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A CHARACTERIZATION OF MAXIMAL SPECTRA IN TERMS OF RELATIONS

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Abstract. *In this paper V.B.Kudryavcev's maximal clases of the functions with delay (maximal spectra) are described in terms of relations.*

Let P_k denote a set of all functions of an algebra of a k-valued logic.

Definition 1. *A spectrum is an infinite sequence*

$$\mathcal{F} = (F_0, F_1, \dots, F_d, \dots)$$

of subsets of P_k.

We shall denote a spectrum by $\mathcal{F} = (F_d)_{d=0,1,2,\dots}$.

Definition 2. *If* $\mathcal{F} = (F_d)_{d=0,1,2,\dots}$ *and* $\mathcal{G} = (G_d)_{d=0,1,2,\dots}$ *are two spectra, we say that a spectrum* $\mathcal{F}$ *is a subspectrum of* $\mathcal{G}$, *denoted by* $\mathcal{F} \subseteq \mathcal{G}$, *if* $F_d \subseteq G_d$ *for every* $d = 0, 1, 2, \dots$.

Definition 3. *Let* $\mathcal{F} = (F_d)_{d=0,1,2,\dots}$ *be a given spectrum. We say that a spectrum* $\tilde{\mathcal{F}} = (\tilde{F}_d)_{d=0,1,2,\dots}$ *is a closure, or a* $\sim$*-closure of the spectrum* $\mathcal{F}$ *if*
(1) $\tilde{F}_0 = \overline{F}_0$, *where* $\overline{F}_0 = [F_0]$,
(2) $\tilde{F}_d = (\tilde{F}_0 \otimes (F_d \otimes \tilde{F}_0)) \cup (\cup_{i=1}^{d-1}(\tilde{F}_i \otimes \tilde{F}_{d-1})$ *for* $d = 1, 2, \dots$,
where for $F, G \subseteq P_k$ *we have*

$$F\otimes G = \{f(g_1(x_{11}, x_{12}, \dots, x_{1m_1}), g_2(x_{21}, x_{22}, \dots, x_{2m_2}), \dots g_n(x_{n1}, x_{n2}, \dots, x_{nm_n})),$$

$$|f \in F, g_j \in G \quad (j = 1, 2, \dots, n)\}.$$

Definition 4. *A spectrum* $\mathcal{F} = (F_d)_{d=0,1,2,\dots}$ *is closed (or* $\sim$*-closed) if* $\mathcal{F} = \tilde{\mathcal{F}}$.

Definition 5. *A spectrum* $\mathcal{F} = (F_d)_{d=0,1,2,\dots}$ *is said to be complete (or* $\sim$*-complete) if*

$$\cup_{d=0}^{\infty} \tilde{F}_d = P_k.$$

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Definition 6. *A spectrum $\mathcal{F} = (F_d)_{d=0,1,2,\ldots}$ is said to be maximal (or $\sim$-maximal) if $\mathcal{F}$ is not complete ($\sim$-complete) and for any $\mathcal{G}$ properly including $\mathcal{F}$, $\mathcal{G}$ is complete ($\sim$-complete).*

Theorem 1. ([1]). *A spectrum $\mathcal{F}$ is complete is and only if it is not a subset of any maximal spectrum.*

From this theorem it follows that it is useful to know maximal spectra.

Let us define the types of spectra (according to [1]).

Definition 7. *A spectrum $\mathcal{F} = (F_d)_{d=0,1,2,\ldots}$ is called of type (A) if there exists a maximal set M in P_k such that $F_d = M$ for all $d = 0, 1, 2, \ldots$.*

Definition 8. *A spectrum $\mathcal{F} = (F_d)_{d=0,1,2,\ldots}$ is called of type (B) is there exist an m-ary polyrelation $\overline{\rho} = (\rho_0, \rho_1, \ldots, \rho_{p-1})$ on $E_k = \{0, 1, 2, \ldots, k-1\}$ with period p, such that*

$$F_d = \cap_{i=0}^{p-1} F(\rho_i, \rho_{i\oplus d})$$

for all $d = 0, 1, 2, \ldots$, where

$$F(\rho_i, \rho_{i\oplus d}) = \{f \in P_k | f(\rho_i) \subseteq \rho_{i\oplus d}\}$$

and $\oplus$ denotes the addition modulo p.

Remind that for a function f and an m-ary relation ρ, an m-ary relation $f(\rho)$ is defined by

$$f(\rho) = \{f(a_0^0, a_0^1, \ldots, a_0^{n-1}), f(a_1^0, a_1^1, \ldots, a_1^{n-1}), \ldots, f(a_{m-1}^0, a_{m-1}^1, \ldots, a_{m-1}^{n-1})$$

$$| f(a_0^0, a_1^0, \ldots, a_{m-1}^0), f(a_0^1, a_1^1, \ldots, a_{m-1}^1), \ldots, f(a_0^{n-1}, a_1^{n-1}, \ldots, a_{m-1}^{n-1}) \in \rho\}.$$

Definition 9. *A spectrum $\mathcal{F} = (F_d)_{d=0,1,2,\ldots}$ is called of type (C) if there exist an m-ary relation ρ and an m-ary diagonal Δ such that*

$$F_0 = F(\rho, \rho)$$

and

$$F_d = F(\rho, \Delta) \quad (d = 1, 2, 3, \ldots).$$

Note that for each fixed value of k there exist only a finite number of spectra of types (A) and (C), but an infinite number of those of type (B).

Theorem 2. ([1]). *A maximal spectrum over E_k is either of type (A), or of type (B) defined by an m-ary polyrelation with $1 \leq m \leq k$ and period $p \geq 2$, or of type (C) defined by m-ary relations with $1 \leq m \leq k$.*

We can associate a spectrum $\{ = (F_d)_{d=0,1,2,\ldots}$ to each set S of functions with delay in the following way:

$$f \in F_d \longleftrightarrow (f, d) \in S \quad (d = 0, 1, 2, \ldots).$$

The converse holds, too: to each spectrum we can associate a set of functions with delay, So, the terms a spectrum and set of functions with delay are mathematically equivalent; hence we can use one term or other depending on the wanted aim.

V.B.Kudryavcev gave all maximal classes in the set $\mathcal{P}_2 = \{(f,d)|f \in P_2, \quad d = 0,1,2,\dots\}$. Here we shall characterize Kudtyavcev's maximal classes (maximal spectra) in terms of relations.

I. Spectra associated to maximal classes $\tilde{L}$, $\tilde{S}$,$\tilde{M}$, $\tilde{T}_0$,and $\tilde{T}_1$ are of type (A) and they are characterized by relations on E_2 which characterize maximal clases L, S, M, T_0, and T_1 in P_2.

1. The spectrum $\mathcal{F}_{\tilde{L}} = (F_d)_{d=0,1,2,\dots}$, where $F_d = L$ for every $d = 0,1,2,\dots$ is characterized by the relation $\rho_L = \{(0,0,0,0), (0,0,1,1), (0,1,0,1), (0,1,1,0), (1,0,0,1), (1,0,1,0), (1,1,0,0), (1,1,1,1)\}$.

2. The spectrum $\mathcal{F}_{\tilde{S}} = (F_d)_{d=0,1,2,\dots}$, where $F_d = S$ for every $d = 0,1,2,\dots$ is characterized by the relation $\rho_S = \{(0,1),(1,0)\}$.

3. The spectrum $\mathcal{F}_{\tilde{M}} = (F_d)_{d=0,1,2,\dots}$, where $F_d = M$ for every $d = 0,1,2,\dots$ is characterized by the relation $\rho_M = \{(0,0),(0,1),(1,1)\}$.

4. The spectrum $\mathcal{F}_{\tilde{T}_0} = (F_d)_{d=0,1,2,\dots}$, where $F_d = T_0$ for every $d = 0,1,2,\dots$ is characterized by the relation $\rho_2 = \{(0)\}$.

5. The spectrum $\mathcal{F}_{\tilde{T}_1} = (F_d)_{d=0,1,2,\dots}$, where $F_d = T_1$ for every $d = 0,1,2,\dots$ is characterized by the relation $\rho_1 = \{(1)\}$.

II. Spectra associated to maximal classes $\tilde{C}$, $\tilde{E}_0$, $\tilde{E}_1$,and $\tilde{H}$ are of type (C).

6. We associate the sectrum $\mathcal{F}_{\tilde{C}} = (F_d)_{d=0,1,2,\dots}$ to the class $\tilde{C}$, where $F_0 = A$ (the set of α-functions) and $F_d = B \cup \Gamma$ (the union of the sets β- and γ-functions). Since there exist the binary relation $\rho = \{(0,1)\}$ and diagonal $\Delta = \{(0,0),(1,1)\}$ such that

$$F_0 = A = F(\rho,\rho)$$

and

$$F_d = B \cup \Gamma = F(\rho,\Delta) \quad (d = 1,2,3,\dots).$$

it follows that the spectrum $\mathcal{F}_C = (F_d)_{d=0,1,2,\dots}$ is of type (C).

7. We associate the spectrum $\mathcal{F}_{\tilde{E}_0} = (F_d)_{d=0,1,2,\dots}$ to the class $\tilde{E}_0$, where $F_0 = A \cup B$ and $F_d = \{0,1\}$ for $d = 1,2,3,\dots$. Since there exist the binary relation $\rho = \{(0,1),(1,1)\}$ and diagonal $\Delta = \{(0,0),(1,1)\}$ such that

$$F_0 = A \cup B = F(\rho,\rho)$$

and

$$F_d = \{0,1\} = F(\rho,\Delta) \quad (d = 1,2,3,\dots).$$

it follows that the spectrum $\mathcal{F}_{\overline{E}_0}$ is of type (C).

8. We associate the spectrum $\mathcal{F}_{\tilde{E}_1} = (F_d)_{d=0,1,2,\dots}$ to the class $\tilde{E}_1$, where $F_0 = A \cup \Gamma$ and $F_d = \{0,1\}$ for $d = 1,2,3\dots$.The obtained spectrum is of type (C) because there exist the binary relation $\rho = \{(0,0),(0,1)\}$ and diagonal Δ such that

$$F_0 = A \cup \Gamma = F(\rho,\rho)$$

and

$$F_d = \{0,1\} = F(\rho, \Delta) \quad (d = 1,2,3,\dots).$$

9. We associate the spectrum $\mathcal{F}_{\tilde{H}} = (F_d)_{d=0,1,2,\dots}$ to the class $\tilde{H}$, where $F_0 = S$ and $F_d = Y$ (the set of even functions) for $d = 1,2,3\dots$. The obtained spectrum is of type (C) because there exist the binary relation $\rho = \{(0,1),(1,0)\}$ and diagonal Δ such that

$$F_0 = S = F(\rho, \rho)$$

and

$$F_d = Y = F(\rho, \Delta) \quad (d = 1,2,3,\dots).$$

III. Spectra associsted to maximal classes $\tilde{W}_r$ and $\tilde{Z}_r$ $(r = 0,1,2,\dots)$ are type (B).

10. We associate the sectrum $\mathcal{F}_{\tilde{W}_r} = (F_d)_{d=0,1,2,\dots}$ to the class $\tilde{W}_r$, $(r = 0,1,2,\dots)$ where $F_{2^r(2q)} = M \quad (q = 0,1,2,\dots,$ $F_{2^r(2q+1)} = \overline{M} = \{f | f \in P_2 \wedge \overline{f} \in M\}$ $(q = 0,1,2,\dots$ and $F_d = \{0,1\}$ $(d \neq p2^r, p = 0,1,2,\dots)$. It is easy to see that there exists the polyrelation $\overline{\rho} = (\rho_0, \rho_1, \dots, \rho_{2^{r+1}-1})$ such that $\rho_0 = \{(0,0),(0,1),(1,1)\}$, $\rho_{2^r} = \rho^{-1}$ and $\rho_s = \Delta$ $(s \neq 0, 2^r)$, where

$$F_d = \cap_{i=0}^{2^{r+1}-1} F(\rho_i, \rho_{i \oplus d}) \quad (d = 0,1,2,\dots)$$

and $\oplus$ denotes the addition modulo 2^{r+1}; so it follows that the obtained spectrum is of type (B).

11. We associate the spectrum $\mathcal{F}_{\tilde{Z}_r} = (F_d)_{d=0,1,2,\dots}$ to the class $\tilde{Z}_r$, $(r = 0,1,2,\dots)$ where $F_{2^r(2q)} = A \quad (q = 0,1,2,\dots,$ $F_{2^r(2q+1)} = \Delta$ (the set of δ-functions) $(q = 0,1,2,\dots$ and $F_d = \emptyset$ $(d \neq p2^r, p = 0,1,2,\dots)$.

Since there exists the polyrelation $\overline{\rho} = (\rho_0, \rho_1, \dots, \rho_{2^{r+1}-1})$ such that $\rho_0 = \{(0,1)\}$, $\rho_{2^r} = \{(1,0)\}$ and $\rho_s = \emptyset$ $(s \neq 0, 2^r)$, where

$$F_d = \cap_{i=0}^{2^{r+1}-1} F(\rho_i, \rho_{i \oplus d}) \quad (d = 0,1,2,\dots)$$

and $\oplus$ denotes the addition modulo 2^{r+1}; so it follows that the obtained spectrum is of type (B).

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