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ON SOME CRITERIA FOR STARLIKENESS IN THE UNIT DISC

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Abstract. *In this paper by using the Ruscheweyh's derivative [3] and certain properties of the classes M_n, defined earlier by the author [4], some new criteria for starlikeness in the unit disc will be given.*

1. Introduction

Let A denote the class of functions analytic in the unit disc $U = \{z : |z| < 1\}$ and normalized with $f(0) = f'(0) - 1 = 0$.

By M_n, $n \in N_0 = N \cup \{0\} = \{0, 1, 2, \dots\}$, we denote the class of functions $f \in A$ defined by the condition

$$(1) \qquad \Re\left\{\frac{D^{n+1}}{D^n f}\right\} < \frac{2n+3}{2n+2}, \quad z \in U,$$

where $D^n f = \big(z/(1-z)^{n+1}\big) * f$ denotes the derivative introduced by Ruscheweyh [3] ("$*$" means the Hadamard product or convolution of two analytic functions).

In [2] it is prooved that $M_{n+1} \subset M_n$ holds for all $n \in N_0$ and that M_n, $n \in N$, is the subclass of univalent functions. Morever, M_n, $n \in N$, is the subclass of starlike functions (S^*) in the unit disc.

Let f and F be analytic in U. Then we say that f is subordinate to F, written by $f \prec F$ of $f(z) \prec F(z)$, if there exists an analytic function $\omega(z)$ in U, such that $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in U$, and $f(z) = F(\omega(z))$.

In this paper we give some criteria for starlikeness in the unit disc by using the Ruscheweyh's derivative, the previous cited facts on the class M_n, and by applying the following well-known Jack's lemma [1].

Lemma. *Let ω be nonconstant and analytic in U with $\omega(0) = 0$. If $|\omega|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0\omega'(z_0) = k\omega(z_0)$, $k \geq 1$.*

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2. Some criteria for starlikeness

Theorem 1. *Let $f \in A$, $\alpha \geq 0$, $\beta < \alpha$, $n \in N_0$, and let*

$$\Re\left\{\alpha\frac{D^{n+2}f}{D^{n+1}f}+\beta\frac{D^{n}f}{D^{n+1}f}\right\}<\alpha+\beta+\frac{\alpha-\beta}{2(n+2)},\quad z\in U. \tag{2}$$

Then f is starlike in U.

Proof. Let us show that condition (2) implies

$$\frac{D^{n+1}f}{D^{n}f}\prec(1-z)\left(1-\frac{n+1}{n+2}z\right)^{-1}. \tag{3}$$

In that sense, let's put

$$\frac{D^{n+1}f}{D^{n}f}=\frac{1-\omega(z)}{1-a\omega(z)},\quad a=\frac{n+1}{n+2}. \tag{4}$$

Evidently $\omega(0)=0$. We want to prove that $|\omega(z)|<1$, $z\in U$. From (4), after taking logarithmical differentiation and using the identity

$$z(D^{m}f)'=(m+1)D^{m+1}f-mD^{m}f,$$

we get

$$\frac{D^{n+2}f}{D^{n+1}f}=\frac{1}{n+2}\left[1+(n+1)\frac{1-\omega(z)}{1-a\omega(z)}-\frac{z\omega'(z)}{1-\omega(z)}+\frac{az\omega'(z)}{1-a\omega(z)}\right], \tag{5}$$

and now

$$\begin{aligned}&\alpha\frac{D^{n+2}f}{D^{n+1}f}+\beta\frac{D^{n}f}{D^{n+1}f}=\\&=\frac{\alpha}{n+2}\left[1+(n+1)\frac{1-\omega(z)}{1-a\omega(z)}-(1-a)\frac{z\omega'(z)}{(1-\omega(z))(1-a\omega(z))}\right]+\\&+\beta\frac{1-a\omega(z)}{1-\omega(z)}.\end{aligned} \tag{6}$$

If not $|\omega(z)|<1$, $z\in U$, then there exists a z_0, $|z_0|<1$ (by Jack's lemma) such that $|\omega(z_0)|=1$, $z_0\omega'(z_0)=k\omega(z_0)$, where $k\geq 1$. If we put $\omega(z_0)=e^{i\theta}$, then from (6) we have

$$\begin{aligned}&\Re\left\{\alpha\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)}+\beta\frac{D^{n}f(z_0)}{D^{n+1}f(z_0)}\right\}=\\&=\frac{\alpha}{n+2}\left[1+\frac{n+1}{a}-\frac{1}{2}+\frac{k-1}{2}\,\frac{1-a^2}{1-2a\cos\theta+a^2}\right]+\beta\frac{1+a}{2}\geq\\&\geq\frac{\alpha}{n+2}\left(\frac{1}{2}+\frac{n+1}{a}\right)+\beta\frac{1+a}{2}=\alpha+\beta+\frac{\alpha-\beta}{2(n+2)},\end{aligned}$$

which is a contradiction to (2). It follows that (3) is true. From (3), we obtain

$$\Re\left\{\frac{D^{n+1}f}{D^n f}\right\} < 2\left(1+\frac{n+1}{n+2}\right)^{-1} = \frac{2n+4}{2n+3} < \frac{2n+3}{2n+2}, \quad z \in U,$$

i.e. $f \in M_n$, $n \in N$. For $n = 0$, from (3) we have

$$\frac{D^1 f}{D^0 f} = \frac{zf'(z)}{f(z)} \prec \frac{1-z}{1-z/2},$$

which gives $\Re\left\{\dfrac{zf'(z)}{f(z)}\right\} > 0$, $z \in U$. Therefore, the statement of Theorem 1 is true. We note that the condition $\beta < \alpha$ is necessary since the expression on the left side of (2) has the value $\alpha + \beta$ for $z = 0$.

Remark 1. If we choose different values α, β and n in Theorem 1, then we may get some criteria for starlikeness. For example, for $n = 0$ we have that the condition

$$\Re\left\{\alpha\left(1+\frac{1}{2}\frac{zf''(z)}{f'(z)}\right)+\beta\frac{f(z)}{zf'(z)}\right\} < \alpha+\beta+\frac{\alpha-\beta}{4}, \quad z \in U. \tag{7}$$

where $\alpha \geq 0$, $\beta < \alpha$, implies $f \in S^*$. The condition (7) we may write in the form

$$\Re\left\{\frac{\alpha}{2}\left(1+\frac{1}{2}\frac{zf''(z)}{f'(z)}\right)+\beta\frac{f(z)}{zf'(z)}\right\} < \frac{3}{4}(\alpha+\beta), \quad z \in U, \tag{8}$$

and from there (for $\alpha = 2$, $\beta = -1$),

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}-\frac{f(z)}{zf'(z)}\right\} < \frac{3}{4}, \quad z \in U,$$

which gives that $f \in S^*$ and $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1-z}{1-z/2}$.

Theorem 2. *Let $f \in A$, $\alpha \geq 0$, $\alpha+\beta \geq 0$ and $n \in N_0$. If*

$$\left|\frac{D^{n+2}f}{D^{n+1}f}-1\right|^{\alpha}\left|\frac{D^{n+1}f}{D^n f}-1\right|^{\beta} < (2n+3)^{-\beta}(2n+4)^{-\alpha}, \quad z \in U, \tag{9}$$

then $f \in S^*$.

PROOF. As in the proof of Theorem 1, we want to show that the relation (3) is satisfied. If we use the same substitution for $D^{n+1}f/D^n f$ as in (4) and apply the same method, we have the same relation (5). Let us show that $|\omega(z)| < 1$, $z \in U$.

On the contrary, by Jack's lemma, there exists a z_0, $|z_0|<1$, such that $|\omega(z_0)|=1$ and $z_0\omega'(z_0)=k\omega(z_0)$, $k\geq 1$. If we put $\omega(z_0)=e^{i\theta}$, then for such z_0 we get

$$\begin{aligned}
&\left|\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)}-1\right|^{\alpha}\left|\frac{D^{n+1}f(z_0)}{D^{n}f(z_0)}-1\right|^{\beta}=\\
&=(n+2)^{-\alpha}\left|(n+1)\frac{(a-1)e^{i\theta}}{1-ae^{i\theta}}+k\frac{(a-1)e^{i\theta}}{(1-ae^{i\theta})(1-e^{i\theta})}\right|^{\alpha}\left|\frac{(a-1)e^{i\theta}}{1-ae^{i\theta}}\right|^{\beta}=\\
&=\frac{(1-a)^{\alpha+\beta}}{(n+2)^{\alpha}}\frac{1}{|1-ae^{i\theta}|^{\alpha+\beta}}\left|n+1+k\frac{1}{1-e^{i\theta}}\right|^{\alpha}\geq\\
&\geq\frac{(1-a)^{\alpha+\beta}}{(n+2)^{\alpha}}\frac{1}{(1+a)^{\alpha+\beta}}(n+1+1/2)^{\alpha}=(2n+3)^{-\beta}(2n+4)^{-\alpha},
\end{aligned}$$

which is a contradiction to (9). Now, as in Theorem 1 we conclude that $f\in S^*$.

Remark 2. If in the previous theorem we choose $n=0$, then we have that the condition (9) has the form

$$\left|\frac{1}{2}\frac{zf''(z)}{f'(z)}\right|^{\alpha}\left|\frac{zf'(z)}{f(z)}-1\right|^{\beta}<3^{-\beta}4^{-\alpha},$$

or equivalently,

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\alpha}\left|\frac{zf'(z)}{f(z)}-1\right|^{\beta}<2^{-\alpha}3^{-\beta}. \tag{10}$$

From (10) we easily obtain that both of the following conditions

$$\left|\frac{zf''(z)}{f'(z)}\left(\frac{zf'(z)}{f(z)}-1\right)\right|<\frac{1}{6},\quad z\in U,$$

and

$$\left|\frac{zf''(z)}{f'(z)}\Big/\left(\frac{zf'(z)}{f(z)}-1\right)\right|<\frac{3}{2},\quad z\in U,$$

imply that $f\in S^*$ and $\frac{zf'(z)}{f(z)}\prec\frac{1-z}{1-z/2}$.

Also, from (10) we have that

$$\left|\frac{zf''(z)}{f'(z)}\right|^{\alpha}\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\alpha}<2^{-\alpha}3^{\alpha-1},\quad z\in U,\quad \alpha\geq 0,$$

implies $f\in S^*$ and $\frac{zf'(z)}{f(z)}\prec\frac{1-z}{1-z/2}$. The similar result was given earlier in [4].

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