FILOMAT (Niš) 13 (1999), 53–57

SOME REMARKS ON (m,n)-RINGS

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Abstract. Among the results of the paper is the following proposition. Let (Q,A,M) be an (m,n)-ring and let $\mathbf{O}$ the $\{1,m\}-$neutral operation of the $m-$group (Q,A). Then for every $i \in \{1,\ldots,n\}$ and for every $a_1^{n-1}, c_1^{m-2} \in Q$ the following equality holds

$$M(a_1^{i-1}, \mathbf{O}(c_1^{m-2}), a_i^{n-1}) = \mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})}\Big|_{j=1}^{m-2}).$$

1. Preliminaries

1.1. Definition: *Let $n \geq 2$ and let (Q,A) be an $n-$groupoid. We say that (Q,A) is a Dörnte $n-$group [briefly: $n-$group] iff is an $n-$semigroup and an $n-$quasigroup as well.*

1.2. Proposition [11]: *Let $n \geq 2$ and let (Q,A) be an $n-$groupoid. Then the following statements are equivalent: (i) (Q,A) is an $n-$group; (ii) there are mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q,\{A,^{-1},\mathbf{e}\})$ [of the type $< n, n-1, n-2 >$].*

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$

(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; *and*

Received September 20, 1999

2000 *Mathematics Subject Classification.* 20N15.

Key words and phrases. $n-$groupoids, $n-$semigroups, $n-$quasigroups, $n-$groups, $\{i,j\}-$neutral operations on $n-$groupoids, inversing operation on $n-$group, $(m,n)-$ring.

A notion of an $n-$group was introduced by W. Dörnte in [1] as a generalization of the notion of a group. See, also $[5-7]$.

(iii) there are mappings $^{-1}$ *and* $\mathbf{e}$ *respectively of the sets* Q^{n-1} *and* Q^{n-2} *into the set* Q *such that the following laws hold in the algebra* $(Q, \{A, ^{-1}, \mathbf{e}\})$ *[of the type* $< n, n-1, n-2 >$*]*

$(\overline{a})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

$(\overline{b})$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

$(\overline{c})$ $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$.

1.3. Remarks: $\mathbf{e}$ is an $\{1, n\}$−neutral operation of n−groupoid (Q, A) iff algebra $(Q, \{A, \mathbf{e}\})$ of type $< n, n-2 >$ satisfies the laws (b) and $(\overline{b})$ from 1.2 *[:[8]]*. The notion of $\{i, j\}$−neutral operation $(i, j \in \{1, \ldots, n\}, i < j)$ of an n−groupoid is defined in a similar way *[:[8]]*. Every n−groupoid has at most one $\{i, j\}$−neutral operation *[:[8]]*. In every n−group, $(n \geq 2)$, there is an $\{1, n\}$−neutral operation *[:[8]]*. There are n−groups without $\{i, j\}$−neutral operations with $\{i, j\} \neq \{1, n\}$*[:[10]]*. In [10], n−groups with $\{i, j\}$−neutral operations, for $\{i, j\} \neq \{1, n\}$ are described. Operation $^{-1}$ from 1.2 *[(c), $(\overline{c})$]* is a generalization of the inversing operation in a group. In fact, if (Q, A) is an n−group, $n \geq 2$, then for every $a \in Q$ and for every sequence a_1^{n-2} over Q is

$$(a_1^{n-2}, a)^{-1} \overset{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2}),$$

where E is an $\{1, 2n-1\}$−neutral operation of the $(2n-1)$−group $(Q, \overset{2}{A})$; $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$ *[:[9]]*. (For $n = 2$, $a^{-1} = \mathsf{E}(a)$; a^{-1} is the inverse element of the element a with respect to the neutral element $\mathbf{e}(\emptyset)$ of the group (Q, A).)

1.4. Proposition [10]: *Let* $n \geq 3$*, let* (Q, A) *be an* n*−group and* $\mathbf{e}$ *its* $\{1, n\}$*−neutral operation. Then the following statements are equivalent:* (i) (Q, A) *is a commutative* n*−group; and* (ii) $\mathbf{e}$ *is an* $\{i, j\}$*−neutral operation of the* n*−group* (Q, A) *for every* $\{i, j\} \subseteq \{1, \ldots, n\}, i < j$.

1.5. Definition: *Let* (Q, A) *be an commutative* m*−group and* $m \geq 2$*. Let also* (Q, M) *be an* n*−groupoid (*n*−semigroup in* [2,3]*) and* $n \geq 2$*. We say that* (Q, A, M) *is an* (m, n)**−ring** *iff for every* $i \in \{1, \ldots, n\}$ *and for every* $a_1^{n-1}, b_1^m \in Q$ *the following equality holds*

$$(o) \qquad M(a_1^{i-1}, A(b_1^m), a_i^{n-1}) = A(\overline{M(a_1^{i-1}, b_j, a_i^{n-1})}\Big|_{j=1}^{m}).$$

A notion of an (m, n)−ring was introduced by G. Čupona in [2] as a generalization of the notion of a ring. See, also[3, 4].

2. Results

2.1. Theorem: *Let (Q,A,M) be an $(m,n)-$ring and let* $\mathbf{O}$ *the* $\{1,m\}-$*neutral operation of the* $m-$*group* (Q,A) *[:1.2,1.3]. Then for every* $i \in \{1,\dots,n\}$ *and for every* $a_1^{n-1}, c_1^{m-2} \in Q$ *the following equality holds*

$$(1) \quad M(a_1^{i-1}, \mathbf{O}(c_1^{m-2}), a_i^{n-1}) = \mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}}^{m-2}).$$

Proof. 1) Let

$$(2) \quad A^{-1}(a_1^{n-1}, x) = y \overset{def}{\Leftrightarrow} A(a_1^{n-1}, y) = x$$

for every $a_1^{n-1}, x, y \in Q$. Then the following statements hold:

1° For every $i \in \{1,\dots,n\}$ and for every $a_1^{n-1}, c_1^{m-2}, x, y \in Q$ the following equality holds

$$M(a_1^{i-1}, A^{-1}(x, c_1^{m-2}, y), a_i^{n-1}) =$$
$$A^{-1}(M(a_1^{i-1}, x, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}}^{m-2}, M(a_1^{i-1}, y, a_i^{n-1})).$$

2° For every $c_1^{m-2}, x \in Q$ the following equality holds
$A^{-1}(x, c_1^{m-2}, x) = \mathbf{O}(c_1^{m-2})$.

Sketch of the proof of 1° :

$A^{-1}(x, c_1^{m-2}, y) = z \Leftrightarrow A(x, c_1^{m-2}, z) = y$;

$M(a_1^{i-1}, A(x, c_1^{m-2}, z), a_i^{n-1}) = M(a_1^{i-1}, y, a_i^{n-1}), \Leftrightarrow$

$$A(M(a_1^{i-1}, x, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}}^{m-2}, M(a_1^{i-1}, z, a_i^{n-1})) =$$
$$M(a_1^{i-1}, y, a_i^{n-1}) \Leftrightarrow$$

$$A^{-1}(M(a_1^{i-1}, x, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}}^{m-2}, M(a_1^{i-1}, y, a_i^{n-1})) =$$
$$M(a_1^{i-1}, z, a_i^{n-1});$$

$$A^{-1}(M(a_1^{i-1}, x, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}}^{m-2}, M(a_1^{i-1}, y, a_i^{n-1})) =$$
$$M(a_1^{i-1}, A^{-1}(x, c_1^{m-2}, y), a_i^{n-1}) \quad [:(2),1.5].$$

Sketch of the proof of 2° :

$A^{-1}(x, c_1^{m-2}, x) = \mathbf{O}(c_1^{m-2}) \Leftrightarrow A(x, c_1^{m-2}, \mathbf{O}(c_1^{m-2})) = x$ *[:(2),1.2,1.3]*.

Finaly, by 1° and 2° we conclude that for every $i \in \{1,\dots,n\}$ and for every $a_1^{n-1}, c_1^{m-2}, x \in Q$ the following series of equalities holds:

$M(a_1^{i-1}, \mathbf{O}(c_1^{m-2}), a_i^{n-1}) = M(a_1^{i-1}, A^{-1}(x, c_1^{m-2}, x), a_i^{n-1}) =$

$A^{-1}(M(a_1^{i-1}, x, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}}^{m-2}, M(a_1^{i-1}, x, a_i^{n-1})) =$

$\mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}^{m-2}})$.

Remarks:

a) For $m = n = 2$: (1) $a \cdot 0 = 0 \cdot a = 0$.

b) $\mathbf{O}$ *is an* $\{i,j\}-$*neutral operation of the* $m-$*group* (Q, A) *for every* $\{i,j\} \subseteq \{1,\dots,m\}, i < j$ *[:1.2-1.5].* □

2.2. Theorem: *Let* (Q, A, M) *be an* $(m,n)-$*ring and let* $-$ *the inversing operation in* $m-$*group* (Q, A). *Then for every* $i \in \{1,\dots,n\}$ *and for every* $a_1^{n-1}, c_1^{m-2}, b \in Q$ *the following equality holds:*

(3) $M(a_1^{i-1}, -(c_1^{m-2}, b), a_i^{n-1}) = -(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}^{m-2}}, M(a_1^{i-1}, b, a_i^{n-1}))$.

Sketch of the proof.

1) $\mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}^{m-2}}) = M(a_1^{i-1}, \mathbf{O}(c_1^{m-2}), a_i^{n-1}) =$
$M(a_1^{i-1}, A(b, c_1^{m-2}, -(c_1^{m-2}, b)), a_i^{n-1}) =$
$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}^{m-2}}, M(a_1^{i-1}, -(c_1^{m-2}, b), a_i^{n-1}))$
*[:*2.1,1.2*]*;

2) $\mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}^{m-2}}) =$
$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}^{m-2}}, M(a_1^{i-1}, -(c_1^{m-2}, b), a_i^{n-1}))$
*[:*1)*]*;

3) $\mathbf{O}(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}^{m-2}}) =$
$A(M(a_1^{i-1}, b, a_i^{n-1}), \overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{j=1}^{m-2}}, -(\overline{M(a_1^{i-1}, c_j, a_i^{n-1})\big|_{i=1}^{m-2}},$
$M(a_1^{i-1}, b, a_i^{n-1})))$ *[:*1.2*]*;

4) (3) *[:*2),3)*]*.

Remark: *For* $m = n = 2$: (3) $a \cdot (-b) = -(a \cdot b)$ □

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