Kragujevac J. Math. 26 (2004) 153–164.

ON NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS BY THE DECOMPOSITION METHOD

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(Received March 12, 2004)

Abstract. We present a reliable algorithm for solving one-dimensional system of nonlinear partial differential equations. We investigate the numerical solution of this problem by using Adomian decomposition method (ADM). The solution is calculated in the form of a series with easily computable components. We prove the convergence of the ADM applied to nonlinear heat equation. Numerical experiments are presented for a set of nonlinear problems from the literature.

1. INTRODUCTION

Over the last twenty years, the Adomian decomposition approach has been applied to obtain formal solutions to a wide class of both deterministic and stochastic PDEs. In recent years, the decomposition method has emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, integral, integro-differential, higher-order ordinary differential equations,partial differential equations (PDEs) and systems [1-8]. These works are summarized in the following. In [1] used the Adomian's technique for solving an elliptical boundary value problem with an auxiliary condition. Adomian et al. [2] solved mathematical models of the dynamic interaction of immune response with a population of bacteria, viruses, antigens or tumor cells have been modelled as systems of nonlinear differential equations or delay-differential equations by the ADM. Abbaoui [3] studied a model of thermic exchanges in a drilling well which was solved with the decomposition method. Ndour et al. [4] presented an example of an interaction model between two species. Guellal et al. [5] used the decomposition method for solving differential systems coming from physics. They gave a comparison between the Runge-Kutta method and the decomposition technique. Abbaoui and Cherruault [6] used the decomposition method for solving the cauchy problem without using the canonical form of Adomian. They also gave a proof of convergence by using a new formulation of the Adomian polynomials and they compared the ADM with the Picard method. In [7], the Adomian's scheme was used for solving differential systems for modelling the HIV immune dynamics. Sanchez et al. [8] investigated the weaknesses of the thin-sheet approximation and proposed a higher-order development allowing to increase the range of convergence and preserving the nonlinear dependence of the variables.

The decomposition method yields rapidly convergent series solutions by using a few iterations for both linear and nonlinear deterministic and stochastic equations. The advantage of this method is that it provides a direct scheme for solving the problem, i.e., without the need for linearization, perturbation, massive computation and any transformation.

The convergence of this method have investigated by Cherruault and co-operators. In [9], Cherruault proposed a new definition of the method and then he insisted that it will become possible to prove the convergence of the decomposition method. In [10], Cherruault and Adomian proposed a new convergence proof of Adomian's method based on properties of convergent series. In [11], Abbaoui et. al., a new approach of decomposition method was obtained in a more natural way than in the classical presentation, was given. Lesnic [12] investigated convergence of Adomian's method to periodic temperature fields in heat conductors.

In this work, we will consider the adaptive numerical approximation of nonlinear evolution equation of the form (reaction-diffusion system):

$$\begin{gathered} u_t - \nabla\left(A\left(x\right)\nabla u\right) = F\left(x,t,u\right),\ x \in \Omega \in \mathbb{R},\ t \in (0,T], \\ u\left(x,0\right) = u_0\left(x\right),\ x \in \Omega, \end{gathered} \tag{1}$$

subject to boundary conditions for the solution vector $u = \left(u_1, u_2, \ldots, u_m\right)^T$ and where $A\left(x\right)$ is the diffusion matrix and the nonlinear vector function $F\left(x,t,u\right)$ describes possible reaction mechanism of the problems. Some assumptions are taken for the data A, F, u_0 in order to assure the existence of a unique local strong solution of the system (1). The reaction-diffusion equations arise in many fields of biology, ecology, chemistry and physics. For instance, the equation are used to describe the dispersive behaviour of cell or animal populations as well as chemical concentrations.

This system can be solved by the different numerical approaches. Some examples are summarized in the following. Comincioli et al. [13] use a wavelet-based method for numerical solution of the system (1). Classical method of lines is considered in [14-16] using a Galerkin-wavelet or collocation-wavelet method for space discretization. In this framework, first the space variable are discretization, usually on a priori selected grid, then the PDE is converted in a system of ODEs, which can be solved by some automatic ODE solver.

Here we are mainly concerned with the performance of the ADM with some numerical test.

The organization of this paper is as the following: We give a brief definition of this method and the proof of convergence in Section 2 and 3, respectively. The accuracy and efficiency of the decomposition method is investigated with numerical illustrations in Section 4. Section 5 consists of a brief conclusions.

2. THE DECOMPOSITION METHOD

The principal algorithm of the Adomian decomposition method when applied to a general nonlinear equation is in the form

$$Lu + Ru + Nu = g. \tag{2}$$

The linear terms are decomposed into $L+R$, while the nonlinear terms are represented by Nu. L is taken as the highest order derivative to avoid difficult integration involving complicated Green's functions, and R is the remainder of the linear operator. L^{-1} is regarded as the inverse operator of L and is defined by a definite integration from 0 to t, i.e.,

$$L^{-1}(.) = \int_0^t \int_0^t (.) \ dt \ dt. \tag{3}$$

If L is a second-order operator, L^{-1} is a two-fold indefinite integral,

$$L^{-1}Lu = u(x,t) - u(x,0) - t\frac{\partial u(x,0)}{\partial t}. \tag{4}$$

Operating on both sides of Eq.(2) with L^{-1} yields

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu, \tag{5}$$

and gives

$$u(x,t) = u(x,0) + t \ u_t(x,0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{6}$$

The decomposition method represents the solution of Eq.(6) as a series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{7}$$

The nonlinear operator, Nu, is decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n. \tag{8}$$

Substituting (7) and (8) into (6), then we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{9}$$

where

$$u_0 = u(x,0) + t\ u_t(x,0) + L^{-1}g. \tag{10}$$

Consequently, it can be written as

$$\begin{aligned} u_1 &= -L^{-1}Ru_0 - L^{-1}A_0, \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1, \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0, \end{aligned} \tag{11}$$

where A_n are Adomian's polynomials of $u_0, u_1, \ldots, u_n$ and are obtained from the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \ldots \tag{12}$$

Eq.(12) gives

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= u_1\frac{d}{du_0}f(u_0), \\ A_2 &= u_2\frac{d}{du_0}f(u_0) + \frac{u_1^2}{2!}\frac{d^2}{du_0^2}f(u_0), \\ A_3 &= u_3\tfrac{d}{du_0}f(u_0) + u_1u_2\tfrac{d^2}{du_0^2}f(u_0) + \tfrac{u_1^3}{3!}\tfrac{d^3}{du_0^3}f(u_0), \\ &\vdots \end{aligned} \tag{13}$$

The accuracy level of the approximation of $u(x,t)$ can be dramatically enhanced by computing components as far as we like. The $n-$term approximant

$$\lim_{n\to\infty} \lim \phi_n = u(x,t) \ \text{ where } \phi_n(x,t) = \sum_{k=0}^{n-1} u_k(x,t), \ n \geq 0, \tag{14}$$

can be used to approximate the solution.

3. CONVERGENCE RESULTS

We consider in the following the hypotheses [17,18]:

$\cdot\,(H_1) \quad (T(u) - T(v), u - v) \geq k\,\|u - v\|^2, k > 0, u, v \in H.$

$\cdot (H_2)$ Whatever may be $M > 0$, there exists a constant $C(M) > 0$ such that for $u, v \in H$ with $\|u\| \leq M, \|v\| \leq M$, we have:

$$(T(u) - T(v), w) \leq C(M) \|u - v\| \ \|w\| \text{ for every } w \in H,$$

where H is a Hilbert space.

Theorem (Sufficient condition of convergence). *If N is Lipschitzian function in H, the Adomian method applied to the following nonlinear heat equation*

$$\frac{\partial}{\partial t}(u) = \frac{\partial^2}{\partial x^2}(u) + f(u),$$

where $f(u)$ is the nonlinear terms.

Proof. We consider the above equation, then we set

$$L(u) = \frac{\partial}{\partial t}(u), \ R(u) = -\frac{\partial^2}{\partial x^2}(u), \ N(u) = -f(u).$$

We have,

$$L(u) = \frac{\partial}{\partial t}(u) = -T(u) = \frac{\partial^2}{\partial x^2}(u) + f(u).$$

This operator T is hemicontinuous. We can the convergence hypothesis (H_1) : i.e. there exists a constant $k > 0$, such that for $u, v \in H$, we have

$$(T(u) - T(v), u - v) \geq k \ \|u - v\|^2,$$

$$T(u) - T(v) = -\frac{\partial^2}{\partial x^2}(u - v) - (f(u) - f(v)),$$

$$(T(u) - T(v), u - v) = \left(-\frac{\partial^2}{\partial x^2}(u - v), u - v\right) - (f(u) - f(v), u - v).$$

But there exists a real $\delta > 0$ such that

$$\left(-\frac{\partial^2}{\partial x^2}(u - v), u - v\right) \geq \delta \ \|u - v\|^2,$$

because

$$\frac{\partial^2}{\partial x^2},$$

is a differential operator in H. In addition,

$$(f(u) - f(v), u - v) \leq \alpha \|u - v\|^2$$

where $\alpha > 0$ is the Lipschitzian constant and therefore

$$(T(u) - T(v), u - v) \geq (\delta - \alpha) \ \|u - v\|^2,$$

and taking $k = \delta - \alpha$, then we obtain hypothesis (H_1).

We can now prove the hypothesis (H_2), i.e. $\forall M > 0$, $\exists C(M) > 0$ such that

$$\|u\| \leq M, \ \|v\| \leq M \Rightarrow (T(u) - T(v), w) \leq C(M) \ \|u - v\| \ \|w\|, \ \forall w \in H.$$

Thus we have

$$\begin{aligned}(T(u) - T(v), w) &\leq \|u - v\| \ \|w\| + \alpha \ \|u - v\| \ \|w\| \\ &\leq C(M) \|u - v\| \ \|w\|,\end{aligned}$$

where $C(M) = 1 + \alpha$. Hence, the hypothesis (H_2) is satisfied.

4. NUMERICAL EXAMPLES

The aim of our work is to present an efficient, robust and reliable method for the solution of nonlinear evolution equations. For illustration purposes we will consider both linear and nonlinear evolution equations in this section. We will show that how the ADM is computationally efficient. To give a clear overview of the methodology, the following test problems will be discussed.

Test problem 1 (Two wavelike solution). We consider the following linear heat equation problem:

$$\begin{aligned}u_t &= u_{xx} + f(x,t), \\ u(x,0) &= f(x),\end{aligned} \tag{15}$$

where $f(x,t)$, $f(x)$ and Dirichlet boundary conditions are chosen so that the exact solution is [16]:

$$\begin{aligned}u(x,t) = &\tanh(10(x-t)+2) - \tanh(10(x-t)+1) \\ &+2\tanh(20(x+2t)-26) - \tanh(29(x+2t)-32).\end{aligned} \tag{16}$$

Proceeding as before, we find the recursive scheme as follows:

$$\begin{aligned}u_0 &= u(x,0) + L_t^{-1}[f(x,t)], \\ &\vdots \\ u_{n+1} &= L_t^{-1} L_x u_n, \quad n \geq 0,\end{aligned} \tag{17}$$

where

$$L_t = \frac{\partial}{\partial t}, \quad L_x = \frac{\partial^2}{\partial x^2}, \quad L_t^{-1}(.) = \int_0^t (.)\ dt. \tag{18}$$

In order to verify numerically whether the proposed methodolgy lead to higher accuracy, we can evaluate the numerical solutions using the n-term approximation (14). Table 1 show the difference of exact solution and approximate solution of the absolute error. We also demonstrate the numerical exact solutions in Figure 1.

x, t	0.01	0.02	0.03	0.04	0.05
0.1	$6.762E-16$	$1.394E-13$	$2.167E-12$	$2.704E-11$	$3.412E-10$
0.2	$1.317E-15$	$2.781E-13$	$4.165E-12$	$5.546E-11$	$6.913E-10$
0.3	$2.197E-15$	$4.196E-13$	$6.268E-11$	$8.343E-11$	$1.046E-09$
0.4	$2.697E-15$	$5.588E-13$	$8.349E-11$	$1.124E-10$	$1.491E-09$
0.5	$3.297E-15$	$6.885E-13$	$1.147E-11$	$1.384E-10$	$1.781E-09$

Table 1. Numerical results for Test problem 1.

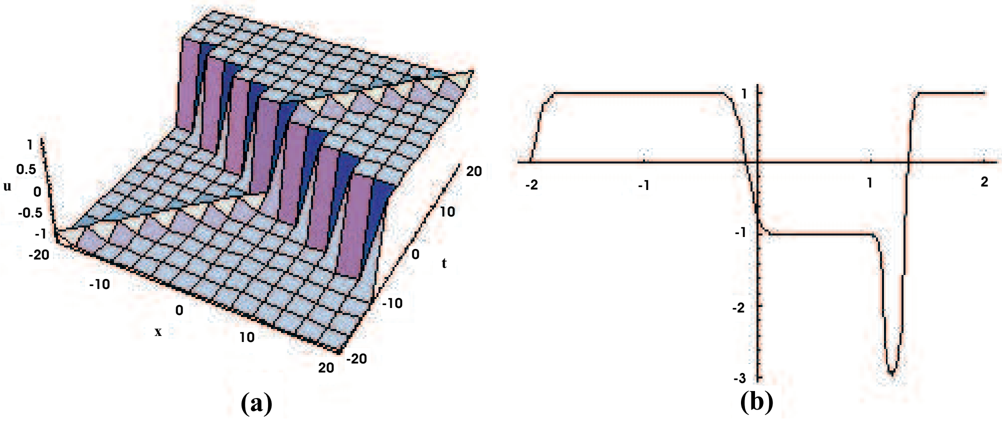

Fig. 1. The surface shows the exact and numerical (ϕ_5) solutions of $u(x,t)$ and its plot at $t = 0$.

Many other iterates could be generated by using *Mathematica*. Table 1 shows the errors obtained upon solving the linear heat equation (15) after normalizing the constants and using only five iterations of the decomposition method. It is to be noted that only 5 iterates were needed to obtain an error of less than $10^{-8}\%$. The overall errors can be made even much smaller by adding new terms of the decomposition.

Test problem 2 (Rational solution). We consider the following the nonlinear reaction-diffusion equation:

$$u_t - u_{xx} = u^2 - (u_x)^2, \tag{19}$$

subject to the initial condition

$$u(x,0) = u_0 = e^x. \tag{20}$$

Since u_0 is known, then the exact solution is obtained the following recursive relation

$$\begin{aligned} u_1 &= L_t^{-1}(L_x u_0) + L_t^{-1}(A_0) - L_t^{-1}(B_0), \\ &\vdots \\ u_{n+1} &= L_t^{-1}(L_x u_n) + L_t^{-1}(A_n) - L_t^{-1}(B_n), \quad n \geq 0, \end{aligned} \tag{21}$$

where A_n and B_n are so-called Adomian polynomials as calculated in [19] according to specific algorithms

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= u_1^2 + 2u_0u_2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ &\vdots \end{aligned} \tag{22}$$

$$\begin{aligned} B_0 &= \left(u_0^2\right)_x, \\ B_1 &= (2u_0u_1)_x, \\ B_2 &= \left(u_1^2 + 2u_0u_2\right)_x, \\ B_3 &= (2u_0u_3 + 2u_1u_2)_x, \\ &\vdots \end{aligned} \tag{23}$$

and so on. To obtain the decomposition solution subject to initial condition given, we first use the equation (19) in an operator form in the same manner as form (6) and then we use (21) to determine the individual terms of the series (7), we get

$$u(x,t) = e^x\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \dots\right), \tag{24}$$

and in a closed form given by

$$u(x,t) = e^{x+t}. \tag{25}$$

This can be justified through substitution.

5. CONCLUSIONS

In this paper, we calculated the exact solution of the reaction-diffusion systems (1) with initial condition by using Adomian decomposition method. We demonstrated that the decomposition procedure is quite efficient to determine solution in closed form by using initial condition. Our present method avoids the tedious work needed by traditional techniques. We got the analytical solution by using only the initial condition in this method. The method avoids the difficulties and massive computational work that usually arise from Parellel techniques, Finite difference method and Crank-Nicolson finite difference method.

It is possible to solve this problem by using both the initial condition and/or the boundary conditions. It will be sufficient to look at the studies [20] to find u_0 by using the boundary conditions.

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