Kragujevac J. Math. 23 (2001) 45–51.

ON WEIGHTED INEQUALITY OF HARDY TYPE FOR HIGHER ORDER DERIVATIVES

Raman Adegoke[a] **and James Adedayo Oguntuase**[b]

[a]*Department of Mathematics and Computer Science, Federal College of Education, Osiele, Abeokuta, Nigeria, and*
[b]*Department of Mathematical Sciences, University of Agriculture Abeokuta, Nigeria*

(Received February 3, 1998)

Abstract. In this paper, we obtain a new weighted inequality of Hardy type for higher order derivatives which generalized the recent result of Stepanov [8].

1. INTRODUCTION

Since Opial [7] results on integral inequalities involving functions and their derivatives was published, a lot of work has been done on it due to its usefulness in the study of differential and integral equations (see for example, Das [2], Levinson [5], Stepanov [8], Imoru [4] and Cheung [1]).

Definition 1. *Let* $(r(x), s(x)) \geq 0,\ x \in \Re$ *and let* $1 \leq p \leq q \leq p' \leq \infty$. *If* $k(x,y) \geq 0$ *is defined on* $\triangle = \left\{(x,y) \in \Re^2 : y < x\right\}$, *then we shall say that the pair of weight functions* $(r(x), s(x))$ *satisfies the* $A(k,p,q)$ *condition with a constant* C *if* *there exists a real number* β, $0 \le \beta \le 1$ *such that for every real number* x,

$$\Big\{\int_x^\infty k(y,x)^{\beta q} r(x)^q dy\Big\}^{1/q}\Big\{\int_{-\infty}^x k(x,y)^{(1-\beta)p'} s(y)^{-p'} dy\Big\}^{1/p'} \le C < \infty.$$

We shall now state and prove some Lemmas needed in the proof of our main result.

Lemma 1. *If* $f^{(j)}(a) = 0$, *for all* $j = 0, 1, 2, \cdots, n-1$, *then*

$$f^{(j)}(x) = \frac{1}{(n-j-1)!}\int_a^x (x-t)^{n-j-1} f^{(n)}(t)dt$$

Proof. This can be obtained by the reverse induction process on j.

Lemma 2. *For any* $f \ge 0$ *and any* $\alpha > 0$ *holds*

$$\int_a^b f(x)\Big[\int_a^x f(t)dt\Big]^\alpha dx = \frac{1}{(\alpha+1)}\Big[\int_a^b f(x)dx\Big]^{\alpha+1}.$$

Proof. Let

$$F(x) = \int_a^x f(t)dt.$$

Then

$$F'(x) = f(x)dx.$$

Therefore

$$\begin{aligned}\int_a^b f(x)\Big[\int_a^x f(t)dt\Big]^\alpha dx &= \int_a^b F(x)^\alpha dF(x)\\ &= \frac{1}{(\alpha+1)}\Big[\int_a^b f(x)dx\Big]^{\alpha+1}.\end{aligned}$$

Hence

$$\int_a^b f(x)\Big[\int_a^x f(t)dt\Big]^\alpha dx = \frac{1}{(\alpha+1)}\Big[\int_a^b f(x)dx\Big]^{\alpha+1}.$$

Lemma 3. *Let* $k(x,y) \geq 0,\ (x,y) \in \triangle = \{(x,y) \in \Re^2 : y < x\}$. *Suppose* $1 \leq p \leq q \leq p'$, *then*

$$\int_{-\infty}^{x} k(x,y)^{(1-\beta)p'} s(y)^{-p'} h(y)^{-p'} dy \;=\; \frac{2p'-q}{3p'-q} h(x)^{-(3p'-q)}.$$

Proof. Let

$$J(x) \;=\; \int_{-\infty}^{x} k(x,y)^{(1-\beta)p'} s(y)^{-p'} h(y)^{-p'} dy.$$

Define h by

$$h(y) \;=\; \left\{ \int_{-\infty}^{y} k(y,z)^{(1-\beta)p'} s(z)^{-p'} dz \right\}^{-\frac{1}{(2p'-q)}}.$$

Hence

$$\begin{aligned} J(x) &= \int_{-\infty}^{x} k(x,y)^{(1-\beta)p'} s(y)^{-p'} \Big[\int_{-\infty}^{y} k(y,z)^{(1-\beta)p'} s(z)^{-p'} dz \Big]^{\frac{p'}{(2p'-q)}} dy \\ &\leq \int_{-\infty}^{x} k(x,y)^{(1-\beta)p'} s(y)^{-p'} \Big[\int_{-\infty}^{y} k(x,z)^{(1-\beta)p'} s(z)^{-p'} \Big]^{\frac{p'}{(2p'-q)}} dy. \end{aligned}$$

Since $k(.,z)$ is nondecreasing and $x > y$ we have

$$\begin{aligned} J(x) &= \frac{2p'-q}{3p'-q} \Big[\int_{-\infty}^{x} k(x,z)^{(1-\beta)p'} s(z)^{-p'} dz \Big]^{\frac{3p'-q}{2p'-q}} \\ &= \frac{2p'-q}{3p'-q} h(x)^{-(3p'-q)} \end{aligned}$$

by Lemma 2 and the proof is complete.

Lemma 4. *If* $k(x,y) \geq 0,\ (x,y) \in \triangle$ *and* $1 \leq p \leq q \leq p' \leq \infty$. *Then*

$$\begin{aligned} \left\{ \int_{y}^{\infty} k(x,y)^{\beta q} r(x)^q h(x)^{-\frac{q(3p'-q)}{p'}} dx \right\}^{q/p} &\leq C^{\frac{p(3p'-q)}{2p'-q}} \Big[\frac{2p'-q}{p'} \Big]^{p/q} \\ &\times \left\{ \int_{y}^{\infty} k(z,y)^{\beta q} r(z)^q dz \right\}^{\frac{-pp'}{q(2p'-q)}}. \end{aligned}$$

Proof. Let

$$\begin{aligned} J(y) &= \int_y^\infty k(x,y)^{\beta q} r(x)^q h(x)^{-\frac{q(3p'-q)}{p'}} \\ &= \int_y^\infty k(x,y)^{\beta q} r(x)^q \left\{\int_{-\infty}^x k(x,z)^{(1-\beta)p'} s(z)^{-p'} dz\right\}^{\frac{q(3p'-q)}{p'(2p'-q)}} . \end{aligned}$$

Since $(r(x), s(x))$ satisfies the $A(k,p,q)$ condition with constant C, we have

$$\left\{\int_{-\infty}^x k(x,z)^{(1-\beta)p'} s(z)^{-p'} dz\right\}^{\frac{q(3p'-q)}{p'(2p'-q)}} \leq C^{\frac{p(3p'-q)}{2p'-q}} \left\{\int_x^\infty k(z,x)^{\beta q} r(z)^q dz\right\}^{\frac{(q-3p')}{2p'-q}} .$$

The fact that $k(z,.)$ is nonincreasing gives

$$\begin{aligned} J(y) &= C^{\frac{q(3p'-q)}{2p'-q}} \int_y^\infty k(x,y)^{\beta q} r(x)^q \left\{\int_x^\infty k(z,x)^{\beta q} r(z)^q dz\right\}^{\frac{(q-3p')}{2p'-q}} \\ &\leq C^{\frac{q(3p'-q)}{2p'-q}} \int_y^\infty k(x,y)^{\beta q} r(x)^q \left\{\int_x^\infty k(z,y)^{\beta q} r(z)^q dz\right\}^{\frac{(q-3p')}{2p'-q}} . \end{aligned}$$

By Lemma 2 we have

$$J(y) = C^{\frac{q(3p'-q)}{2p'-q}} \Big[\frac{2p'-q}{p'}\Big] \left\{\int_y^\infty k(z,y)^{\beta q} r(z)^q dz\right\}^{\frac{-p'}{(2p'-q)}} .$$

Hence

$$J(y)^{p/q} = C^{\frac{p(3p'-q)}{2p'-q}} \Big[\frac{2p'-q}{p'}\Big]^{p/q} \left\{\int_y^\infty k(z,y)^{\beta q} r(z)^q dz\right\}^{\frac{-pp'}{q(2p'-q)}} .$$

This completes the proof of the Lemma.

2. THE MAIN RESULT

Theorem 1. *Let f be a function that vanish at a or b together with its derivatives up to and including $(j-1)$. Suppose $r(x)$ and $s(x)$ are nonnegative weight functions such that $(r(x), s(x))$ satisfies*

$$\Big[\int_a^\infty k(x-a)^{(j-1)\beta q} r(x)^q dx\Big]^{1/q} \Big[\int_{-\infty}^a k(x-a)^{(j-1)(1-\beta)p'} s(x)^{-p'} dx\Big]^{1/p'} \leq B < \infty. \tag{1}$$

Furthermore, if $1 \le p \le q \le p' \le \infty$ *and* $2p' - q > 0$, *then*

$$\left\{\int_a^b \Big[r(x)f(x)\Big]^q dx\right\}^{1/q} \;\le\; C\left\{\int_a^b \Big[s(x)f^{(j)}(x)\Big]^p dx\right\}^{1/p}, \tag{2}$$

where

$$C \;=\; B\frac{1}{(k-1)!}\frac{(2p'-q)^{1/p}}{(3p'-q)}\frac{(2p'-q)^{1/q}}{p'}.$$

Proof. Let f be supported on (a,b) and define h by

$$h(y) \;=\; \Big[\int_y^\infty k(y-z)^{(j-1)(1-\beta)p'} s(z)^{-p'} dz\Big]^{-\frac{1}{(2p'-q)}}.$$

Then

$$\begin{aligned} I &= \left\{\int_a^b \Big[r(x)f(x)\Big]^q dx\right\}^{1/q} \\ &= \frac{1}{(j-1)!}\left\{\int_a^b r(x)^q \Big[\int_a^x (x-y)^{(j-1)} f^{(j)}(y)dy\Big]^p dx\right\}^{1/q} \end{aligned}$$

by Lemma 1 and

$$\begin{aligned} I &= \frac{1}{(j-1)!}\left\{\int_a^b r(x)^q \Big[\int_a^x (x-y)^{\beta(j-1)} f^{(j)}(y)s(y)h(y)\right. \\ &\quad \left.\times (x-y)^{(j-1)(1-\beta)} s(y)^{-1} h(y)^{-1} dy\Big]^q dx\right\}^{1/q} \\ &\le \frac{1}{(j-1)!}\left\{\int_a^b r(x)^q \Big[\int_a^x (x-y)^{(j-1)\beta p}\Big[f^{(j)}(y)s(y)h(y)\Big]^p dy\Big]^{q/p}\right. \\ &\quad \left.\times \Big[\int_a^x (x-y)^{(j-1)(1-\beta)p'} s(y)^{-p'} h(y)^{-p'} dy\Big]^{q/p'} dx\right\}^{1/q} \end{aligned}$$

by Holder's inequality. Hence

$$I \le \frac{1}{(j-1)!}\left\{\int_a^b r(x)^q \Big[\int_a^x (x-y)^{(j-1)\beta p}\Big[f^{(j)}(y)s(y)h(y)\Big]^p dy\Big]^{q/p}\right.$$

$$\left.\times \quad \Big[\int_a^x (x-y)^{(j-1)(1-\beta)p'} s(y)^{-p'} \Big[\int_y^\infty (x-y)^{(j-1)(1-\beta)p'} s(z)^{-p'} dz\Big]^{\frac{p'}{(2p'-q)}} dy\Big]^{q/p'} dx\right\}^{1/q}$$

$$
\begin{aligned}
&\leq \frac{1}{(j-1)!}\Big\{\int_a^b r(x)^q\Big[\int_a^x (x-y)^{(j-1)\beta p}\Big[f^{(j)}(y)s(y)h(y)\Big]^p dy\Big]^{q/p}\\
&\times \Big[\int_a^x (x-y)^{(j-1)(1-\beta)p'}s(y)^{-p'}\Big[\int_a^y (x-z)^{(j-1)(1-\beta)p'}s(z)^{-p'}dz\Big]^{\frac{p}{(2p'-q)}}dy\Big]^{q/p'}dx\Big\}^{1/q}.
\end{aligned}
$$

Lemma 3 and the fact that $k(.,z)$ is nondecreasing gives

$$
\begin{aligned}
I &\leq \frac{1}{(j-1)!}\Big\{\int_a^b r(x)^q\Big[\int_a^x (x-y)^{(j-1)\beta p}\Big[f^{(j)}(y)s(y)h(y)\Big]^p dy\Big]^{q/p}\\
&\quad\times\Big[\frac{(2p'-q)}{(3p'-q)}h(x)^{-(3p'-q)}\Big]^{1/p'}dx\Big\}^{1/q}\\
&= \frac{1}{(j-1)!}\Big[\frac{(2p'-q)}{(3p'-q)}\Big]^{1/p'}\Big\{\int_a^b r(x)^q\Big[\int_a^x (x-y)^{(j-1)\beta p}\Big[f^{(j)}(y)s(y)h(y)\Big]^p dy\Big]^{q/p}\\
&\quad\times h(x)^{-\frac{q(3p'-q)}{p'}}dx\Big\}^{q/p}.
\end{aligned}
$$

By Minkowski's inequality we have

$$
\begin{aligned}
I &\leq \frac{1}{(j-1)!}\Big[\frac{(2p'-q)}{(3p'-q)}\Big]^{1/p'}\Big\{\int_a^b\Big[\int_a^x (x-y)^{(j-1)\beta q}r(x)^q h(x)^{-\frac{q(3p'-q)}{p'}}dx\Big]^{p/q}\\
&\quad\times\Big[f^{(j)}(y)s(y)h(y)\Big]^p dy\Big\}^{1/p}.
\end{aligned}
$$

By Lemma 4 we have

$$
\begin{aligned}
I &\leq \frac{1}{(j-1)!}\Big[\frac{(2p'-q)}{(3p'-q)}\Big]^{1/p'}\Big[\frac{(2p'-q)}{p'}\Big]^{p/q}C^{\frac{(3p'-q)}{2p'-q}}\Big\{\int_a^b\Big[f^{(j)}(y)s(y)\Big]^p\\
&\times\Big[\int_a^y (x-y)^{(j-1)(1-\beta)p'}s(z)^{-p'}dz\Big]^{-\frac{p}{2p'-q}}\Big[\int_y^a (z-y)^{(j-1)\beta q}r(z)^q dz\Big]^{-\frac{pp'}{q(2p'-q)}}dy\Big\}^{1/p'}.
\end{aligned}
$$

Since $(r(x),s(x))$ satisfies equation (1), then we have

$$
\Big[\int_a^y (x-y)^{(j-1)(1-\beta)p'}s(z)^{-p'}dz\Big]^{-\frac{p}{2p'-q}} \leq C^{\frac{pp'}{2p'-q}}\Big[\int_a^y (z-y)^{(j-1)\beta q}r(z)^q dz\Big]^{-\frac{pp'}{q(2p'-q)}}
$$

$$
\leq \frac{1}{(j-1)!}\Big[\frac{(2p'-q)}{(3p'-q)}\Big]^{1/p'}\Big[\frac{(2p'-q)}{p'}\Big]^{1/q}C\Big\{\int_a^b\Big[f^{(j)}(y)s(y)\Big]^p\Big\}^{1/p}.
$$

Hence

$$
\Big\{\int_a^b\Big[r(x)f(x)\Big]^q dx\Big\}^{1/q} \leq C\Big\{\int_a^b\Big[s(x)f^{(j)}(x)\Big]^p dx\Big\}^{1/p}.
$$

Remark 1. If we set $p = q = 2$ in Theorem 1., then we shall obtain

$$\left\{\int_a^b \left[r(x)f(x)\right]^2 dx\right\}^{1/2} \leq C\left\{\int_a^b \left[s(x)f^{(j)}(x)\right]^2 dx\right\}^{1/2}$$

which is a recent result obtained by Stepanov [8].

Acknowledgements: The authors are grateful to the referee for his useful comments which have improved the final presentation of this paper.

References

[1] W. S. Cheung, *Some new Opial-type inequalities*, Mathematika **37** (1990), 136–142.

[2] K. M. Das, *An inequality similar to Opial's inequality*, Proc. Amer. Math. Soc. **22** (1969), 258–261.

[3] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1959.

[4] C. O. Imoru, *Weighted norm inequalities of Opial and Hardy-type for higher order derivatives*, I.C.T.P. preprints IC/90/461, 1990.

[5] N. Levinson, *On an inequality of Opial and Beesack*, Proc. Amer. Math. Soc. **22** (1964), 555–566.

[6] J. A. Oguntuase, R. Adegoke, *Weighted norm inequality for some integral operators*, Coll. Sci. Papers Fac. Sci. Kragujevac **21** (1999), 55–62.

[7] Z. Opial, *Sur une inegalite*, Ann. Polon Math. **8** (1960), 29–32.

[8] V. D. Stepanov, *A weighted inequality of Hardy-type for higher order derivatives*, Trud. Math. Inst. Steklov **187** (1989), 178–190.