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# DERIVATIONS OF $BE$-ALGEBRAS FROM HYPER $BE$-ALGEBRAS

M. HAMIDI[1] AND A. BORUMAND SAEID[2]

Abstract. In this paper, we investigate some results in ($BE$-algebras) dual $BCK$-algebras and hyper ($BE$-algebras) dual $K$-algebras. We show that by a set $X$, we can construct a hyper ($BE$-algebra) dual $K$-algebra. By concept of ($BE$-algebras) dual $BCK$-algebras and fundamental relation on hyper ($BE$-algebras) dual $K$-algebras the notion of fundamental ($BE$-algebras) dual $BCK$-algebras is introduced. We prove that any ($BE$-algebra) dual $BCK$-algebra is a fundamental ($BE$-algebra) dual $BCK$-algebra, in practical, any infinite set converts to fundamental ($BE$-algebra) dual $BCK$-algebra of itself.

## 1. Introduction

In 1966, Imai and Iseki [2, 3] introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. As a generalization of a $BCK$-algebra, H. S. Kim and Y. H. Kim introduced the notion of a $BE$-algebra and investigated several properties [4].

The hyper algebraic structure theory was introduced in 1934 [5], by F. Marty at the 8th congress of Scandinavian mathematicians. Hyperstructures have many applications to several sectors of both pure and applied sciences. A. Radfar et al. introduced the notion of hyper $BE$–algebra, dual hyper $K$-algebra and investigate some properties [6]. Furthermore, they showed that under some condition hyper $BE$-algebras are equivalent to dual hyper $K$-algebras. Fundamental relations are one of the main tools in algebraic hyperstructures theory. In [1], M. Hamidi et al. introduced the concept of fundamental relation on hyper $BE$-algebras and hyper $K$-algebras.

*Key words and phrases.* (Hyper $BE$-algebra) dual hyper $K$-algebra, fundamental ($BE$-algebra) dual $BCK$-algebra.

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Furthermore, they showed that quotient of any dual hyper $K$-algebra on a regular relation is a hyper $BE$-algebra and this quotient on any good strongly regular relation is a dual $BCK$-algebra. They introduced "$\delta$" as a relation on (hyper $BE$-algebras) dual hyper $K$-algebras and "$\delta^*$" as a transitive closure of "$\delta$" such that is the smallest equivalence relation that contains "$\delta$".

In this paper, we need a relation for connecting category of ($BE$-algebras) dual $BCK$-algebras and category of (hyper $BE$-algebras) dual hyper $K$-algebras. For this, firstly we show that by any nonempty set can construct a hyper $BE$-algebra and dual hyper $K$-algebra. We introduce the notion of fundamental ($BE$-algebra) dual $BCK$-algebra via fundamental relation $\delta^*$ on (hyper $BE$-algebra) dual hyper $K$-algebra. We show that any nonempty set converts to a ($BE$-algebra) dual $BCK$-algebra such that is isomorphic to a fundamental ($BE$-algebra) dual $BCK$-algebra and especially any infinite ($BE$-algebra) dual $BCK$-algebra is isomorphic to fundamental ($BE$-algebra) dual $BCK$-algebra of itself. Moreover, we find some relations between category of ($BE$-algebras) dual $BCK$-algebras and (hyper $BE$-algebras) dual hyper $K$-algebras.

## 2. Preliminaries

**Definition 2.1.** [4, 7] An algebra $(X; *, 1)$ of type $(2, 0)$ is called a $BE$-algebra if following axioms hold:

(BE1) $x * x = 1$;
(BE2) $x * 1 = 1$;
(BE3) $1 * x = x$;
(BE4) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

We introduce a relation "$\leq$"on $X$ by $x \leq y$ if and only if $x * y = 1$. The $BE$–algebra $(X; *, 1)$ is said to be commutative, if for all $x, y \in X$, $(x * y) * y = (y * x) * x$.

**Definition 2.2.** [7] An algebra $(X; *, 1)$ of type $(2, 0)$ is called a dual $BCK$-algebra if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(dBCK1) $x * y = y * x = 1 \Rightarrow x = y$;
(dBCK2) $(x * y) * ((y * z) * (x * z)) = 1$;
(dBCK3) $x * ((x * y) * y) = 1$.

The dual $BCK$-algebra $(X; *, 1)$ is said to be commutative, if for all $x, y \in X$, $(x * y) * y = (y * x) * x$.

**Theorem 2.1.** [7] *By every nonempty set, can construct a commutative dual $BCK$-algebra.*

**Definition 2.3.** [1, 6] Let $H$ be a nonempty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation. Then $(H; \circ, 1)$ is called a hyper $BE$–algebra, if it satisfies the following axioms:

(HBE$_1$) $x < 1$ and $x < x$;

(HBE$_2$) $x \circ (y \circ z) = y \circ (x \circ z)$;
(HBE$_3$) $x \in 1 \circ x$;
(HBE$_4$) $1 < x$ implies $x = 1$, for all $x, y, z \in H$.

$(H; \circ, 1)$ is called a dual hyper $K$-algebra if satisfies $(HBE_1)$, $(HBE_2)$ and the following axioms:
(DHK$_1$) $x \circ y < (y \circ z) \circ (x \circ z)$;
(DHK$_4$) $x < y$ and $y < x$ imply that $x = y$, for all $x, y, z \in H$,
where the relation "$<$" is defined by $x < y$ if and only if $1 \in x \circ y$. For any two nonempty subsets $A$ and $B$ of $H$, we define $A < B$ if and only if there exist $a \in A$ and $b \in B$ such that $a < b$ and $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. The dual hyper $K$-algebra $(X; \circ, 1)$ is called a weak commutative dual hyper $K$-algebra, if for any $x, y \in X$, $((x \circ y) \circ y) \bigcap ((y \circ x) \circ x) \neq \emptyset$.

**Theorem 2.2.** [6] *Let $H$ be a hyper $BE$-algebra. Then*
(i) $A \circ (B \circ C) = B \circ (A \circ C)$;
(ii) $A < A$;
(iii) $1 < A$ *implies* $1 \in A$;
(iv) $x < y \circ x$;
(v) $x < y \circ z$ *implies* $y < x \circ z$;
(vi) $x < (x \circ y) \circ y$;
(vii) $z \in x \circ y$ *implies* $x < z \circ y$;
(viii) $y \in 1 \circ x$ *implies* $y < x$, *for all* $x, y, z \in H$ *and* $A, B, C \subseteq H$.

**Theorem 2.3.** [1] *Let $(X; \circ, 1)$ be a (hyper $BE$-algebra) dual hyper $K$-algebra and $R$ be an equivalence relation on $X$. Then, $R$ is a regular relation on $X$ if and only if $(X/R; *, \overline{1})$ is a hyper $BE$-algebra, where for any $\overline{x}, \overline{y} \in X/R$*

$$\overline{x} * \overline{y} = \{\overline{z} | z \in x \circ y\}$$

*and a binary relation "$<$" on $X/R$ by*

$$\overline{x} < \overline{y} \Leftrightarrow \overline{1} \in \overline{x} * \overline{y}.$$

**Theorem 2.4.** [1] *Let $(X; \circ, 1)$ be a (hyper $BE$-algebra) dual hyper $K$-algebra and $R$ be an equivalence relation on $X$. If $R$ is a strongly regular relation on $X$, then $(X/R; *, \overline{1})$ is a $BE$-algebra.*

## 3. Constructing of ($BE$-algebras) Dual $BCK$-algebras and (Hyper $BE$-algebras) Dual Hyper $K$-algebras

In this section, we construct a commutative ($BE$-algebra) dual $BCK$-algebra and a weak commutative (hyper $BE$-algebra) dual hyper $K$-algebra from arbitrary set. We show that the sets with the same cardinal number convert to isomorphic $BE$-algebras, dual $BCK$-algebras, hyper $BE$-algebras and dual hyper $K$-algebras.

**Theorem 3.1.** *Let $(X, *, 1)$ be a (BE-algebra) dual BCK-algebra. If $Y$ is a set and $|X| = |Y|$, then there exists a binary operation "$*'$" and $1'$ on $Y$, such that $(Y, *', 1')$ is a (BE-algebra) dual BCK-algebra and $(X, *, 1) \cong (Y, *', 1')$.*

*Proof.* Since $|X| = |Y|$, then there exists a bijection $\varphi : X \to Y$. For any $y_1, y_2 \in Y$, we define a binary operation "$*'$" on Y as follows:

$$y_1 *' y_2 = \varphi(x_1 * x_2),$$

where $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$ and $x_1, x_2 \in X$. It is easy to show that $*'$ is well-defined. Let $y_1 = y_1'$ and $y_2 = y_2'$. Since $\varphi$ is a bijection, then there exist unique elements $x_1, x_2, x_1', x_2' \in X$ such that $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$, $y_1' = \varphi(x_1')$, $y_2' = \varphi(x_2')$ and so $x_1 = x_1'$, $x_2 = x_2'$. Hence, $y_1 *' y_2 = \varphi(x_1 * x_2) = \varphi(x_1' * x_2') = y_1' *' y_2'$. We can show that $(Y, *', 1')$ is a ($BE$-algebra) dual $BCK$-algebra. Let $x_1, x_2 \in X$. Then

$$\varphi(x_1 * x_2) = \varphi(x_1) *' \varphi(x_2)$$

and $\varphi(1) = \varphi(x * x) = \varphi(x) *' \varphi(x) = 1'$. Therefore, $\varphi$ is homomorphism and then is isomorphism. □

**Theorem 3.2.** *Let $(X, \circ, 1)$ be a (hyper BE-algebra) dual hyper K-algebra. If $Y$ is a set and $|X| = |Y|$, then there exists a binary hyperoperation "$\circ'$" and $1'$ on $Y$, such that $(Y, \circ', 1')$ is a (hyper BE-algebra) dual hyper K-algebra and $(X, \circ, 1) \simeq (Y, \circ', 1')$.*

*Proof.* Since $|X| = |Y|$, then there exists a bijection $\varphi : X \to Y$. For any $y_1, y_2 \in Y$, we define a binary hyperoperation "$\circ'$" on Y as follows:

$$y_1 \circ' y_2 = \varphi(x_1 \circ x_2),$$

where $y_1 = \varphi(x_1)$, $y_2 = \varphi(x_2)$ and $x_1, x_2 \in X$. It is easy to show that $\circ'$ is well-defined. Now, we define $1' = \varphi(1)$. We can show that $(Y, \circ', 1')$ is a (hyper $BE$-algebra) dual hyper $K$-algebra.

Let $x_1, x_2 \in X$. Then

$$\varphi(x_1 \circ x_2) = \varphi(x_1) \circ' \varphi(x_2).$$

Therefore, $\varphi$ is homomorphism and then is isomorphism. □

**Theorem 3.3.** *Let $(X, \star, 1)$ and $(Y, \star', 1')$ be dual BCK-algebras. Then there exists a binary hyperoperation "$\circ$" and "$1''$"on $X \times Y$, such that $(X \times Y, \circ, 1'')$ is a dual hyper K-algebra.*

*Proof.* Let $(X, \star, 1)$ and $(Y, \star', 1')$ be two dual $BCK$-algebras. For any $(x_1, y_1), (x_2, y_2)$ $\in X \times Y$, we define the binary hyperoperation "$\circ$" on $X \times Y$ as follows:

$$(x_1, y_1) \circ (x_2, y_2) = \{(x_1 \star x_2, y_2), (x_1 \star x_2, y_1 \star' y_2)\}$$

and $1'' = (1, 1')$. First, we show that the hyperoperation "$\circ$" is well defined.

Let $(x_1, y_1) = (x_1', y_1')$ and $(x_2, y_2) = (x_2', y_2')$. Then,

$$\begin{aligned}(x_1, y_1) \circ (x_2, y_2) &= \{(x_1 \star x_2, y_2), (x_1 \star x_2, y_1 \star' y_2)\} \\ &= \{(x_1' \star x_2', y_2'), (x_1' \star x_2', y_1' \star' y_2')\}\end{aligned} \tag{3.1}$$

$$=(x_1', y_1') \circ (x_2', y_2').$$

Now, we define a binary relation "$<''$" on $X \times Y$ as follows:

$$(x, y) <'' (z, w) \text{ if and only if } (1, 1') \in (x, y) \circ (z, w).$$

We show that for any $(x, y), (z, w) \in X \times Y$,

$$(x, y) <'' (z, w) \text{ if and only if } x < z \text{ and } y <' w. \tag{3.2}$$

For this, let $(x, y) <'' (z, w)$. Then $(1, 1') \in (x, y) \circ (z, w) = \{(x \star z, w), (x \star z, y \star' w)\}$ and so $(1, 1') = (x \star z, w)$ or $(1, 1') = (x \star z, y \star' w)$. If $(1, 1') = (x \star z, w)$, then $x < z$ and $y <' w = 1'$. If $(1, 1') = (x \star z, y \star' w)$, then $x < z$ and $y <' w$. Therefore, in any cases, we have, $x < z$ and $y <' w$.

Conversely, let $x < z$ and $y <' w$. Then $x \star z = 1$ and $y \star' w = 1'$. Hence $(1, 1') \in (x, y) \circ (z, w)$. Therefore, $(x, y) <'' (z, w)$.

Now, we will show that $(X \times Y, \circ, 1'')$ is a dual hyper $K$-algebra. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$. Then we have the following.

(HBE1) By (3.2), $(x_1, y_1) <'' (x_1, y_1)$ and $(x_1, y_1) <'' (1, 1')$.

(HBE2)

$$\begin{aligned}
&(x_1, y_1) \circ ((x_2, y_2)) \circ (x_3, y_3)) \\
=&(x_1, y_1) \circ \{(x_2 \star x_3, y_3), (x_2 \star x_3, y_2 \star' y_3)\} \\
=&\{(x_1 \star (x_2 \star x_3), y_3), (x_1 \star (x_2 \star x_3), y_1 \star' y_3), (x_1 \star (x_2 \star x_3), y_2 \star' y_3), \\
&(x_1 \star (x_2 \star x_3), y_1 \star' (y_2 \star' y_3))\} \\
=&\{(x_2 \star (x_1 \star x_3), y_3), (x_2 \star (x_1 \star x_3), y_1 \star' y_3), (x_2 \star (x_1 \star x_3), y_2 \star' y_3), \\
&(x_2 \star (x_1 \star x_3), y_2 \star' (y_1 \star' y_3))\} \\
=&(x_2, y_2) \circ \{(x_1 \star x_3, y_3), (x_1 \star x_3, y_1 \star' y_3)\} \\
=&(x_2, y_2) \circ ((x_1, y_1) \circ (x_3, y_3)).
\end{aligned}$$

(HBE3) $(1, 1') \circ (x, y) = \{(1 \star x, y), (1 \star x, 1' \star' y)\} = \{(x, y)\}$.

(HBE4) If $(1, 1') <'' (x, y)$, then by (3.2), $(x, y) = (1, 1')$.

(DHK3)

$$\begin{aligned}
&((x_2, y_2) \circ (x_3, y_3)) \circ ((x_1, y_1) \circ (x_3, y_3)) \\
=&\{(x_2 \star x_3, y_3), (x_2 \star x_3, y_2 \star' y_3)\} \circ \{(x_1 \star x_3, y_3), (x_1 \star x_3, y_1 \star' y_3)\} \\
=&\{((x_2 \star x_3) \star (x_1 \star x_3), y_3), ((x_2 \star x_3) \star (x_1 \star x_3), y_3 \star' y_3), \\
&((x_2 \star x_3) \star (x_1 \star x_3), y_1 \star' y_3)), ((x_2 \star x_3) \star (x_1 \star x_3), y_3 \star' (y_1 \star' y_3))), \\
&((x_2 \star x_3) \star (x_1 \star x_3), (y_2 \star' y_3) \star' y_3)), ((x_2 \star x_3) \star (x_1 \star x_3), (y_2 \star' y_3) \star' (y_1 \star' y_3)\}.
\end{aligned}$$

Since $(x_1, y_1) \circ (x_2, y_2) = \{(x_1 \star x_2, y_2), (x_1 \star x_2, y_1 \star' y_2)\}$, we get

$$(x_1, y_1) \circ (x_2, y_2) <'' ((x_2, y_2) \circ (x_3, y_3)) \circ ((x_1, y_1) \circ (x_3, y_3)).$$

(DHK4) Let $(x_1, y_1) <'' (x_2, y_2)$ and $(x_2, y_2) <'' (x_1, y_1)$. Then by (3.2), $(x_1, y_1) = (x_2, y_2)$. Therefore, $(X \times Y, \circ, 1'')$ is a dual hyper $K$-algebra. □

**Corollary 3.1.** *Let $(X, \star, 1)$ and $(Y, \star', 1')$ be BE-algebras. Then there exists a binary hyperoperation "$\circ$" and "$1''$" on $X \times Y$, such that $(X \times Y, \circ, 1'')$ is a hyper BE-algebra.*

*Example* 3.1. Consider the dual $BCK$-algebras $(\{1,2,3,4\}, \star, 1)$ and $(\{a,b\}, \star', a)$ as follows:

| $\star$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 1 | 1 | 3 | 4 |
| 3 | 1 | 2 | 1 | 4 |
| 4 | 1 | 2 | 3 | 1 |

,

| $\star'$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $a$ | $a$ |

.

Now for any $x, y, w$ we set $w_x = (w, x)$, $\overline{w}_x = \{(w, x)\}$, $\overline{w}_{x,y} = \{(w, x), (w, y)\}$ and define a hyperoperation "$\circ$" on $\{1,2,3,4\} \times \{a,b\}$ as follows:

| $\circ$ | $1_a$ | $1_b$ | $2_a$ | $2_b$ | $3_a$ | $3_b$ | $4_a$ | $4_b$ |
|---|---|---|---|---|---|---|---|---|
| $1_a$ | $\overline{1}_a$ | $\overline{1}_b$ | $\overline{2}_a$ | $\overline{2}_b$ | $\overline{3}_a$ | $\overline{3}_b$ | $\overline{4}_a$ | $\overline{4}_b$ |
| $1_b$ | $\overline{1}_a$ | $\overline{1}_{a,b}$ | $\overline{2}_a$ | $\overline{2}_{a,b}$ | $\overline{3}_a$ | $\overline{3}_{a,b}$ | $\overline{4}_a$ | $\overline{4}_{a,b}$ |
| $2_a$ | $\overline{1}_a$ | $\overline{1}_b$ | $\overline{1}_a$ | $\overline{1}_b$ | $\overline{3}_a$ | $\overline{3}_b$ | $\overline{4}_a$ | $\overline{4}_b$ |
| $2_b$ | $\overline{1}_a$ | $\overline{1}_{a,b}$ | $\overline{1}_a$ | $\overline{1}_{a,b}$ | $\overline{3}_a$ | $\overline{3}_{a,b}$ | $\overline{4}_a$ | $\overline{4}_{a,b}$ |
| $3_a$ | $\overline{1}_a$ | $\overline{1}_b$ | $\overline{2}_a$ | $\overline{2}_b$ | $\overline{1}_a$ | $\overline{1}_b$ | $\overline{4}_a$ | $\overline{4}_b$ |
| $3_b$ | $\overline{1}_a$ | $\overline{1}_{a,b}$ | $\overline{2}_a$ | $\overline{2}_{a,b}$ | $\overline{1}_a$ | $\overline{1}_{a,b}$ | $\overline{4}_a$ | $\overline{4}_{a,b}$ |
| $4_a$ | $\overline{1}_a$ | $\overline{1}_b$ | $\overline{2}_a$ | $\overline{2}_b$ | $\overline{3}_a$ | $\overline{3}_b$ | $\overline{1}_a$ | $\overline{1}_b$ |
| $4_b$ | $\overline{1}_a$ | $\overline{1}_{a,b}$ | $\overline{2}_a$ | $\overline{2}_{a,b}$ | $\overline{3}_a$ | $\overline{3}_{a,b}$ | $\overline{1}_a$ | $\overline{1}_{a,b}$ |

.

Then it is easy to see that $(\{1,2,3,4\} \times \{a,b\}, \circ, 1_a)$ is a dual hyper $K$-algebra.

**Corollary 3.2.** *Let $(X, \star, 1)$ be a (BE-algebra) dual BCK-algebra. Then for any arbitrary distinct elements $a, b$:*
(i) *there exists a binary hyperoperation "$\circ$", and constant "$1'$" on $X \times \{a,b\}$, such that $(X \times \{a,b\}, \circ, 1')$ is a (hyper BE-algebra) dual hyper K-algebra;*
(ii) *if $X$ is infinite, then there exists a binary hyperoperation "$\circ$", a constant "$1$" on $X$, $\circ'$ and $1'$ on $X \times \{a,b\}$ such that $(X, \circ, 1)$ and $(X \times \{a,b\}, \circ', 1')$ are (hyper BE-algebra) dual hyper K-algebras and $(X \times \{a,b\}, \circ', 1') \cong (X, \circ, 1)$.*

**Theorem 3.4.** *By every nonempty set, we can construct a weak commutative dual hyper K-algebra.*

*Proof.* Let $X$ be an arbitrary nonempty set. If $|X| \leq 2$, is clear. But if $|X| > 2$, for any $x, y \in X$, we define a binary operation "$\circ$" on $X$ as follows:

$$x \circ y = \begin{cases} \{x_0, p\}, & \text{if } x = y, \\ \{y\}, & \text{otherwise,} \end{cases}$$

and

$$x < y \text{ if and only if } \ x_0 \in x \circ y,$$

where $x_0$ and $p$ are fixed elements of $X$. Now, we show that $(X, \circ, x_0)$ is a dual hyper $K$-algebra.

(HBE1) Let $x \in X$. Then by definition of $\circ$, $x_0 \in x \circ x$ and $x_0 \in x \circ x_0$ and so $x < x_0$ and $x < x$.
(HBE2) Let $x, y, z \in X$. We consider the following cases.
Case 1: $x, y, z$ are distinct. Then, $x \circ (y \circ z) = x \circ \{z\} = \{z\} = y \circ \{z\} = y \circ (x \circ z)$.
Case 2: $x = y \neq z$. Then, $x \circ (y \circ z) = x \circ \{z\} = \{z\} = y \circ \{z\} = y \circ (x \circ z)$.
Case 3: $x = z \neq y$. Then, $x \circ (y \circ z) = x \circ \{z\} = \{x_0, p\} = y \circ \{x_0, p\} = y \circ (x \circ z)$.
Case 4: $y = z \neq x$. Then, $x \circ (y \circ z) = x \circ \{x_0, p\} = \{x_0, p\} = y \circ \{z\} = y \circ (x \circ z)$.
Case 5: $x = y = z$. Then, $x \circ (y \circ z) = x \circ \{x_0, p\} = \{x_0, p\} = y \circ \{x_0, p\} = y \circ (x \circ z)$.
(DHK3) Let $x, y, z \in X$. We have the following cases.
Case 1: $x, y, z$ are distinct. Then, $x \circ y = \{y\} < \{x_0, p\} = \{z\} \circ \{z\} = (y \circ z) \circ (x \circ z)$.
Case 2: $x = y \neq z$. Then, $x \circ y = \{x_0, p\} < \{x_0, p\} = \{z\} \circ \{z\} = (y \circ z) \circ (x \circ z)$.
Case 3: $x = z \neq y$. Then, $x \circ y = \{y\} < \{x_0, p\} = \{z\} \circ \{x_0, p\} = (y \circ z) \circ (x \circ z)$.
Case 4: $y = z \neq x$. Then, $x \circ y = \{y\} < \{z\} = \{x_0, p\} \circ \{z\} = (y \circ z) \circ (x \circ z)$.
Case 5: $x = y = z$. Then, $x \circ y = \{x_0, p\} < \{x_0, p\} = \{x_0, p\} \circ \{x_0, p\} = (y \circ z) \circ (x \circ z)$.
(DHK4) Let $x, y \in X$. Then by definition of $\circ$, $x_0 \in x \circ y = x \circ y$ implies that $x = y$.
Moreover, for any $x, y \in X, x_0 \in (x \circ y) \circ y \cap (y \circ x) \circ x \neq \emptyset$.

Therefore, $(X, \circ, x_0)$ is a weak commutative dual $K$-algebra. □

*Example* 3.2. Let $X = \{1, a, b, c, d, e\}$. Define the hyperoperation "$\circ$" as follows:

| $\circ$ | 1 | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| 1 | $\{1, e\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ | $\{e\}$ |
| $a$ | $\{1, e\}$ | $\{1, e\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ | $\{e\}$ |
| $b$ | $\{1, e\}$ | $\{a\}$ | $\{1, e\}$ | $\{c\}$ | $\{d\}$ | $\{e\}$ |
| $c$ | $\{1, e\}$ | $\{a\}$ | $\{b\}$ | $\{1, e\}$ | $\{d\}$ | $\{e\}$ |
| $d$ | $\{1, e\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{1, e\}$ | $\{e\}$ |
| $e$ | $\{1, e\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ | $\{1, e\}$ |

Then $(X; \circ, 1)$ is a weak commutative dual hyper $K$–algebra.

**Corollary 3.3.** *By every nonempty set, we can construct a weak commutative hyper $BE$-algebra.*

## 4. Fundamental ($BE$-algebras) dual $BCK$-algebras

In this section, by using the notion of fundamental relation, we define the concept of fundamental ($BE$-algebra) dual $BCK$-algebra and we prove that any ($BE$-algebra) dual $BCK$-algebra is a fundamental ($BE$-algebra) dual $BCK$-algebra.

Now, in the following, we apply the concept of $\delta^*$ relation on (hyper $BE$–algebra) dual hyper $K$–algebra which is studied in [1]. Let $(X; \circ, 1)$ be a (hyper $BE$-algebra) dual hyper $K$-algebra and $A$ be a subset of $X$. $\mathcal{L}(A)$ will denote the set of all finite combinations of elements $A$ with $\circ$ and $\bigodot_{i=1}^{n} a_i = a_1 \circ a_2 \circ \ldots a_n$.

**Definition 4.1.** Let $(X;\circ,1)$ be a (hyper $BE$–algebra) dual hyper $K$–algebra. Consider

$$\delta_1 = \{(x,x)|x \in X\}$$

and for every integer $n \geq 1$, $\delta_n$, define as follows:

$$x\delta_n y \Leftrightarrow \exists(a_1,a_2,\ldots,a_n) \in X^n, \exists u \in \mathcal{L}(a_1,a_2,\ldots,a_n)\ \{x,y\} \subseteq u.$$

Obviously, for every $n \geq 1$, the relations $\delta_n$ are symmetric, the relation $\delta = \bigcup_{n\geq 1} \delta_n$ is a reflexive and symmetric relation. Let $\delta^*$ be the transitive closure of $\delta$ (the smallest transitive relation such that contains $\delta$).

*Example* 4.1. Let $X = \{1,a,b,c\}$. Then $(X,\circ,1)$ is a weak dual hyper $K$-algebra as follows:

| $\circ$ | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1$ | $\{1,a\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ |
| $a$ | $\{1,a\}$ | $\{1,a\}$ | $\{b\}$ | $\{c\}$ |
| $b$ | $\{1,a\}$ | $\{a\}$ | $\{1,a\}$ | $\{c\}$ |
| $c$ | $\{1,a\}$ | $\{a\}$ | $\{b\}$ | $\{1,a\}$ |

.

Now, $\delta = \{(1,1),(a,a),(b,b),(c,c),(1,a),(a,1)\}$ and clearly $\delta^* = \delta$.

**Theorem 4.1.** [1] *Let $(X;\circ,1)$ be a (hyper $BE$-algebra) weak commutative dual hyper $K$-algebra. Then $\delta^*$ is a (strongly regular) good strongly regular on $X$.*

**Theorem 4.2.** [1] *Let $(X;\circ,1)$ be a (hyper $BE$-algebra) weak commutative dual hyper $K$-algebra. Then $(X/\delta^*;*,\overline{1})$ is a ($BE$-algebra) commutative dual $BCK$-algebra.*

**Lemma 4.1.** *Let $(X,\circ_X,1_X)$ and $(Y,\circ_Y,1_Y)$ be (hyper $BE$-algebras) dual hyper $K$-algebras. Then*
(i) *there exists a hyperoperation "$\circ_{X\times Y}$" and "$1_{X\times Y}$" on $X\times Y$ such that $(X\times Y,\circ_{X\times Y}, 1_{X\times Y})$ is a (hyper $BE$-algebras) dual hyper $K$-algebra;*
(ii) *for any $(x,y),(x',y') \in X\times Y$, we have $(x,y)\delta^*_{X\times Y}(x',y')$ if and only if $x\ \delta^*_X\ x'$ and $y\ \delta^*_Y\ y'$.*

*Proof.* (i) For any $(x,y),(x',y') \in X\times Y$, define $\circ_{X\times Y}$ on $X\times Y$ as follows:

$$(x,y)\circ_{X\times Y}(x',y') = (x\circ_X x')\times(y\circ_Y y')$$

and $1_{X\times Y} = (1_X,1_Y)$. It is easy to verify that $(X\times Y,\circ_{X\times Y},1_{X\times Y})$ is a (hyper $BE$-algebra) dual hyper $K$-algebra.
(ii) Let $(x,x') \in X^2$ and $(y,y') \in Y^2$. Then $x\delta^*_X x'$ and $y\delta^*_Y y'$ if and only if there exist $u \in \mathcal{L}(X)$ and $v \in \mathcal{L}(Y)$ such that $\{x,x'\} \subseteq u$ and $\{y,y'\} \subseteq v$, if and only if $\{(x,y),(x',y')\} \in u\times v$ if and only if $(x,y)\delta^*_{X\times Y}(x',y')$. □

*Example* 4.2. Let $X = \{1,2,3\}$ and $Y = \{a,b\}$. Then $(X;\circ_1,1)$ and $(Y;\circ_2,1)$ are dual hyper $K$-algebras by the following tables:

| $\circ_1$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $\{1,3\}$ | $\{2\}$ | $\{3\}$ |
| 2 | $\{1,3\}$ | $\{1,3\}$ | $\{3\}$ |
| 3 | $\{1,3\}$ | $\{2\}$ | $\{1,3\}$ |

,

| $\circ_2$ | $a$ | $b$ |
|---|---|---|
| $a$ | $\{a,b\}$ | $\{b\}$ |
| $b$ | $\{a,b\}$ | $\{a,b\}$ |

.

Now we define a hyperoperation "$\circ$" on $\{1,2,3\}\times\{a,b\}$ as follows:

| $\circ$ | $(1,a)$ | $(1,b)$ | $(2,a)$ | $(2,b)$ | $(3,a)$ | $(3,b)$ |
|---|---|---|---|---|---|---|
| $(1,a)$ | $A\times B$ | $A\times\{b\}$ | $\{2\}\times B$ | $\{(2,b)\}$ | $\{3\}\times B$ | $\{(3,b)\}$ |
| $(1,b)$ | $A\times B$ | $A\times\{a,b\}$ | $\{2\}\times B$ | $\{2\}\times B$ | $\{3\}\times B$ | $\{3\}\times B$ |
| $(2,a)$ | $A\times B$ | $A\times\{b\}$ | $A\times B$ | $A\times\{b\}$ | $\{3\}\times B$ | $\{(3,b)\}$ |
| $(2,b)$ | $A\times B$ | $A\times B$ | $A\times B$ | $A\times B$ | $\{3\}\times B$ | $\{3\}\times B$ |
| $(3,a)$ | $A\times B$ | $A\times\{b\}$ | $\{2\}\times B$ | $\{(2,b)\}$ | $A\times B$ | $A\times\{b\}$ |
| $(3,b)$ | $A\times B$ | $A\times B$ | $\{2\}\times B$ | $\{2\}\times B$ | $A\times B$ | $A\times B$ |

,

where $A=\{1,3\}$ and $B=\{a,b\}$. Clearly $(\{1,2,3\}\times\{a,b\},\circ,(1,a))$ is a dual hyper $K$-algebra.

**Theorem 4.3.** *Let $(X,\circ_X,1_X)$ and $(Y,\circ_Y,1_Y)$ be (hyper BE-algebras) dual hyper K-algebras. Then*

$$\frac{(X\times Y,\circ_{X\times Y},1_{X\times Y})}{\delta^*_{X\times Y}}\cong\frac{(X,\circ_X,1_X)}{\delta^*_X}\times\frac{(Y,\circ_Y,1_Y)}{\delta^*_Y}.$$

*Proof.* Define the mapping:

$$\varphi:\left(\frac{(X\times Y,\circ_{X\times Y})}{\delta^*_{X\times Y}},*\right)\to\left(\frac{(X,\circ_X)}{\delta^*_X}\times\frac{(K,\circ_Y)}{\delta^*_Y},*\right),$$

by $\varphi(\delta^*_{X\times Y}(x,y))=(\delta^*_X(x),\delta^*_Y(y))$. Firstly, we show that $\varphi$ is well defined and one to one. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$. Then, by Lemma 4.1, $\delta^*_{X\times Y}(x_1,y_1)=\delta^*_{X\times Y}(x_2,y_2)$ if and only if $\delta^*_X(x_1)=\delta^*_X(x_2)$ and $\delta^*_Y(y_1)=\delta^*_Y(y_2)$ if and only if $\varphi(\delta^*_{X\times Y}(x_1,y_1))=\varphi(\delta^*_{X\times Y}(x_2,y_2))$. Secondly, we show that $\varphi$ is homomorphism. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$. Then, by Lemma 4.1, for any $c\in\delta^*_X(x_1)\circ_X\delta^*_X(x_2)$ and $d\in\delta^*_Y(y_1)\circ_Y\delta^*_Y(y_2)$ we have

$$\begin{aligned}\varphi(\delta^*_{X\times Y}(x_1,y_1)*(\delta^*_{X\times Y}(x_2,y_2))=&\varphi(\delta^*_{X\times Y}(c,d))=(\delta^*_X(c),\delta^*_Y(d))\\=&(\delta^*_X(x_1),\delta^*_Y(y_1))*(\delta^*_X(x_2),\delta^*_Y(y_2))\\=&\varphi(\delta^*_{X\times Y}(x_1,y_1))*\varphi(\delta^*_{X\times Y}(x_2,y_2))\end{aligned}$$

and $\varphi(\delta^*_{X\times Y}(1_X,1_Y))=(\delta^*_X(1_X),\delta^*_Y(1_Y))$. Clearly, $\varphi$ is a bijection and hence, $\varphi$ is an isomorphism. □

**Theorem 4.4.** *Let $(X,\circ,1)$, $(Y,\circ',1')$ be (hyper BE-algebras) dual hyper K-algebras and $f:(X,\circ,1)\to(Y,\circ',1')$ be a homomorphism. Then the following statements are satisfied:*

(i) *for any $x,y\in X$, $x\delta^*y$ implies that $f(x)\delta^*f(y)$;*

(ii) *if $f$ is an injective, then for any $x, y \in X$, $f(x)\delta^* f(y)$ implies that $x\delta^* y$;*
(iii) *if $f$ is a bijection, then for any $x, y \in X$, $x\delta^* y$ if and only if $f(x)\delta^* f(y)$;*
(iv) *if $f$ is a bijection, then for any $x \in X$, $f(\delta^*(x)) = \delta^*(f(x))$.*

*Proof.* (i) Let $x, y \in X$. Since $x\delta^* y$, then there exists $u \in \mathcal{L}(X)$, such that $\{x, y\} \subseteq u$. Now, for a homomorphism $f : (X, \circ, 1) \to (Y, \circ', 1')$ we have $\{f(x), f(y)\} = f\{x, y\} \subseteq f(u) \in \mathcal{L}(Y)$. Therefore, $f(x)\delta^* f(y)$.
(ii) For $x, y \in X$, since $f(x)\delta^* f(y)$, there exists $v \in \mathcal{L}(Y)$, such that $\{f(x), f(y)\} \subseteq v$. Since $f : (X, \circ, 1) \to (Y, \circ', 1')$ is injective, we get,

$$\{x, y\} = \{f^{-1}(f(x)), f^{-1}(f(y))\} = f^{-1}\{f(x), f(y)\} \subseteq f^{-1}(v) \in \mathcal{L}(X).$$

Therefore, $x\delta^* y$.
(iii) By (i) and (ii), the proof is straightforward.
(iv) Let $x \in X$. Then we have $f(\delta^*(x)) = \bigcup_{y \in \delta^*(x)} f(y) = \bigcup_{y\delta^* x} f(y)$. By (iii), for any $x, y \in X$, $x\delta^* y$ if and only if $f(x)\delta^* f(y)$. Therefore,

$$\begin{aligned} f(\delta^*(x)) &= \bigcup_{y \in \delta^*(x)} f(y) = \bigcup_{y\delta^* x} f(y) \\ &= \bigcup_{f(y)\delta^* f(x)} f(y) = \bigcup_{f(y) \in \delta^*(f(x))} f(y) \\ &= \delta^*(f(x)). \end{aligned}$$

□

*Example* 4.3. Let $X = \{1, a, b, c, d\}$ and $Y = \{x, y, z, t\}$. Then $(X; \circ_1, 1)$ and $(Y; \circ_2, 1)$ are hyper $BE$-algebras by the following tables:

| $\circ_1$ | $1$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $1$ | $\{1,d\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ |
| $a$ | $\{1,d\}$ | $\{1,d\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ |
| $b$ | $\{1,d\}$ | $\{a\}$ | $\{1,d\}$ | $\{c\}$ | $\{d\}$ |
| $c$ | $\{1,d\}$ | $\{a\}$ | $\{b\}$ | $\{1,d\}$ | $\{d\}$ |
| $d$ | $\{1,d\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{1,d\}$ |

,

| $\circ_2$ | $x$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $x$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{t\}$ |
| $y$ | $\{x\}$ | $\{x\}$ | $\{z\}$ | $\{t\}$ |
| $z$ | $\{x\}$ | $\{y\}$ | $\{x\}$ | $\{t\}$ |
| $t$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x\}$ |

.

Now, we define a map $f : (X; \circ_1, 1) \to (Y; \circ_2, x)$ by $f(c) = z$, $f(d) = t$ and $f(1) = f(a) = f(b) = x$. Clearly $f$ is a homomorphism, but is not injective and $f(b) \in \delta^*(f(1))$, but $b \notin \delta^*(1)$.

**Theorem 4.5.** *Let $(X, \circ, 1)$ be a (hyper $BE$-algebra) dual hyper $K$-algebra. If $Y$ is a set and $|X| = |Y|$, then there exists a binary hyperoperation "$\circ'$" "$1'$" on $Y$, such that $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong \left(\frac{(Y,\circ',1')}{\delta^*}, *\right)$.*

*Proof.* Since $|X| = |Y|$, then by Theorem 3.2, there exists a binary hyperoperation "$\circ'$", and $1'$ on Y such that $(Y, \circ', 1')$ is a (hyper $BE$-algebras) dual hyper $K$-algebra. Moreover, there exists an isomorphism $f : (X, \circ, 1) \to (Y, \circ', 1')$, such that $f(1) = 1'$. Now, we define the map $\varphi : \left(\frac{(X,\circ,1)}{\delta^*}, *\right) \to \left(\frac{(Y,\circ',1')}{\delta^*}, *\right)$ by $\varphi(\delta^*(x)) = \delta^*(f(x))$. First,

we show that for any $x_1, x_2 \in X$, $\varphi(\delta^*(x_1) * \delta^*(x_2)) = \varphi(\delta^*(x_1)) * \varphi(\delta^*(x_2))$. For any $x_1, x_2 \in X$ we have

$$\begin{aligned} \varphi(\delta^*(x_1) * \delta^*(x_2)) =& \varphi(\delta^*(x_1 \circ x_2)) = \delta^*(f(x_1 \circ x_2)) \\ =& \delta^*(f(x_1) \circ' f(x_2)) = \delta^*(f(x_1)) * \delta^*(f(x_2)) \\ =& \varphi(\delta^*(x_1)) * \varphi(\delta^*(x_2)). \end{aligned} \tag{4.1}$$

Since $f$ is bijection, then $\varphi$ is a bijection. Now, we show that $\varphi$ is well-defined. Let $y_1, y_2 \in Y$. Then there exist unique elements $x_1, x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Now, by Equation (4.1) and Theorem 4.4, $\varphi(\delta^*(x_1)) = \varphi(\delta^*(x_2))$ if and only if $\delta^*(f(x_1)) = \delta^*(f(x_2))$ if and only if $\delta^*(x_1) = \delta^*(x_2)$. Therefore, $\varphi$ is well-defined and one to one and by Equation (4.1), is a homomorphism. Hence $\varphi$ is an isomorphism. Therefore, $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong \left(\frac{(Y,\circ',1')}{\delta^*}, *\right)$. □

**Corollary 4.1.** *Let $(X, \circ, 1)$ and $(Y, \circ', 1')$ be isomorphic (hyper $BE$-algebras) dual hyper $K$-algebra. Then $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong \left(\frac{(Y,\circ',1')}{\delta^*}, *\right)$.*

**Definition 4.2.** A ($BE$-algebra) dual $BCK$-algebra $(X, \star, 1)$, is called a (*fundamental $BE$-algebra*) *fundamental dual $BCK$-algebra, if there exists a nontrivial (hyper $BE$-algebra) dual hyper $K$-algebra $(Y, \circ', 1')$, such that $\left(\frac{(Y,\circ',1')}{\delta^*}, *\right) \cong (X, \star, 1)$. In other words, it is equal to the fundamental of nontrivial (hyper $BE$-algebra) dual hyper $K$-algebra up to isomorphic.*

*Example* 4.4. Let $X = \{1, 2, 3\}$. Then $(X, \star, 1)$ is a $BE$-algebra as follows:

| $\star$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 1 | 1 | 3 |
| 3 | 1 | 2 | 1 |

.

We construct a dual hyper $K$-algebra $(Y = \{a, b, c, d\}, \circ, a)$ as follows:

| $\circ$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $\{a, b\}$ | $\{b\}$ | $\{c\}$ | $\{d\}$ |
| $b$ | $\{a, b\}$ | $\{a, b\}$ | $\{c\}$ | $\{d\}$ |
| $c$ | $\{b, a\}$ | $\{b\}$ | $\{a, b\}$ | $\{d\}$ |
| $d$ | $\{a, b\}$ | $\{b\}$ | $\{c\}$ | $\{a, b\}$ |

.

Clearly, $\delta^*(a) = \{a, b\}$, $\delta^*(c) = \{c\}$ and $\delta^*(d) = \{d\}$ and so $\frac{Y}{\delta^*} = \{\{a, b\}, \{c\}, \{d\}\}$. Now, we have

| $*$ | $\delta^*(a)$ | $\delta^*(c)$ | $\delta^*(d)$ |
|---|---|---|---|
| $\delta^*(a)$ | $\delta^*(a)$ | $\delta^*(c)$ | $\delta^*(d)$ |
| $\delta^*(c)$ | $\delta^*(a)$ | $\delta^*(a)$ | $\delta^*(d)$ |
| $\delta^*(d)$ | $\delta^*(a)$ | $\delta^*(c)$ | $\delta^*(a)$ |

.

Clearly, $\left(\frac{Y}{\delta^*}, *, \delta^*(a)\right)$ is a dual $BCK$-algebra and $(X, \star, 1) \cong \left(\frac{Y}{\delta^*}, *, \delta^*(a)\right)$. Therefore, $(X, \star, 1)$ is a fundamental dual $BCK$-algebra.

*Remark* 4.1. We know that on any ($BE$-algebra) dual $BCK$-algebra $(X, \star, 1)$, if define a binary hyperoperation $''\circ''$ as $x \circ y = \{x \star y\}$ such that is a singleton, then $(X, \circ, 1)$ is a *trivial* (hyper $BE$-algebra) dual hyper $K$-algebra. Therefore, its fundamental ($BE$-algebra) dual $BCK$-algebra is isomorphic to $(X, \star, 1)$. In the following, we define nontrivial (hyper $BE$-algebra) dual hyper $K$-algebra such that its fundamental ($BE$-algebra) dual $BCK$-algebra, be isomorphic to given ($BE$-algebra) dual $BCK$-algebra $(X, \star, 1)$.

**Theorem 4.6.** *Every dual $BCK$-algebra is isomorphic to a fundamental dual $BCK$-algebra.*

*Proof.* Let $(X, \star, 1)$ be a dual $BCK$-algebra. Then by Theorem 3.3, for any dual $BCK$-algebra $(Y, \star', 1')$, $(X \times Y, \circ, (1, 1'))$ is a dual hyper $K$-algebra. First, we show that for any $(a, b) \in X \times Y$, $\delta^*(a, b) = \{(a, x) \mid x \in Y\}$. For this let, $u = \bigodot_{i=1}^{n}(x_i, y_i) \in \mathcal{L}(X \times Y)$, where $(x_i, y_i) \in X \times Y$. We have

$$u = \bigodot_{i=1}^{n}(x_i, y_i) = \left\{ (\bigodot_{i=1}^{n} x_i, y_i), |x_i \in X, y_i \in Y \right\}.$$

Then we obtain $u = \{(a, y_i) | a \in X \text{ is fixed and } y_i \in Y\}$. Hence, for any $(a, b), (c, d) \in X \times Y$, $(a, b)\delta^*(c, d)$ if and only if $a = c$. Now, we define the map

$$\varphi : \left( \frac{(X \times Y, \circ, (1, 1'))}{\delta^*}, * \right) \to (X, \star, 1)$$

by $\varphi(\delta^*(x, y)) = x$. It is clear that $\delta^*(x, y) = \delta^*(x', y')$ if and only if $x = x'$ if and only if $\varphi(\delta^*(x, y)) = \varphi(\delta^*(x', y'))$. Then, $\varphi$ is well defined and one to one. In follow, we show that $\varphi$ is a homomorphism. For this we have,

$$\varphi(\delta^*(x, y) * \delta^*(x', y')) = \varphi(\delta^*(x \star x', y)) = x \star x' = \varphi(\delta^*(x, y)) \star \varphi(\delta^*(x', y')).$$

Clearly, $\varphi$ is onto. Therefore, $\varphi$ is an isomorphism. □

*Remark* 4.2. (i) The (hyper $BE$-algebras) dual hyper $K$-algebra $(X \times Y, \circ, 1)$ is called the *associated* (*hyper $BE$-algebras*) *dual hyper $K$-algebra* to $X$ via $Y$ (or shortly associated (hyper $BE$-algebras) dual hyper $K$-algebra) and denote by $X_Y$.
(ii) The mapping $\varphi : X \to X_Y$ by $\varphi(x) = (x, 1)$ is an embedding.

**Theorem 4.7.** *Let $(X, *, 1_X)$ and $(Y, *', 1_Y)$ be isomorphic dual $BCK$-algebras. Then, for any dual $BCK$-algebra $(Z, *'', 1_Z)$, $X_Z$ and $Y_Z$ are isomorphic dual hyper $K$-algebras.*

*Proof.* Let $f : (X, *, 1_X) \to (Y, *', 1_Y)$ be an isomorphism. Define map $\theta : (X_Z, \circ, 1) \to (Y_Z, \circ, 1)$ by $\theta(x, y) = (f(x), y)$, where $x \in X$, $y \in Y$. Clearly $\theta$ is a bijection, now we show that $\theta$ is a homomorphism. Let $(x_1, y), (x_2, y') \in X_Z$. Then,

$$\begin{aligned}\theta((x_1, y) \circ (x_2, y')) =& \theta(\{(x_1 * x_2, y), (x_1 * x_2, y')\}) \\ =& \{\theta(x_1 * x_2, y), \theta(x_1 * x_2, y')\} = \{(f(x_1 * x_2), y), (f(x_1 * x_2), y')\}\end{aligned}$$

$$\begin{aligned}&=\{(f(x_1)\ast' f(x_2),y),(f(x_1)\ast' f(x_2),y')\}\\&=(f(x_1),y)\circ(f(x_2),y'))\\&=\theta((x_1,y))\circ\theta((x_2,y')).\end{aligned}$$

Therefore, $\theta$ is an isomorphism and $(X_Z,\circ,1)\cong(Y_Z,\circ,1)$. □

**Corollary 4.2.** *Every $BE$-algebra is isomorphic to a fundamental $BE$-algebra.*

*Example* 4.5. Let $X=\{1,2,3\}$. Then $(X,\star,1)$ is a $BE$-algebra as follows:

| $\star$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 1 | 1 | 3 |
| 3 | 1 | 2 | 1 |

.

By Example 4.4, we saw that $(X,\star,1)$ is a fundamental dual $BCK$-algebra. Now, by Theorem 4.6, we construct a new dual hyper $K$-algebra. Consider the dual $BCK$-algebras $(\{1,2,3\},\star,1)$ and $(\{a,b\},\star',a)$ as follows:

| $\star$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 1 | 1 | 3 |
| 3 | 1 | 2 | 1 |

, | $\star'$ | $a$ | $b$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $a$ | $a$ |

.

Now we define a hyperoperation "$\circ$" on $\{1,2,3\}\times\{a,b\}$ as follows:

| $\circ$ | $(1,a)$ | $(1,b)$ | $(2,a)$ | $(2,b)$ | $(3,a)$ | $(3,b)$ |
|---|---|---|---|---|---|---|
| $(1,a)$ | $\{(1,a)\}$ | $A$ | $\{(2,a)\}$ | $B$ | $\{(3,a)\}$ | $C$ |
| $(1,b)$ | $A$ | $\{(1,b)\}$ | $B$ | $\{(2,b)\}$ | $C$ | $\{(3,b)\}$ |
| $(2,a)$ | $\{(2,a)\}$ | $B$ | $\{(3,a)\}$ | $C$ | $\{(3,a)\}$ | $C$ |
| $(2,b)$ | $B$ | $\{(2,b)\}$ | $C$ | $\{(3,b)\}$ | $C$ | $\{(3,b)\}$ |
| $(3,a)$ | $\{(3,a)\}$ | $C$ | $\{(3,a)\}$ | $C$ | $\{(3,a)\}$ | $C$ |
| $(3,b)$ | $C$ | $\{(3,b)\}$ | $C$ | $\{(3,b)\}$ | $C$ | $\{(2,b)\}$ |

,

where $A=\{(1,a),(1,b)\},B=\{(2,a),(2,b)\}$ and $C=\{(3,a),(3,b)\}$. Then it is easy to see that $(Z=\{1,2,3\}\times\{a,b\},\circ,(1,a))$ is a dual hyper $K$-algebra. We have $\delta^*((1,a))=\{(1,a),(1,b)\}$, $\delta^*((2,a))=\{(2,a),(2,b)\}$ and $\delta^*((3,a))=\{(3,a),(3,b)\}$. Hence $\frac{Z}{\delta^*}=\{\delta^*((1,a)),\delta^*((2,a)),\delta^*((3,a))\}$ and obtain the following table:

| $*$ | $\delta^*(1,a)$ | $\delta^*(2,a)$ | $\delta^*(3,a)$ |
|---|---|---|---|
| $\delta^*(1,a)$ | $\delta^*(1,a)$ | $\delta^*(2,a)$ | $\delta^*(3,a)$ |
| $\delta^*(2,a)$ | $\delta^*(1,a)$ | $\delta^*(1,a)$ | $\delta^*(3,a)$ |
| $\delta^*(3,a)$ | $\delta^*(1,a)$ | $\delta^*(2,a)$ | $\delta^*(1,a)$ |

.

Clearly, $\left(\frac{Z}{\delta^*},*,\delta^*((0,a))\right)$ is a dual $BCK$-algebra and $(X,\star,1)\cong\left(\frac{Z}{\delta^*},*,\delta^*((0,a))\right)$. Therefore, $(X,\star,1)$ is a fundamental dual $BCK$-algebra.

**Corollary 4.3.** *By every nonempty set, we can construct a fundamental* ($BE$*-algebra*) *dual $BCK$-algebra.*

*Proof.* By Theorem 2.1, there exists a binary operation "$\star$" and "$x_0$" such that $(X, \star, x_0)$ is a dual $BCK$-algebra. By Theorem 4.6, $(X, \star, x_0)$ is a fundamental dual $BCK$-algebra. □

**Theorem 4.8.** *Let $(X, \star, 1)$ be any finite ($BE$-algebra) dual $BCK$-algebra. Then for any binary hyperoperation "$\circ$", and constant "$1'$" on $X$, such that $(X, \circ, 1')$ is a (hyper $BE$-algebra) dual hyper $K$-algebra, there is not any isomorphic between $(X, \star, 1)$ and $\left(\frac{(X,\circ,1')}{\delta^*}, *\right)$, that is $(X, \star, 1) \ncong \left(\frac{(X,\circ,1')}{\delta^*}, *\right)$.*

*Proof.* Let $(X, \star, 1)$ be a finite ($BE$-algebra) dual $BCK$-algebra, $|X| = n$, "$\circ$" be a hyperoperation, and "$1'$" constant on $X$, such that $(X, \circ, 1')$ be a (hyper $BE$-algebra) dual hyper $K$-algebra. Then there exist $x, y \in X$ such that $|x \circ y| \geq 2$. Hence, there are $m, n \in x \circ y$ such that $\delta^*(m) = \delta^*(n)$. Since $\frac{X}{\delta^*} = \{\delta^*(x) \mid x \in X\}$, then, $|\frac{X}{\delta^*}| < n = |X|$. Therefore, $\left(\frac{(X,\circ,1')}{\delta^*}, *\right) \ncong (X, \star, 1)$. □

**Theorem 4.9.** *Let $(X, \star, 1)$ be a finite ($BE$-algebra) dual $BCK$-algebra such that $|X| \geq 2$. Then, there exists a hyperoperation "$\circ$" on $X$ such that $(X, \circ, 1)$ is a dual hyper $K$-algebra and $|(X, \star, 1)| = |((X, \circ, 1)/\delta^*, \delta^*(1), *)| + 1$.*

*Proof.* Let $p \in X$ be a fixed element of $X$. Then by Theorem 3.4, $(X, \circ, 1)$ is a dual hyper $K$-algebra as follows:

$$x \circ y = \begin{cases} \{1, p\}, & \text{if } x = y, \\ \{y\}, & \text{otherwise,} \end{cases}$$

where $x, y \in X$. Clearly $\delta^*(1) = \{1, p\}$ and for any $x \in X, \delta^*(x) = \{x\}$. Therefore, $|(X, \star, 1)| = |((X, \circ, 1)/\delta^*, \delta^*(1), *)| + 1$. □

*Remark* 4.3. Now, in the follow we try to show that for any infinite set $X$, there exists an operation "$\star$", constant 1 and a hyperoperation "$\circ$" on $X$, such that $(X, \star, 1)$ is an ($BE$-algebra) dual $BCK$-algebra and $(X, \circ, 1)$ is a (hyper $BE$-algebra) dual hyper $K$-algebra. Moreover, $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong (X, \star, 1)$.

**Theorem 4.10.** *Let $X$ be an infinite set. Then there exists an operation "$\star$", constant "$1$" and a binary hyperoperation "$\circ$" on $X$ such that $\left(\frac{(X,\circ,1)}{\delta^*}, *\right) \cong (X, \star, 1)$. That is, $X$ is a fundamental dual $BCK$-algebra of itself.*

*Proof.* Let $X$ be an infinite set. Then for any arbitrary set $\{a, b\}$, by Corollary 3.2, there exists a binary hyperoperation "$\circ$", a constant "1" on $X$, $\circ'$ and $1'$ on $X \times \{a, b\}$ such that $(X, \circ, 1)$ and $(X \times \{a, b\}, \circ', 1')$ are (hyper $BE$-algebra) dual hyper $K$-algebra and $(X \times \{a, b\}, \circ', 1') \cong (X, \circ, 1)$ and so

$$\frac{(X \times \{a, b\}, \circ', (1, a))}{\delta^*} \cong \frac{(X, \circ, 1)}{\delta^*}. \tag{4.2}$$

First, we show that for any $(m,t) \in X \times \{a,b\}$, $\delta^*(m,t) = \{(m,a),(m,b)\}$. For this let $u = \bigodot_{i=1}^{n}(m_i,n_i) \in \mathcal{L}(X \times \{a,b\})$, where $(m_i,n_i) \in X \times \{a,b\}$. We have

$$u = \bigodot_{i=1}^{n}(m_i,n_i) = \left\{ (\bigodot_{i=1}^{n} m_i, a), (\bigodot_{i=1}^{n} m_i, b) \right\}.$$

Now, we obtain $u = \{(m,a),(m,b) | m \in X \text{ is fixed}\}$. Hence, for any $(m,t),(n,s) \in X \times \{a,b\}$,

$$(m,t)\delta^*(n,s) \Leftrightarrow m = n.$$

Moreover, by Theorem 2.1, there exists an operation $\star$ and constant 1 such that $(X,\star,1)$ is a dual $BCK$-algebra.

Now, we define the map $\varphi : \left(\frac{(X\times\{a,b\},\circ',(1,a))}{\delta^*}, *\right) \to (X,\star,1)$ by $\varphi(\delta^*(m,t)) = m$. It is clear that $\delta^*(m,t) = \delta^*(m',s)$ if and only if $m = m'$ if and only if $\varphi(\delta^*(m,t)) = \varphi(\delta^*(m',s))$. Then, $\varphi$ is well defined and one to one.

Now, we show that $\varphi$ is a homomorphism. For this we have,

$$\varphi(\delta^*(m,t) * \delta^*(m',s)) = \varphi(\delta^*(m \star m',t)) = m \star m' = \varphi(\delta^*(m,t)) \star \varphi(\delta^*(m',s)).$$

Clearly, $\varphi$ is onto. Hence, $\varphi$ is an isomorphism and so

$$\left(\frac{(X \times \{a,b\},\circ',(1,a))}{\delta^*}, *\right) \cong (X,\star,1). \tag{4.3}$$

Therefore, by (4.2) and (4.3), we have

$$(X,\star,1) \cong \frac{(X \times \{a,b\},\circ',(1,a))}{\delta^*} \cong \frac{(X,\circ,1)}{\delta^*}.$$

□

**Corollary 4.4.** *Any infinite set is isomorphic to fundamental $BE$-algebra of itself.*

## 5. Categorical Relations on Dual Hyper $K$-algebras and Dual $BCK$-algebras

In this section we need to connect ($BE$-algebras) dual $BCK$-algebras and (hyper $BE$-algebras) dual hyper $K$-algebras, so apply the results of previous sections. First, briefly introduce the category of dual hyper $K$-algebras (dual $BCK$-algebras). Category $\mathcal{DH}_k$ ($\mathcal{D}_{bck}$), objects: $(X,\circ,1),(Y,\circ',1'),\ldots$ that are dual hyper $K$-algebras ($(X,\star,1),(Y,\star',1'),\ldots$ that are dual $BCK$-algebras). Arrows: $f,g,\ldots$ that are homomorphisms. For two categories $\mathcal{DH}_k$ and $\mathcal{D}_{bck}$, define a categorical morphism as follows:

$$F : \mathcal{DH}_k \to \mathcal{D}_{bck} \text{ by } F(X) = \left(\frac{X}{\delta^*}, *, \delta^*(1)\right), \tag{5.1}$$

where $(X, \circ, 1)$ is a dual hyper $K$-algebra and for any homomorphism $f : (X, \circ, 1) \to (Y, \circ', 1')$, we define

$$F(f) : \left(\frac{X}{\delta^*}, *, \delta^*(1)\right) \to \left(\frac{Y}{\delta^*}, *, \delta^*(1')\right) \text{ by } F(f) = \delta^*(f). \tag{5.2}$$

By Corollary 4.1, $F$ is well-defined and we have the next result.

**Theorem 5.1.** *$F$ is a functor of $\mathcal{DH}_k$ to $\mathcal{D}_{bck}$.*

*Proof.* For any object $(X, \circ, 1)$ of $\mathcal{DH}_k$, by (5.1), $F(X) = \left(\frac{X}{\delta^*}, *, \delta^*(1)\right)$ is a dual $BCK$-algebra and then $F(X)$ is an object in $\mathcal{D}_{bck}$. Now, we show that for any morphism $f : (X, \circ, 1) \to (Y, \circ', 1'), Ff$ is a morphism in $\mathcal{D}_{bck}$. Let $\delta^*(x), \delta^*(y) \in \frac{X}{\delta^*}$. Then, by (5.2),

$$\begin{aligned} Ff(\delta^*(x) * \delta^*(y)) =& Ff(\delta^*(x \circ y)) = \delta^*(f(x \circ y)) \\ =& \delta^*(f(x) \circ' f(y)) = Ff(\delta^*(x)) * Ff(\delta^*(y)). \end{aligned}$$

Hence, if $g : (X, \circ, 1) \to (Y, \circ', 1')$ and $f : (Y, \circ', 1') \to (Z, \circ'', 1'')$ are morphisms in $\mathcal{DH}_k$, then $F(g) : \left(\frac{X}{\delta^*}, *, \delta^*(1)\right) \to \left(\frac{Y}{\delta^*}, *, \delta^*(1')\right)$ by $F(g) = \delta^*(g)$ and $F(f) : (\frac{Y}{\delta^*}, *, \delta^*(1')) \to (\frac{Z}{\delta^*}, *, \delta^*(1''))$ by $F(f) = \delta^*(f)$ are morphisms in $\mathcal{D}_{bck}$. Now,

$$F(f) \circ F(g) = F(f)(F(g)) = F(f)(\delta^*(g)) = \delta^*(f \circ g) = F(f \circ g).$$

Moreover, for $1 : X \to X$, $F(1) = \delta^*(1)$ and then for any $x \in X$,

$$F(1)(x) = \delta^*(1(x)) = \delta^*(x) = 1_{FX}(x).$$

Therefore, $F$ is a functor of $\mathcal{DH}_k$ to $\mathcal{D}_{bck}$. □

*Remark* 5.1. If $X$ is a dual $BCK$-algebra and

$$\mathcal{B}(X) = \{S \in \mathcal{D}_{bck} |\ \text{X is a fundamental dual } BCK\text{-algebra of S}\},$$

by Theorem 4.6, $\mathcal{B}(X) \neq \emptyset$.

Now, for $\mathcal{DH}_k$ and $\mathcal{D}_{bck}$ and any dual $BCK$-algebras $(X, *, 1)$ and $S = \{a, b\}$, define a categorical morphism, as follows:

$$U : \mathcal{D}_{bck} \to \mathcal{DH}_k \text{ by } U(X) = X_S \tag{5.3}$$

and for any dual $BCK$-algebra homomorphism $f : (X, *, 1) \to (Y, *', 1')$ define

$$U(f) : X_S \to Y_S \text{ by } U(f) = (f, 1). \tag{5.4}$$

By Theorem 4.7, $U$ is well defined and now, we have the next result.

**Theorem 5.2.** *$U$ is a faithful functor of $\mathcal{D}_{bck}$ to $\mathcal{DH}_k$.*

*Proof.* For any object $(X, *, 1)$ of $\mathcal{D}_{bck}$, by Theorem 4.6 and 5.3, $U(X) = X_S$ is a dual hyper $K$-algebra and then $U(X)$ is an object in $\mathcal{DH}_k$. Now, we show that for any morphism $f : (X, \circ, 1) \to (Y, \circ', 1'), Uf$ is a morphism in $\mathcal{DH}_k$. Let $(x_1, y_1), (x_2, y_2) \in X_S$. Now, by (5.4),

$$Uf((x_1, y_1) \circ'' (x_2, y_2)) = (f, 1)((x_1, y_1) \circ'' (x_2, y_2))$$

$$=(f(x_1), y_1) \circ'' (f(x_2), y_2) = Uf((x_1, y_1)) \circ'' Uf((x_2, y_2)).$$

Hence, if $g : (X, *, 1) \to (Y, *', 1')$ and $f : (Y, *', 1') \to (Z, *'', 1'')$ are morphisms in $\mathcal{D}_{bck}$, then $U(g) : (X_S, \circ'', 1'') \to (Y_S, \circ'', 1'')$ by $U(g) = (g, 1)$ and $U(f) : (Y_S, \circ'', 1) \to (Z_S, \circ'', 1)$ by $U(f) = (f, 1)$ are morphisms in $\mathcal{DH}_k$. Now,

$$\begin{aligned}(U(f) \circ U(g))(x, y) =& U(f)(U(g))(x, y) = U(f)(g(x), y)\\ =&(f(g(x)), y) = (f \circ g, 1)(x, y) = U(f \circ g)(x, y).\end{aligned}$$

Moreover, for $1 : X \to X$, $U(1) = (1, 1)$ and then for any $(x, y) \in X_S$,

$$U(1)(x, y) = (1, 1)(x, y) = (x, y) = 1_{FX}(x, y).$$

Therefore, $U$ is a functor of $\mathcal{D}_{bck}$ to $\mathcal{DH}_k$. Now, let $(X, *, 1)$ and $(Y, *, 1')$ be objects in $\mathcal{D}_{bck}$, $f_1, f_2 : X \to Y$ be parallel arrows of $\mathcal{D}_{bck}$ and $U(f_1) = U(f_2)$. Then, for any $(x, y) \in X_S$, $U(f_1)(x, y) = U(f_2)(x, y)$ implies that $f_1(x) = f_2(x)$ and so $f_1 = f_2$. Therefore, $U$ is a faithful functor. □

**Theorem 5.3.** *On Objects of* $\mathcal{D}_{bck}$, $F \circ U = 1$.

*Proof.* For any object $(X, *, 1)$ in $\mathcal{D}_{bck}$ by Theorem 4.6, (5.1) and (5.3)

$$(F \circ U)(X, *, 1) = F(X_S, \circ, 1) = \left(\frac{(X_S, \circ, 1)}{\delta^*}, *\right) \cong (X, *, 1).$$ □

**Theorem 5.4.** *For functors* $1, F \circ U : \mathcal{D}_{bck} \to \mathcal{D}_{bck}$ *there exists a transformation* $\tau : 1 \to F \circ U$ *such that is natural.*

*Proof.* For two functors 1 (identity) and $F \circ U$ of $\mathcal{D}_{bck}$ to $\mathcal{D}_{bck}$, define a map $\tau : 1 \to F \circ U$ as follows:

$$\tau : 1(X) \to (F \circ U)(X) \text{ by } \tau(x) = \delta^*(x, 1). \tag{5.5}$$

Now, for any dual $BCK$-algebra homomorphism $f : X \to X'$, consider the following diagram:

$$\begin{array}{ccc} 1(X) & \xrightarrow{\tau_X} & (F \circ U)(X) \\ {\scriptstyle 1(f)}\downarrow & & \downarrow{\scriptstyle F \circ U(f).} \\ 1(X') & \xrightarrow{\tau_{X'}} & (F \circ U)(X') \end{array}$$

For any $r \in X$ by (5.4) and (5.5), we have

$$\begin{aligned}((F \circ U)(f) \circ \tau)x =& (F \circ U)f(\tau(x))\\ =&(F \circ U)(f)(\delta^*(x, 1))\\ =&(\delta^*(f(x)), 1) = \tau_{X'}(f(x))\\ =&\tau_{X'}(1(f)x) = (\tau_{X'} \circ 1(f))x.\end{aligned}$$

Therefore, $\tau : 1 \to (F \circ U)$ is a natural transformation. □

**Theorem 5.5.** *For two functors* $1, U \circ F : \mathcal{DH}_k \to \mathcal{DH}_k$ *there exists a transformation* $\upsilon : 1 \to U \circ F$ *such that is natural.*

*Proof.* For two functors 1 (identity) and $U \circ F$ of $\mathcal{DH}_k$ to $\mathcal{DH}_k$, define a map $\upsilon : 1 \to U \circ F$ as follows:

$$\upsilon : 1(X) \to (U \circ F)(X) \text{ by } \upsilon(x) = (\delta^*(x), 1). \tag{5.6}$$

Now, for morphism homomorphism $f : X \to X'$, consider the following diagram:

$$\begin{array}{ccc} 1(X) & \xrightarrow{\nu_X} & (U \circ F)(X) \\ {\scriptstyle 1(f)}\downarrow & & \downarrow{\scriptstyle U \circ F(f).} \\ 1(X') & \xrightarrow{\nu_{X'}} & (U \circ F)(X') \end{array}$$

For any $r \in X$ by (5.2) and (5.6), we have

$$\begin{aligned} ((U \circ F)(f) \circ \upsilon)x =& (U \circ F)f(\upsilon(x) = (U \circ F)f((\delta^*(x), 1)) \\ =& (\delta^*(f(x)), 1) \\ =& \upsilon_{X'}(f(x)) = \upsilon_{X'}(1(f)x) = (\upsilon_{X'} \circ 1(f))x. \end{aligned}$$

Therefore, $\upsilon : 1 \to (U \circ F)$ is a natural transformation. □

## 6. Conclusion

In the present paper, on any set, we construct a ($BE$-algebra) dual $BCK$-algebra, (hyper $BE$-algebra) dual hyper $K$-algebra. We introduce the notion of fundamental ($BE$-algebra) dual $BCK$-algebra via the fundamental relation $\delta^*$ and investigated some of their useful properties. Moreover,
(i) by every nonempty set, we can construct a (hyper $BE$-algebra) dual hyper $K$-algebra;
(ii) by every nonempty set, we can construct a ($BE$-algebra) dual $BCK$-algebra such that is isomorphic to a fundamental ($BE$-algebra) dual $BCK$-algebra;
(iii) By every nonempty set, we can construct a ($BE$-algebra) dual $BCK$-algebra such that is isomorphic to fundamental ($BE$-algebra) dual $BCK$-algebra of itself;
(iv) We define functors between these categories and construct a natural transformation between their combinations and identity functor.

In our future work, we should be get more results in (hyper $BE$-algebras) dual hyper $K$–algebras and its application.

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[1]Department Of Mathematics,
University Of Payame Noor,
Tehran, Iran
*Email address*: m.hamidi@pnu.ac.ir

[2] Dept. of Pure Math.
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, Iran
*Email address*: arsham@uk.ac.ir