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NONEXPANSIVE MAPPINGS AND CONVEX SEQUENCES

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Abstract. The paper studies the convergence of one convex sequence to a fixed point of nonexpansive mapping on $g-$orbitally complete normed spaces with $\lambda-$uniform convex sphere

1. INTRODUCTION, SOME NEW NOTIONS AND THE MAIN RESULT

Let X be metric space. For mapping $f : E \to E$, $E \subset X$ we say that it is nonextensive if

$$d(f(x), f(y)) < d(x, y) \tag{1}$$

Nonextensive mappings have been widely studied in relation to existantial fixed point in normed spaces, e.g. by Browder [1], Karlovitz [2], Söneberg [9] , Kirk [3], Reinermann [8], Opial [7], etc. A considerable contribution to calculating a fixed point of nonextensive compact operator $f : E \to E$, where E is a closed, limited and convex subset of normed space X, over the sequence

$$x_{n+1} = \frac{x_n + f(x_n)}{2},$$

has been given by Krasnoselskij [4].

This paper introduces notions like $g-$orbital completeness of space and spaces with $\lambda-$uniform convex sphere.

Definition. *Normed space X is considered to be a space with $\lambda-$uniform convex sphere if for every $\varepsilon > 0$ there is $\delta > 0$ so that for all $x, y \in X$ and $\| x - y \| > \varepsilon$ it is true that*

$$\| \lambda x + (1-\lambda) y \| \leq (1-\delta) \max\{\| x \|, \| y \|\}, \lambda \in (0,1) \tag{2}$$

Let there be a mapping $f : E \to E$, $E \subset X$ where X is normed space. Let us define a function

$$g(x, f(x)) = (\lambda_1 + \lambda_2 + \ldots + \lambda_p)^{-1} (\lambda_1 x + \lambda_2 f(x) + \ldots + \lambda_p f^{p-1}(x))$$

$\lambda_i \in R, \lambda_i > 0, i = 1, 2, \ldots, p.$

The set

$$Og(x, f) = \{g_0(x, f(x)), g_1(x, f(x)), g_2(x, f(x)), \ldots\}$$

where $g_0(x, f(x)) = x$ and $g_n(x, f(x)) = g(g_{n-1}(x, f(x)), f(g_{n-1}(x, f(x))))$, is called a sequence of convex orbits given by the function g. If each Cauchy's sequence of $Og(x, f)$ converges to X then space X is $g-$orbitally complete.

Introduction of convex sequences for calculating fixed points of certain mapping is justified by the fact that it is not always possible to reach a fixed point [6].

The point x of a convex set $E \subset X$, where X is normed linear vector space, it is called an extremal point, if $x = \lambda x_1 + (1-\lambda) x_2$, $\lambda \in (0, 1)$, $x_1, x_2 \in E$, then follows $x_1 = x_2 = x$.

Lemma. *Let $f : E \to E$ be completely continual linear operator, where E is a limited subset of normed space X, and J is a set of solutions of equation $x = f(x)$ in E.*

Let there be

$$R(J,\alpha) = \{\alpha |\, x \in E, d(x,J) \geq \alpha\},\ \alpha > 0.$$

Then for every $x \in R(J,\alpha)$ *and every* $\alpha > 0$ *there is* $\varepsilon = \varepsilon(\alpha) > 0$ *so that*

$$\parallel f(x) - x \parallel > \varepsilon,$$

and the set of solutions J *of equation* $x = f(x)$ *is convex.*

The proof of this lemma can be found in [5].

Theorem. *Let there be a complete conctinual operator* $f = E \rightarrow E$ *where* E *is a closed, limited and convex subset of normed space* X *with* $\lambda-$*uniformly convex sphere. If* O *is a extremal point of the set* E*, and for* $p \geq 3$*,* X *is* $g-$*orbitally complete space, and for all* $x, y \in E$ *operator* f *satisfies the condition*

$$\parallel f(x) - f(y) \parallel \leq \parallel x - y \parallel,$$

then the sequence

$$x_n = \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \left(\sum_{i=1}^{p} \lambda_i f^{i-1}(x_{n-1})\right),$$

for $n \in N$*,* $\lambda_i \in R$*,* $\lambda_i \geq 0$*,* $i = 1, 2, \ldots, p$ *and for arbitrary* $x_0 \in E$ *, converges at least to one solution of the equation* $x = f(x)$

Proof. We introduce the following mark

$$J = \{x \mid x \in E,\ x = f(x)\}$$

On the basis of nonexpansiveness of operator f and by definition of a sequence $\{x_n\}_{n \in N}$ we obtain that

$$\begin{aligned} d(x_{n+1}, J) &= \inf_{y \in J} \parallel x_{n+1} - y \parallel \leq \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \cdot \inf_{y \in J} \sum_{i=1}^{p} \lambda_i \parallel f^{i-1}(x_n) - f^{i-1}(y) \parallel \\ &\leq \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \cdot \inf_{y \in J} \sum_{i=1}^{p} \lambda_i \parallel x_n - y \parallel \\ &= \inf_{y \in J} \sum_{i=1}^{p} \lambda_i \parallel x_n - y \parallel \\ &= d(x_n, y) \end{aligned}$$

Consequently, the sequence $\{d(x_n, J)\}$, $n \in N$, nonincreasing.

Suppose now that for certain $\alpha > 0$, $x_1, x_2, \ldots, x_k \in R(J, \alpha)$. On the basis of the previous lemma it follows that for this α there is $\varepsilon(\alpha)$ so that

$$\| f(x_i) - x_i \| > \varepsilon(\alpha), \; i = 1, 2, \ldots, k.$$

Since space X with $\lambda-$uniformly convex sphere, for each $y \in J$ we obtain:

$$\begin{aligned}
\| x_2 - y \| &= \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \left\| \sum_{i=1}^{p} \lambda_i f^{i-1}(x_1) - \sum_{i=1}^{p} \lambda_i y \right\| \\
&\leq \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} (\lambda_1 + \lambda_2) \left\| \tfrac{\lambda_1}{\lambda_1+\lambda_2}(x_1 - y) + \tfrac{\lambda_2}{\lambda_1+\lambda_2}(fx_1 - fy) \right\| + \\
&\quad + \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \sum_{i=3}^{p} \lambda_i \| f^{i-1}(x_1) - f^{i-1}(y) \| \\
&\leq \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} (\lambda_1 + \lambda_2)(1-\delta) \max\{\| x_1 - y \|, \| fx_1 - fy \|\} + \\
&\quad + \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \sum_{i=3}^{p} \lambda_i \| x_1 - y \| \\
&\leq \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \left((\lambda_1 + \lambda_2)(1-\delta) + \sum_{i=1}^{p} \lambda_i\right) \cdot 2M,
\end{aligned}$$

where

$$M = \sup_{t \in E} \| t \|$$

Similarly, we prove that

$$\| x_k - y \| \leq 2M \cdot \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \cdot \left((\lambda_1 + \lambda_2)(1-\delta) + \sum_{i=3}^{p} \lambda_i\right)^{k-1}$$

then it follows that

$$d(x_k, J) \leq 2M \cdot \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \cdot \left((\lambda_1 + \lambda_2)(1-\delta) + \sum_{i=3}^{p} \lambda_i\right)^{k-1}$$

Since $x_i \in R(J, \alpha)$, then also $\| f(x_i) - x_i \| \geq \varepsilon$ for $i = 1, 2, \ldots, k$, and consequently

$$\varepsilon \leq \| f(x_i) - x_i \| \leq \| f(x_i) - f(y) \| + \| y - x_i \| \leq 2 \| x_i - y \|$$

that is $\| x_i - y \| \geq \frac{\varepsilon}{2}$ for $i = 1, 2, \ldots, k,\ y \in E$

Now, we have

$$2M \cdot \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \cdot \left((\lambda_1 + \lambda_2)(1-\delta) + \sum_{i=3}^{p} \lambda_i\right)^{k-1} \geq \frac{\varepsilon}{2}$$

This inequality is valid if

$$k < 1 + (\ln 4M - \ln \varepsilon) \cdot \left(-\ln \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \cdot \left((\lambda_1 + \lambda_2)(1-\delta) + \sum_{i=3}^{p} \lambda_i\right)\right)^{-1}$$

Since the sequence $\{d(x_n, J)\}$, $n \in N$ is nonincreasing, for

$$n > 1 + (\ln 4M - \ln \varepsilon) \cdot \left(-\ln \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \cdot \left((\lambda_1 + \lambda_2)(1-\delta) + \sum_{i=3}^{p} \lambda_i\right)\right)^{-1}$$

it is valid that

$$d(x_n, J) < \alpha,$$

then it follows

$$\lim_{n \to \infty} d(x_n J) = 0. \tag{3}$$

Based on the relation (3) it follows that for every $\beta > 0$ there is n_0 so that $d(x_{n_0}, J) < \frac{\beta}{2}$,which implies that for certain $y_0 \in J$ it is true $d(x_{n_0}, y) < \frac{\beta}{2}$

For $m_1, m_2 > n_0$ there are inequalities

$$\| x_{m_1} - x_{m_2} \| \leq \| x_{m_1} - y_0 \| + \| y_0 - x_{m_2} \| \leq \frac{\beta}{2} + \frac{\beta}{2} = \beta$$

Therefore, the sequence $\{x_n\}_{n \in N}$ is Cauchy's and since space X is g-orbitally complete, then it is convergent in E. Let there be $\lim_{n \to \infty} x_n = \xi$

From the relation

$$\lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \left(\sum_{i=1}^{p} \lambda_i\right)^{-1} \sum_{i=1}^{p} \lambda_i f^{i-1}(x_n) = \xi$$

we get that

$$\sum_{i=1}^{p} \lambda_i \left(\xi - f^{i-1}(\xi)\right) = 0$$

Since 0 is extremal point it is valid that $\xi = f(\xi)$ so sequence $\{x_n\}_{n \in N}$ converges to at least one solution of equation $x = f(x)$.

This completes our proof.

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