} \quad \text{and} \quad \widetilde{E} = E \cup \bigcup\nolimits_{j\in J_E} (t_j, \sigma(t_j)). \tag{2.2}$$

Proposition 2.1 ([3])**.** *Let $A \subset \mathbb{T}$. Then A is a Δ-measurable if and only if, A is Lebesgue measurable.*

In this case the following properties hold for every Δ-measurable set A.

1. *If $b \notin A$, then*

$$\mu_\Delta(A) = \mu_L(A) + \sum_{j\in J_A} \mu(t_j); \tag{2.3}$$

2. *$\mu_\Delta(A) = \mu_L(A)$ if and only if $b \notin A$ and A has no right-scattered point.*

Theorem 2.1 ([3])**.** *Let $E \subset \mathbb{T}$ be a Δ-measurable such that $b \notin E$, let $\widetilde{E}$ be the set defined in (2.2), let $\varphi : \mathbb{T} \to \overline{\mathbb{R}}$ be a Δ-measurable function and $\widetilde{\varphi} : [a, b] \to \overline{\mathbb{R}}$ be the extension of φ to $[a, b]$. Then, φ is Lebesgue Δ-integrable on E if and only if $\widetilde{\varphi}$ is Lebesgue integrable on $\widetilde{E}$ and we have*

$$\int_E \varphi(t)\,\Delta t = \int_{\widetilde{E}} \widetilde{\varphi}(t)\,dt = \int_E \varphi(t)\,dt + \sum_{j\in J_E} \mu(t_j)\,\varphi(t_j). \tag{2.4}$$

We state some of their properties whose proofs can be found in [7, 8].

Definition 2.3 ([7])**.** Let $p \in [1, +\infty)$. Then, the set $L^p_\Delta(\mathbb{T}, \mathbb{R})$ is a Banach space together with the norm defined for every $\varphi \in L^p_\Delta(\mathbb{T}, \mathbb{R})$ as

$$\|\varphi\|_{L^p_\Delta(\mathbb{T},\mathbb{R})} = \left(\int_{[a,b)\cap\mathbb{T}} |\varphi(s)|^p \Delta s\right)^{\frac{1}{p}}.$$

We denote by:

$$C^1(\mathbb{T}, \mathbb{R}) := \left\{\varphi : \mathbb{T} \to \mathbb{R} : \varphi \text{ is } \Delta\text{-differentiable on } \mathbb{T}^k \text{ and } \varphi^\Delta \in C\left(\mathbb{T}^k, \mathbb{R}\right)\right\},$$
$$C^1_{rd}(\mathbb{T}, \mathbb{R}) := \left\{\varphi : \mathbb{T} \to \mathbb{R} : \varphi \text{ is } \Delta\text{-differentiable on } \mathbb{T}^k \text{ and } \varphi^\Delta \in C_{rd}\left(\mathbb{T}^k, \mathbb{R}\right)\right\}.$$

Theorem 2.2 ([8])**.** *Let $p \in [1, \infty)$, then, we have the following properties:*

1. *$C_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T}, \mathbb{R})$;*
2. *$L^p_\Delta(\mathbb{T}, \mathbb{R})$ is dense in $L^1_\Delta(\mathbb{T}, \mathbb{R})$;*
3. *$C^1_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $C(\mathbb{T}, \mathbb{R})$.*

Theorem 2.3 ([7])**.** *Let $p \in [1, +\infty)$. The set $W^{1,p}(\mathbb{T}, \mathbb{R})$ is a Banach space together with the norm defined for every $\varphi \in W^{1,p}(\mathbb{T}, \mathbb{R})$ as*

$$\|\varphi\|_{W^{1,p}(\mathbb{T},\mathbb{R})} = \|\varphi\|_{L^p_\Delta(\mathbb{T},\mathbb{R})} + \left\|\varphi^\Delta\right\|_{L^p_\Delta(\mathbb{T},\mathbb{R})}.$$

3. Main Results

In this section, assume that $\mathbb{T}$ is bounded with $a := \min \mathbb{T}$ and $b := \max \mathbb{T}$ and for simplification, we note

$$[c, d)_\mathbb{T} = [c, d) \cap \mathbb{T} \quad \text{and} \quad [c, d]_\mathbb{T} = [c, d] \cap \mathbb{T}, \quad \text{for all } c, d \in \mathbb{T}.$$

Remark 3.1. $C(\mathbb{T}, \mathbb{R})$ and $C_{rd}(\mathbb{T}, \mathbb{R})$ are Banach spaces together with the norm defined by

$$\|\varphi\|_\infty := \sup_{t\in[a,b]_\mathbb{T}} |\varphi(t)|.$$

Set

$$I := \{j \in J : \rho(t_j) = t_j\}.$$

To derive main results in this section, we need the following lemma.

Lemma 3.1. *Let $p \in [1, +\infty[$, $C(\mathbb{T}, \mathbb{R})$ is dense in $C_{rd}(\mathbb{T}, \mathbb{R})$ provided with the induced topology of $L^p_\Delta(\mathbb{T}, \mathbb{R})$.*

Proof. For all $i \in I$, we defined r_i by $r_i = \{t_j : t_j < t_i\}$. Let $(v^i_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$v^i_n = \frac{t_i - r_i}{(b - a)\, 2^n} \mu(t_i), \quad \text{for all } i \in I.$$

Then, for all $i \in I$, we have $(v_n^i)_n \in (r_i, t_i)$. Let $(t_n^i)_{n\in\mathbb{N}}$ be a sequence on time scale $\mathbb{T}$ defined by

$$t_n^i = \inf \left[t_i - v_n^i, t_i\right)_{\mathbb{T}}, \quad \text{for all } n \in \mathbb{N},\ i \in I. \tag{3.1}$$

Let $\varphi \in C_{rd}(\mathbb{T}, \mathbb{R})$, we consider the sequence function $(\varphi_n)_{n\in\mathbb{N}}$ given by

$$\varphi_n(t) = \begin{cases} \varphi(t_i) + \dfrac{\varphi(t_i) - \varphi(t_n^i)}{t_i - t_n^i}(t - t_i), & \text{if } t \in [t_n^i, t_i]_{\mathbb{T}} \text{ for all } i \in I, \\ \lim\limits_{t\to b^-} \varphi(t), & \text{if } t = b, \\ \varphi(t), & \text{if not.} \end{cases}$$

Set $t \in [t_n^i, t_i]_{\mathbb{T}}$, for all $i \in I$, which implies that

$$\begin{aligned} |\varphi_n(t) - \varphi(t)| &\le |\varphi(t_i)| + |\varphi(t)| + \left|\varphi(t_i) - \varphi\left(t_n^i\right)\right| \left|\frac{t - t_i}{t_i - t_n^i}\right| \\ &\le 2\|\varphi\|_\infty + \left|\varphi(t_i) - \varphi\left(t_n^i\right)\right| \\ &\le 4\|\varphi\|_\infty . \end{aligned}$$

Finally, we get that $|\varphi_n(t) - \varphi(t)| \le 4\|\varphi\|_\infty$ for all $t \in [a, b)_{\mathbb{T}}$. It is clear that $(\varphi_n)_n$ is continuous in $\mathbb{T}$. Now, we show that $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in $L_\Delta^p(\mathbb{T}, \mathbb{R})$. In particular, we have

$$\begin{aligned} \int_{[a,b)_{\mathbb{T}}} |\varphi_n(t) - \varphi(t)|^p \Delta t &= \int_{A_n} |\varphi_n(t) - \varphi(t)|^p \Delta t \le 4^p \|\varphi\|_\infty^p \int_{A_n} \Delta t \\ &= 4^p \|\varphi\|_\infty^p \mu_\Delta(A_n), \end{aligned}$$

with $A_n = \bigcup\limits_{i\in I} [t_n^i, t_i)_{\mathbb{T}}$, for all $n \in \mathbb{N}$. From (2.3), we have

$$\begin{aligned} \mu_\Delta(A_n) &= \lambda(A_n) + \sum_{i\in I} \sum_{t\in[t_n^i, t_i)_{\mathbb{R}}} \mu(t) \\ &\le \sum_{i\in I} \lambda\left(\left[t_n^i, t_i\right[\right) + \sum_{i\in I}\left(t_i - t_n^i\right) \\ &\le 2\sum_{i\in I}\left(t_i - t_n^i\right) \le \sum_{i\in I} v_n^i \\ &\le \sum_{i\in I} \frac{t_i - r_i}{(b-a)2^n}\mu(t_i) \le \frac{b-a}{2^{n-1}}. \end{aligned} \tag{3.2}$$

Therefore, we obtain

$$\|\varphi_n - \varphi\|_{L_\Delta^p(\mathbb{T},\mathbb{R})}^p \le 4^p \|\varphi\|_\infty^p \frac{b-a}{2^{n-1}}, \quad \text{for all } n \in \mathbb{N}.$$

The proof is complete. □

Remark 3.2. $C^1(\mathbb{T}, \mathbb{R})$ and $C_{rd}^1(\mathbb{T}, \mathbb{R})$ are Banach spaces together with the norm defined by

$$\|\varphi\|_1 := \|\varphi\|_\infty + \|\varphi^\Delta\|_\infty.$$

Let us define a second type of extension for a function φ on $[a,b]$. We introduce the following function

$$\overline{\varphi}(t) := \begin{cases} \varphi(t), & \text{if } t \in \mathbb{T}, \\ \dfrac{\varphi(\sigma(t_j)) - \varphi(t_j)}{\mu(t_j)}(t - t_j) + \varphi(t_j), & \text{if } t \in (t_j, \sigma(t_j)) \text{ for all } j \in J. \end{cases} \tag{3.3}$$

Lemma 3.2. *If $\varphi : [a,b] \to \mathbb{R}$ belongs to $C^1(a,b)$, then $\varphi_{|\mathbb{T}}$ belongs to $C^1_{rd}(\mathbb{T},\mathbb{R})$.*

Proof. We note $\psi = \varphi_{|\mathbb{T}}$, then ψ is $\Delta-$differentiable on $\mathbb{T}^k$, and ψ^Δ is given by

$$\psi^\Delta(t) = \begin{cases} \varphi'(t), & \text{if } t \in \mathbb{T}^k \backslash \mathcal{R}, \\ \dfrac{\varphi(\sigma(t_j)) - \varphi(t_j)}{\mu(t_i)}, & \text{if } t = t_j \in \mathbb{T}^k \text{ for all } j \in J. \end{cases}$$

Now, we show that ψ^Δ is rd-continuous. Let $t \in \mathbb{T}^k$ a left-dense or a right-dense point and prove that

$$\lim_{s \to t} \psi^\Delta(s) = \varphi'(t).$$

Since $\varphi \in C^1(a,b)$, then for all $\varepsilon > 0$, there exists $\alpha > 0$, such that

$$\left|\varphi'(s) - \varphi'(t)\right| \le \varepsilon, \quad \text{for all } s \in (t-\alpha, t+\alpha). \tag{3.4}$$

We define ξ on $(t-\alpha, t+\alpha)$ by $\xi(s) = \varphi(s) - \varphi(t)(t-s)$. By (3.4) we have $\left|\xi'(s)\right| \le \varepsilon$, for all $s \in (t-\alpha, t+\alpha)$. Then ξ is an ε-Lipschitz function on $(t-\alpha, t+\alpha)$, so we get

$$\left|\varphi'(\tau) - \frac{\varphi(\tau) - \varphi(s)}{\tau - s}\right| < \varepsilon, \qquad \text{for all } s, \tau \in (t-\alpha, t+\alpha) \text{ and } \tau \ne s.$$

And we have $\lim_{s \to t} \sigma(s) = t$. There exists $\gamma > 0$, such that $|\sigma(s) - t| \le \varepsilon$, for all $s \in (t-\gamma, t+\gamma) \cap \mathbb{T}$. Put $\delta = \min(\alpha, \gamma)$ for all $s \in (t-\delta, t+\delta) \cap \mathbb{T}$. We consider the following two cases.

If s is right-dense, then

$$\left|\varphi'(\tau) - \psi^\Delta(s)\right| = |\varphi'(\tau) - \varphi'(s)| \le \varepsilon.$$

If s is right-scattered, one has $\sigma(s), s \in (t-\delta, t+\delta) \cap \mathbb{T}$, then

$$\left|\varphi'(\tau) - \psi^\Delta(s)\right| = \left|\varphi'(\tau) - \frac{\varphi(\sigma(s)) - \varphi(s)}{\sigma(s) - s}\right| \le \varepsilon.$$

Finally, we obtain that ψ^Δ is a continuous function at right-dense points in $\mathbb{T}$, and its left-sided limits exist at left dense points in $\mathbb{T}$. □

Lemma 3.3. *Let $p \in [1, +\infty[$, $C^1(\mathbb{T},\mathbb{R})$ is dense in $C^1_{rd}(\mathbb{T},\mathbb{R})$ provided with the induced topology of $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$.*

Proof. Let $\varphi \in C^1_{rd}(\mathbb{T},\mathbb{R})$, we define $P_{i,n}$ by

$$P_{i,n}(t) = \varphi(t_i) + \varphi^\Delta(t_i)(t - t_i) + \alpha h_2(t, t_i) + \beta h_3(t, t_i), \quad \text{for all } t \in \left[t^i_n, t_i\right]_{\mathbb{T}},$$

where $(t_n^i)_{n\in\mathbb{N}}$ is defined in (3.1) and $(h_k)_k$ are polynomials defined in [1], we choose α and β such that

$$P_{i,n}\left(t_n^i\right)=\varphi\left(t_n^i\right) \quad \text{and} \quad P_{i,n}^{\Delta}\left(t_n^i\right)=\varphi^{\Delta}\left(t_n^i\right), \quad \text{for all } i\in I, n\in\mathbb{N}. \tag{3.5}$$

Then α and β is the solution of the following system

$$\begin{cases} \alpha h_2\left(t_i^n,t_i\right)+\beta h_3\left(t_i^n,t_i\right)=\varphi\left(t_i^n\right)-\varphi\left(t_i\right)-\varphi^{\Delta}\left(t_i\right)h_1\left(t_i^n,t_i\right), \\ \alpha h_1\left(t_i^n,t_i\right)+\beta h_2\left(t_i^n,t_i\right)=\varphi^{\Delta}\left(t_i^n\right)-\varphi^{\Delta}\left(t_i\right). \end{cases}$$

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$\varphi_n(t)=\begin{cases} P_{i,n}(t), & \text{if } t\in[t_n^i,t_i]_{\mathbb{T}} \text{ for all } i\in I, \\ \lim\limits_{t\to b^-}\varphi(t), & \text{if } t=b, \\ \varphi(t), & \text{if not.} \end{cases}$$

By (3.5), we conclude that φ_n is Δ-differentiable on $\mathbb{T}^k$ and $\left(\varphi_n^{\Delta}\right)$ is continuous in $\mathbb{T}^k$. For all $i\in I$, we get

$$\begin{aligned}\int_{[t_n^i,t_i[\cap\mathbb{T}}\left|\varphi_n(t)-\varphi(t)\right|\Delta t\leq&\int_{[t_n^i,t_i[\cap\mathbb{T}}\left(\left|\varphi(t)\right|+\left|\varphi(t_i)\right|+\left|\varphi^{\Delta}(t_i)\right|h_1(t,t_i)\right)\Delta t\\&+\int_{[t_n^i,t_i[\cap\mathbb{T}}\left|\alpha h_2(t,t_i)+\beta h_3(t,t_i)\right|\Delta t\\\leq&2\left\|\varphi\right\|_{\infty}h_1\left(t_i,t_i^n\right)+\left\|\varphi^{\Delta}\right\|_{\infty}h_1\left(t_i,t_i^n\right)\\&+\left|\alpha h_3\left(t_i^n,t_i\right)+\beta h_4\left(t_i^n,t_i\right)\right|\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}&\int_{[t_n^i,t_i[\cap\mathbb{T}}\left|\varphi_n^{\Delta}(t)-\varphi^{\Delta}(t)\right|\Delta t\\&\leq\int_{[t_n^i,t_i[\cap\mathbb{T}}\left(\left|\varphi^{\Delta}(t)\right|+\left|\varphi^{\Delta}(t_i)\right|+\left|\alpha h_1(t,t_i)+\beta h_2(t,t_i)\right|\right)\Delta t\\&\leq2\left\|\varphi^{\Delta}\right\|_{\infty}h_1\left(t_i,t_i^n\right)+\left|\alpha h_2\left(t_i^n,t_i\right)+\beta h_3\left(t_i^n,t_i\right)\right|.\end{aligned}\tag{3.7}$$

For all $i\in I$, we define $\eta_{k,i,n}$ on $[t_n^i,t_i)_{\mathbb{T}}$ by

$$\eta_{k,i,n}(s)=\alpha h_k(s,t_i)+\beta h_{k+1}(s,t_i), \quad \text{for all } k\in\mathbb{N}.$$

Hence, we deduce that

$$\eta_{k,i,n}^{\Delta}(s)=\alpha h_{k-1}(s,t_i)+\beta h_k(s,t_i)=\eta_{k-1,i,n}(s), \quad \text{for all } s\in\left[t_n^i,t_i\right)_{\mathbb{T}^k}, \tag{3.8}$$

by (3.8), we get

$$\left|\eta_{k,i,n}(s)\right|\leq\int_s^{t_i}\left|\eta_{k-1,i,n}(\tau)\right|\Delta\tau, \quad \text{for all } k\in\mathbb{N}, s\in\left[t_n^i,t_i\right)_{\mathbb{T}^k}. \tag{3.9}$$

Since, $\left|\eta_{1,i,n}(s)\right|\leq\left|\eta_{1,i,n}\left(t_i^n\right)\right|$ for all $s\in[t_n^i,t_i)_{\mathbb{T}}$, using the inequality (3.9), we find

$$\left|\eta_{2,i,n}(s)\right|\leq\left(t_i-t_i^n\right)\left|\eta_{1,i,n}\left(t_i^n\right)\right|, \quad \text{for all } s\in\left[t_n^i,t_i\right)_{\mathbb{T}} \tag{3.10}$$

and

$$\left|\eta_{3,i,n}(s)\right| \leq (t_i - t_i^n)^2 \left|\eta_{1,i,n}(t_i^n)\right|, \quad \text{for all } s \in \left[t_n^i, t_i\right)_{\mathbb{T}}. \tag{3.11}$$

By (3.10), we obtain

$$\begin{aligned} \left|\alpha h_2(t_i^n, t_i) + \beta h_3(t_i^n, t_i)\right| &\leq (t_i - t_i^n) \left|\eta_{1,i,n}(t_i^n)\right| \\ &\leq (t_i - t_i^n) \left|\varphi^{\Delta}(t_i^n) - \varphi^{\Delta}(t_i)\right| \\ &\leq 2 (t_i - t_i^n) \left\|\varphi^{\Delta}\right\|_{\infty}, \end{aligned} \tag{3.12}$$

and by (3.11), we have

$$\begin{aligned} \left|\alpha h_3(t_i^n, t_i) + \beta h_4(t_i^n, t_i)\right| &\leq (t_i - t_i^n)^2 \left|\eta_{1,i,n}(t_i^n)\right| \\ &\leq (t_i - t_i^n)^2 \left|\varphi^{\Delta}(t_i^n) - \varphi^{\Delta}(t_i)\right| \\ &\leq 2 \left\|\varphi^{\Delta}\right\| (t_i - t_i^n)^2. \end{aligned} \tag{3.13}$$

Substituting (3.13) in (3.6), we get

$$\begin{aligned} \int_{[a,b)_{\mathbb{T}}} \left|\varphi_n(t) - \varphi(t)\right| \Delta t &\leq \left(2\left\|\varphi\right\|_{\infty} + \left\|\varphi^{\Delta}\right\|_{\infty}\right) \sum_{i\in I} (t_i - t_i^n) + \left\|\varphi^{\Delta}\right\|_{\infty} \sum_{i\in I} (t_i - t_i^n)^2 \\ &\leq \frac{b-a}{2^n}\left(2\left\|\varphi\right\|_{\infty} + (b-a+1)\left\|\varphi^{\Delta}\right\|_{\infty}\right). \end{aligned} \tag{3.14}$$

It follows from (3.12) and (3.7), that

$$\int_{[a,b)_{\mathbb{T}}} \left|\varphi_n^{\Delta}(t) - \varphi^{\Delta}(t)\right| \Delta t \leq 4\left\|\varphi^{\Delta}\right\|_{\infty} \sum_{i\in I} (t_i - t_i^n) \leq \frac{b-a}{2^{n-2}} \left\|\varphi^{\Delta}\right\|_{\infty}. \tag{3.15}$$

By inequality (3.14) and (3.15), we obtain that $(\varphi_n)_n$ converges to φ in $W_{\Delta}^{1,1}(\mathbb{T},\mathbb{R})$. Finally, by Hölder's inequality, we conclude that $(\varphi_n)_n$ converges to φ in $W_{\Delta}^{1,p}(\mathbb{T},\mathbb{R})$. □

Remark 3.3. Let E, F, G be three spaces such that $E \subset F \subset G$ and (G,τ) is a topological space.

1) If F is dense in (G,τ) and E is dense in (F,τ), then E is dense in (G,τ).
2) If E is dense in G, then F is dense in G.

The following theorem is a new generalization of the Theorem 2.2.

Theorem 3.1. *Let $p \in [1,+\infty[$, then $C(\mathbb{T},\mathbb{R})$ is dense in $L_{\Delta}^{p}(\mathbb{T},\mathbb{R})$.*

Proof. Let $p \in [1,+\infty[$, we have $C(\mathbb{T},\mathbb{R}) \subset C_{rd}(\mathbb{T},\mathbb{R}) \subset L_{\Delta}^{p}(\mathbb{T},\mathbb{R})$. By Lemma 3.1 and Theorem 2.2, hence $C_{rd}(\mathbb{T},\mathbb{R})$ is dense in $L_{\Delta}^{p}(\mathbb{T},\mathbb{R})$ and $C(\mathbb{T},\mathbb{R})$ is dense in $C_{rd}(\mathbb{T},\mathbb{R})$ provided with the induced topology of $L_{\Delta}^{p}(\mathbb{T},\mathbb{R})$. Then, by Remark 3.3, we obtain $C(\mathbb{T},\mathbb{R})$ is dense in $L_{\Delta}^{p}(\mathbb{T},\mathbb{R})$. □

The following results are consequences of Theorem 3.2.

Proposition 3.1. *Let $p \in [1, +\infty[$, then $C^1_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T}, \mathbb{R})$.*

Proof. Let $p \in [1, +\infty[$. By Theorem 2.2, we have $C^1_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $C(\mathbb{T}, \mathbb{R})$, then $C^1_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $C(\mathbb{T}, \mathbb{R})$ provided with the induced topology of $L^p_\Delta(\mathbb{T}, \mathbb{R})$, and we have $C(\mathbb{T}, \mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T}, \mathbb{R})$, by Remark 3.3 , we conclude $C(\mathbb{T}, \mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T}, \mathbb{R})$. □

As a proposition of the previous result, we deduce the following corollary.

Corollary 3.1. *Let $p \in [1, +\infty)$, then $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$ is dense in $C(\mathbb{T}, \mathbb{R})$.*

Proof. We have $C^1_{rd}(\mathbb{T}, \mathbb{R}) \subset W^{1,p}_\Delta(\mathbb{T}, \mathbb{R}) \subset C(\mathbb{T}, \mathbb{R})$, by Theorem 2.2, $C^1_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $C(\mathbb{T}, \mathbb{R})$. Therefore, Remark 3.3 implies that $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$ is dense in $C(\mathbb{T}, \mathbb{R})$. □

In the same way, we find the following corollary.

Corollary 3.2. *Let $p \in [1, +\infty)$, then $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$ is dense in $C_{rd}(\mathbb{T}, \mathbb{R})$.*

Corollary 3.3. *Let $p \in [1, +\infty)$, then $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T}, \mathbb{R})$.*

The next result show that spaces $C^1_{rd}(\mathbb{T}, \mathbb{R})$ and $C^1(\mathbb{T}, \mathbb{R})$ are dense in $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$.

Theorem 3.2. *Let $p \in [1, +\infty)$, $C^1_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$.*

Proof. Let $\varphi \in W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$, by Corollary 3.9 in [7], we have $\overline{\varphi} \in W^{1,p}(a, b)$. Since $C^1((a, b))$ is dense in $W^{1,p}(a, b)$, then there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \in C^1(a, b)$ that converges to $\overline{\varphi}$ in $W^{1,p}(a, b)$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$\varphi_n = \psi_{n|\mathbb{T}}, \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 3.2, we get $(\varphi_n)_n \in C^1_{rd}(\mathbb{T}, \mathbb{R})$. Now we show that $(\varphi_n)_n$ converges to φ in $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$, we have

$$\left\|(\psi_n - \overline{\varphi})_{|\mathbb{T}}\right\|_{W^{1,p}_\Delta(\mathbb{T},\mathbb{R})} = \|\varphi_n - \varphi\|_{W^{1,p}_\Delta(\mathbb{T},\mathbb{R})},$$

by Corollary 3.10 in [7], there exists a constant $C > 0$ which only depends on $(b - a)$ such that

$$\|\varphi_n - \varphi\|_{W^{1,p}_\Delta(\mathbb{T},\mathbb{R})} \leq C \|\psi_n - \overline{\varphi}\|_{W^{1,p}((a,b))},$$

we prove that $(\varphi_n)_n$ converges to φ in $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$. □

Theorem 3.3. *Let $p \in [1, +\infty[$, then $C^1(\mathbb{T}, \mathbb{R})$ is dense in $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$.*

Proof. Let $p \in [1, +\infty[$. We have $C^1(\mathbb{T}, \mathbb{R}) \subset C^1_{rd}(\mathbb{T}, \mathbb{R}) \subset W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$. By Lemma 3.3 and Theorem 3.2, hence $C^1_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$ and $C^1(\mathbb{T}, \mathbb{R})$ is dense in $C^1_{rd}(\mathbb{T}, \mathbb{R})$ provided with the induced topology of $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$. Then, by Remark 3.3, we obtain $C^1(\mathbb{T}, \mathbb{R})$ is dense in $W^{1,p}_\Delta(\mathbb{T}, \mathbb{R})$. □

4. Conclusion

Finally, we give a diagrams that summarizes the main results

$$\begin{array}{ccccc}
C^1_{rd}(\mathbb{T},\mathbb{R}) & & C_{rd}(\mathbb{T},\mathbb{R}) & & \\
\downarrow & & \downarrow & & \\
W^{1,p}_{\Delta}(\mathbb{T},\mathbb{R}) & \longrightarrow & L^p_{\Delta}(\mathbb{T},\mathbb{R}) & \longrightarrow & L^1_{\Delta}(\mathbb{T},\mathbb{R}) \\
\uparrow & & \uparrow & & \\
C^1(\mathbb{T},\mathbb{R}) & & C(\mathbb{T},\mathbb{R}) & &
\end{array}$$

For $\mathbb{T}$ is bounded and $p \in [1, +\infty)$.

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